COLLEGE ALGEBRA AND TRIGONOMETRY

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CHAPTER OVERVIEW

1: Algebra Review

This College Algebra text will cover a combination of classical algebra and analytic geometry, with an introduction to the transcendental exponential and log arithmic functions. There are also chapters devoted to the theory of functions, combinatorics and trigonometry. If mathematics is the language of science, then algebra is the grammar of that language. Like grammar, algebra provides a structure to mathematical notation, in addition to its uses in problem solving and its ability to change the appearance of an expression without changing the value.

1.1: Algebraic Simplification

When algebraic techniques are presented as skills in isolation, they are much simpler to understand and practice. However the problem solving process in any context involves deciding which skills to use when. Most College Algebra students will have practiced problems in the form:

$$\begin{gathered}(x+7)(x-2)=?\\ \text{or}\\ (2x+1)^2=?\end{gathered}\tag{1.1.1}$$

The problems in this section deal with a combination of these processes which are often encountered as parts of more complex problems.

Example 1.1.1

Simplify:

$$3(x-1)(2x+5)-(x+4)^2$$

Solution

In this example, the simplification involves two expressions: $3(x-1)(2x+5)$ and $(x+4)^2$. The $(x+4)^2$ is preceded by a negative (or subtraction) sign. This textbook will often treat $-x$ and $+(-x)$ as equivalent statements, since subtraction is defined as the addition of a negative.

We will simplify each expression separately and then look to combine like terms.

$$3(x-1)(2x+5)-(x+4)^2=3\left(2x^2+3x-5\right)-\left(x^2+8x+16\right)$$

Notice that the results of both multiplications remain inside of parentheses. This is because each one has something that must be distributed.

In the case of $\left(2x^2+3x-5\right)$, there is a 3 which must be distributed, resulting in $6x^2+9x-15$. In the case of $\left(x^2+8x+16\right)$ there is a negative sign or -1 which must be distributed, resulting in $-x^2-8x-16$. It is important in these situations that the negative sign be distributed to all terms in the parentheses.

So

$$\begin{aligned}3(x-1)(2x+5)-(x+4)^2&=3\left(2x^2+3x-5\right)-\left(x^2+8x+16\right)\\&=6x^2+9x-15-x^2-8x-16\\&=5x^2+x-31\end{aligned}$$

Example 1.1.2

Simplify:

$$2(x+3)^2-4(3x-1)(x+2)$$

Solution

This example shows some of the same processes as the previous example. There are again two expressions that must be simplified, each of which has a coefficient that must be distributed. It is often helpful to wait until after multiplying the binomials before distributing the coefficient. However, as is often true in mathematics, there are several different approaches that may be taken in simplifying this problem.

If someone prefers to first distribute the coefficient before multiplying the binomials, then the coefficient must only be distributed to ONE of the binomials, but not both. For example, in multiplying $3*2*5=30$, we can first multiply $2*5=10$

and then $3 * 10 = 30$. Each factor is multiplied only once.
In the example above we can proceed as we did with the previous example:

$$\begin{aligned} 2(x+3)^2-4(3x-1)(x+2) &= 2\left(x^2+6x+9\right)-4\left(3x^2+5x-2\right) \\ &= 2x^2+12x+18-12x^2-20x+8 \\ &= -10x^2-8x+26 \end{aligned}$$

Or, we can choose to distribute the 4 first:

$$\begin{aligned} 2(x+3)^2-4(3x-1)(x+2) &= 2\left(x^2+6x+9\right)-(12x-4)(x+2) \\ &= 2x^2+12x+18-\left(12x^2+20x-8\right) \\ &= 2x^2+12x+18-12x^2-20x+8 \\ &= -10x^2-8x+26 \end{aligned}$$

Or, we can distribute the 4 as a negative. If we do this, then the sign in front of the parentheses will be positive:

$$\begin{aligned} 2(x+3)^2-4(3x-1)(x+2) &= 2\left(x^2+6x+9\right)+(-12x+4)(x+2) \\ &= 2x^2+12x+18+\left(-12x^2-20x+8\right) \\ &= -10x^2-8x+26 \end{aligned}$$

Distributing the 2 in front of the squared binomial must also be handled carefully if you choose to do this. If you distribute the 2 before squaring the $(x+3)$, then the 2 will be squared as well. If you choose to distribute the 2, the $(x+3)^2$ must be written out as $(x+3)(x+3)$

$$\begin{aligned} 2(x+3)^2-4(3x-1)(x+2) &= 2(x+3)(x+3)-4(3x-1)(x+2) \\ &= (2x+6)(x+3)-4\left(3x^2+5x-2\right) \\ &= 2x^2+12x+18-12x^2-20x+8 \\ &= -10x^2-8x+26 \end{aligned}$$

Most examples in this text will distribute the coefficients as the last step before combining like terms for a final answer.

Example 1.1.3

Simplify:

$$3x[5-(2x+7)]+(3x-2)^2-(x-5)(x+4)$$

This example has three expressions that should be simplified separately before combining like terms. In the first expression $3x[5-(2x+7)]$, we should simplify inside the brackets before distributing the $3x$

$$\begin{aligned} & 3x[5-(2x+7)]+(3x-2)^2-(x-5)(x+4) \\ = & \ 3x[5-2x-7]+(3x-2)^2-(x-5)(x+4) \\ = & \ 3x[-2x-2]+(3x-2)(3x-2)-\left(x^2-x-20\right) \\ = & \ -6x^2-6x+\left(9x^2-12x+4\right)-x^2+x+20 \\ = & \ -6x^2-6x+9x^2-12x+4-x^2+x+20 \\ = & \ 2x^2-17x+24 \end{aligned}$$

Exercises 1.1.1

Simplify each expression.

1. $(x-2)[2x-2(3+x)]-(x+5)^2$
2. $3x^2-[7x-2(2x-1)(3-x)]$
3. $(a+b)^2-(a+b)(a-b)-\left[a(2b-2)-\left(b^2-2a\right)\right]$

4. $5x-3(x-2)(x+7)+3(x-2)^2$
5. $(m+3)(m-1)-(m-2)^2+4$
6. $(a-1)(a-2)-(a-2)(a-3)+(a-3)(a-4)$
7. $2a^2-3(a+1)(a-2)-[7-(a-1)]^2$
8. $2(x-5)(3x+1)-(2x-1)^2$
9. $6y+(3y+1)(y+2)-(y-3)(y-8)$
10. $6x-4(x+10)(x-1)+(x+1)^2$

Answers

1. $-x^2-16x-13$
2.
3. $3b^2$
4.
5. $6m-3$
6.
7. $-2a^2+19a-58$
8.
9. $2y^2+24y-22$
10.

1.2: Factoring

This section will review three of the most common types of factoring - factoring out a Greatest Common Factor, Trinomial Factoring and factoring a Difference of Squares.

Greatest Common Factor

Factoring out a greatest common factor essentially undoes the distributive multiplication that often occurs in mathematical expressions. This factor may be monomial or polynomial, but in these examples, we will explore monomial common factors.

In multiplying $3xy^2(5x-2y)=15x^2y^2-6xy^3$ the monomial term $3xy^2$ is multiplied or distributed to both terms inside the parentheses. The process of **factorization** undoes this multiplication.

Example 1.2.1

Factor

$$7x^2+14x$$

Solution

This expression has two terms. The coefficients share a common factor of 7 and the only variable involved in this expression is x. The highest power of the variable that is shared by both terms is x^1, so this is the power of x that can be factored out of both terms. The greatest common factor is $7x$

$$7x^2+14x=7x(x+2)$$

It isn't necessary to find the greatest common factor right away. In more complicated problems, the factoring can be accomplished in pieces, similar in fashion to reducing fractions.

Example 1.2.2

Factor

$$42x^2y^6+98xy^3-210x^3y^2$$

Solution

This expression has three terms. It's not immediately clear what the greatest common factor of the coefficients is, but they're all even numbers, so we could at least divide them all by 2. The $98xy^3$ term has an x^1, which means that this is the hightest power of x that we could factor out of all the terms. The $210x^3y^2$ has a y^2, which is the highest power of y that can be factored out of all the terms. So we can at least proceed with these factors:

$$\begin{aligned}42x^2y^6+98xy^3-210x^3y^2 &= 2xy^2*21xy^4+2xy^2*49y-2xy^2*105x^2\\ &= 2xy^2\left(21xy^4+49y-105x^2\right)\end{aligned}$$

Now, we didn't try very hard to find the greatest common factor in the beginning of this problem, so it's important that we continue to question whether or not there are any remaining common factors. The 21 and 49 clearly share a common factor of 7, so it would make sense to see if 105 is divisible by 7 as well. If we divide 105 by 7, we see that $105=7^*15$. So, we can also factor out a common factor of 7 from the remaining terms in the parentheses.

$$\begin{aligned}2xy^2\left(21xy^4+49y-105x^2\right) &= 2xy^2\left(7*3xy^4+7*7y-7*15x^2\right)\\ &= 7*2xy^2\left(3xy^4+7y-15x^2\right)\\ &= 14xy^2\left(3xy^4+7y-15x^2\right)\end{aligned}$$

Trinomial Factoring $(a=1)$

Trinomial factoring undoes the multiplication of two binomials, and it comes in two flavors - simple and complex. The simplest form of trinomial factoring involves a trinomial expression in the form ax^2+bx+c in which the value of a is 1 This makes the task of factorization simpler than if the value of a is not 1.

Example 1.2.3

Factor $x^2 + 7x + 10$

Solution

In this example, the value of a is 1, which makes this type of trinomial factoring a little less difficult that it would otherwise be. Whether or not the value of a is 1 the fundamental issue that governs this type of factoring is the $+$ or $-$ sign of the constant term. In this problem, the constant term is positive. That means that we need to find factors of 10 that add up to 7. This is relatively straightforward:

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

A companion problem to this one is $x^2 - 7x + 10$. Notice that, in this case, the sign of the constant term is still positive, which means that we still need factors of 10 that add up to 7. This means we still need to use 2 and 5. However, in this case, instead of the +10 being produced from a multiplication of (+2)(+5) it is the result of multiplying $(-2)(-5)$. This is what makes the 7 in the second example negative:

$$x^2 - 7x + 10 = (x - 2)(x - 5)$$

Example 1.2.4

Factor $x^2 + 3x - 10$

Solution

In this case, the sign of the constant term is negative. That means that we need to find factors of 10 that have a difference of 3. This is still 5 and 2.

$$x^2 + 3x - 10 = (x - 2)(x + 5)$$

The multiplication of the (-2) and the (+5) produce the (-10) and the fact that the 2 and 5 have opposite signs creates the difference that gives us $(+3)$. A companion problem to this one is $x^2 - 3x - 10$. In this case, the sign of the constant term is still negative, which means that we still need factors of 10 that have a difference of 3. This means we still need to use 2 and 5. However in this case, instead of the (+3) as the coefficient of the middle term, we'll need a (-3) . To do this we simply reverse the signs of the 2 and 5 from the previous problem:

$$x^2 - 3x - 10 = (x + 2)(x - 5)$$

Now the (+2)(-5) gives us (-10), but the $+2x - 5x$ gives us $(-3x)$ instead of $(+3x)$.

Example 1.2.5

Factor $x^2 + 11x - 42$

Solution

In this problem, the sign of the constant term is negative. That means that we need factors of 42 that have a difference of 11. A systematic exploration of all the factor pairs of 42 can help us to find the correct pair:

1	42
2	21
3	14
6	7

Here, we can see that the factors 3 and 14 have a difference of 11. This means that we will use these factors in our answer: $(x \quad 3)(x$
$14)$. In determining how to place the $+$ and $-$ signs in the parentheses, we can refer back to the original problem: $x^2+11x-42$. If we want a difference of $(+11x)$, then we'll need to have $a(+14)$ and $a(-3)$

$$x^2+11x-42=(x-3)(x+14)$$

Example 1.2.6

Factor $x^2+28x+96$

Solution

In this problem, the sign of the constant term is positive. That means that we need factors of 96 that add up to 28. A systematic exploration of all the factor pairs of 96 can help us to find the correct pair:

1	96
2	48
3	32
4	24
6	16
8	12

Here, we can see that the factors 4 and 24 add up to 28. This means that we will use these factors in our answer: $(x \quad 4)(x \quad 24)$. In determining how to place the $+$ and $-$ signs in the parentheses, we can refer back to the original problem: $x^2+28x+96$. If we want 4 and 24 to add up to $(+28)$, then they should both be positive:

$$x^2+28x+96=(x+4)(x+24)$$

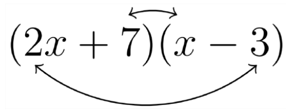
In building the charts of factor pairs in the previous two problems, nothing more difficult than dividing the constant term by the numbers $1, 2, 3, 4, 5, 6, \ldots$and so on can help you to find the full list of factor pairs. If you don't get a whole number when dividing - for instance $96 \div 5 = 19.2$, then this number is not included in the list of factor pairs.

Trinomial Factoring $(a \neq 1)$

If the value of a is not 1, this means that, if the trinomial is factorable, at least one of its binomial factors also has a coefficient other than 1. For instance:

$$(2x+7)(x-3) = 2x^2 + 1x - 21$$

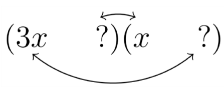
If we were to try undo this multiplication through the process of trinomial factoring, we should look to the sign of the constant term. In this example, the sign is negative. This still means that we will need to find factor pairs that produce a difference of $(+1x)$ as the middle term. However, in this scenario, it is not just the factors of 21 that are involved in producing the $(+1x)$, but the combination of the factors of 21 and the factors of the leading coefficient 2. The middle term $(+1x)$ comes from the multiplication of the $(2x)(-3)$ and the multiplication of (+7)$(+1x)$



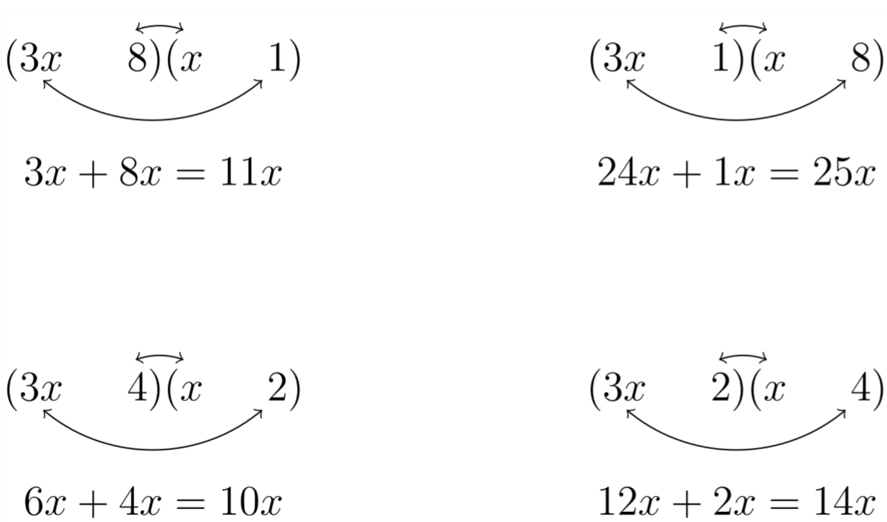
$$\begin{aligned}(2x+7)(x-3) &= 2x^2 - 6x + 7x - 21 \\ &= 2x^2 + x - 21\end{aligned}$$

In trying to factor a trinomial like $2x^2 + x - 21$, we need to take this into consideration. For example, if we were to factor $3x^2 - 10x + 8$, we should first still look to the sign of the constant term, which, in this case, is positive. That means we want factor pairs that will add up to 10. But we have to take into consideration the interaction of the factors of the 3 with the factors of the 8. The 3 is a prime number, which means that we don't have a choice - it can only be split up into $3 * 1$, so we can start:

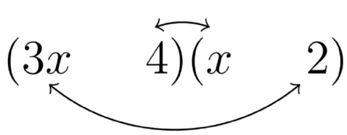
$$\text{Factor } 3x^2 + 10x + 8 \tag{1.2.1}$$



Our options for filling in the question marks will come from the factors of 8 , either $8 * 1$ or $4 * 2$. The process is by trial and error:



We can see that the choice above:



gives us the required $10x$ as the middle term. since the original problem was $3x^2 + 10x + 8$ we'll want to fill in the signs as both positive:

$$3x^2 + 10x + 8 = (3x+4)(x+2) \tag{1.2.2}$$

A second method for handling this type of factoring depends on how the factors of the leading coefficient and constant term interact with each other to produce the middle term. In this process, given the problem $3x^2+10+8$, we can multiply the first and last coefficient and then look at the factor pairs of the product:

$$3*8=24 \tag{1.2.3}$$

1	24
2	12
3	8
4	6

We can see that the factor pair of 24 that adds up to 10 is $6*4$. We proceed by splitting the $10x$ into $6x+4x$ and then factor by grouping. If you are uncomfortable with factoring by grouping, then this is probably not a good method to try. However, if you are comfortable with factoring by grouping, the rest of the process is relatively straightforward:

$$3x^2+10x+8=3x^2+6x+4x+8 \tag{1.2.4}$$

We then factor a common factor from the first two terms and the last two terms separately, and then factor out the common binomial factor of $(x+2)$

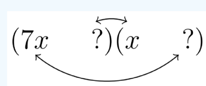
$$\begin{aligned}3x^2+10x+8 &= 3x^2+6x+4x+8\\ &= 3x(x+2)+4(x+2)\\ &= (x+2)(3x+4)\end{aligned}$$

Example 1.2.7

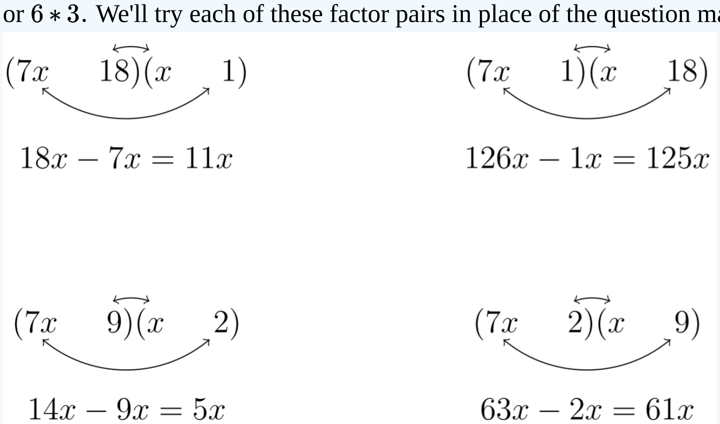
Factor $7x^2-5x-18$

Solution

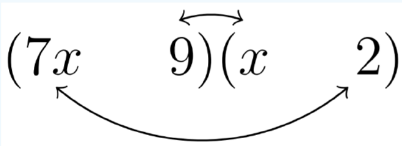
In this example, the sign of the constant term is negative, which means that we'll need factor pairs that produce a difference of 5. The leading coefficient is 7, which is prime, so, again, the only way to split up the 7 is $7*1$



The options for filling in the question marks come from the factors of 18, for which there are three possibilities: $18*1, 9*2$, or $6*3$. We'll try each of these factor pairs in place of the question marks:



The choice above:



gives us the required $5x$ as the middle term. since we're looking for a $(-5x)$, we'll make the 14 negative and the 9 positive: $-14x + 9x = -5x$

$$7x^2 - 5x - 18 = (7x + 9)(x - 2)$$

If we want to try the other method for factoring $7x^2 - 5x - 18$, we would multiply $7 * 18 = 126$, and then work to find factor pairs of 126 that have a difference of 5

1	126
2	63
3	42
6	21
7	18
9	14

Here, the last factor pair, $9 * 14$, has a difference of 5. So then we proceed to factor by grouping:

$$\begin{aligned} 7x^2 - 5x - 18 &= 7x^2 + 9x - 14x - 18 \\ &= x(7x + 9) - 2(7x + 9) \\ &= (7x + 9)(x - 2) \end{aligned}$$

Notice that when the -2 was factored out from the last two terms $-14x - 18$, we ended up with $-2(7x + 9)$, because

$(-2)*(+9)=-18$. This is also important because in order to factor out the common binomial factor of $(7x+9)$, this binomial must be exactly the same in both terms.

Difference of Squares

Factoring a difference of squares is actually a special form of trinomial factoring. If we consider a trinomial of the form ax^2+bx+c, where c is a perfect square and negative, we will find something interesting about the possible values of b that make the trinomial factorable.
Example

$$\text{Consider } x^2+bx-36 \tag{1.2.5}$$

For this expression to be factorable, the middle coefficient b would need to be equal to the difference of any of the factor pairs of 36. If we look at the possible factor pairs, we see the following:

1	36
2	18
3	12
4	9
6	6

This means that the possible values for b that would make this expression factorable are:

$$\begin{gathered}
36-1=35 \rightarrow x^2+35x-36=(x+36)(x-1) \\
18-2=16 \rightarrow x^2+16x-36=(x+18)(x-2) \\
12-3=9 \rightarrow x^2+9x-36=(x+12)(x-3) \\
9-4=5 \rightarrow x^2+5x-36=(x+9)(x-4) \\
6-6=0 \rightarrow x^2+0x-36=x^2-36=(x+6)(x-6)
\end{gathered} \tag{1.2.6}$$

As we see, factoring x^2-36 means that the factors of the perfect square $36=6*6$ will cancel each other out leaving $0x$ in the middle. If there is a perfect square as the leading coefficient, then this number should be square rooted as well:

$$16x^2-25=(4x+5)(4x-5) \tag{1.2.7}$$

In the example above, the $+20x$ and $-20x$ as the middle terms cancel each other out leaving just $16x^2-25$

These three types of factoring can also be combined with each other as we see in the following examples.

Example 1.2.6

Factor $2x^2-50$

Solution

This is not a trinomial because it doesn't have three terms. It is also not a difference of squares because 2 and 50 are not perfect squares. However, there is a common factor of 2 which we can factor out:

$$2x^2-50=2\left(x^2-25\right) \tag{1.2.8}$$

The expression inside the parentheses is a difference of squares and should be factored:

$$2x^2-50=2\left(x^2-25\right)=2(x+5)(x-5) \tag{1.2.9}$$

Example 1.2.7

Factor $24-2x-x^2$

Solution

Here the sign of the x^2 term is negative. For this problem we can factor out $a-1$ and proceed as we did with the previous problems in which the leading coefficient was positive or we can factor it as it is:

$$24-2x-x^2=-\left(x^2+2x-24\right)=-(x+6)(x-4) \tag{1.2.10}$$

If we want to factor it as it is, we should be aware that the constant term is positive and the quadratic term is negative, which means that we will want the factors of 24 to have a difference of 2

$$24-2x-x^2=(6+x)(4-x) \tag{1.2.11}$$

Example 1.2.8

Factor $6x^2+12x+6$

Solution

First, we notice that this expression has a common factor of 6. If we factor out the 6, then we should be left with an easier problem:

$$6x^2+12x+6=6\left(x^2+2x+1\right)=6(x+1)(x+1)=6(x+1)^2 \tag{1.2.12}$$

Exercises 1.2.1

Factor each expression completely.

1) $8a^2b^3+24a^2b^2$
2) $19x^2y-38x^2y^3$
3) $13t^8+26t^4-39t^2$
4) $5y^5+25y^4-20y^3$
5) $45m^4n^5+36mn^6+81m^2n^3$
6) $125x^3y^5+60x^4y^4-85x^5y^2$

Factor each trinomial into the product of two binomials.

7) a^2+3a+2

8) $y^2-8y-48$
9) $x^2-6x-27$
10) $t^2-13t+42$
11) $m^2+3m-54$
12) $x^2+11x+24$

Factor completely. Remember to look first for a common factor. If the polynomial is prime, state this.
13) a^2-9
14) y^2-121
15) $-49+k^2$
16) $-64+t^2$
17) $6x^2-54$
18) $25y^2-4$
19) $200-2a^2$
20) $3m^2-12$
21) $98-8k^2$
22) $-80w^2+45$
23) $5y^2-80$
24) $-4a^2+64$
25) $8y^2-98$
26) $24a^2-54$
27) $36k-49k^3$
28) $16y-81y^3$

Factor each trinomial completely. Remember to look first for a common factor. If the polynomial is prime, state this.
29) $3y^2-15y+16$
30) $8a^2-14a+3$
31) $9x^2-18x+8$
32) $6a^2-17a+12$
33) $2x^2+7x+6$
34) $2m^2+13m-18$
35) $20y^2+22y+6$
36) $36x^2+81x+45$
37) $24a^2-42a+9$
38) $48x^2-74x-10$

Factor each expression completely.
39) $30+7y-y^2$
40) $45+4a-a^2$
41) $24-10x-x^2$
42) $36-9x-x^2$
43) $84-8x-x^2$
44) $72-6a-a^2$
45) $6y^2+24y+15$
46) $10y^2-75y+35$
47) $20ax^2-36ax-8a$

Answer

1) $8a^2b^2(b+3)$
3) $13t^2\left(t^6+2t^2-3\right)$
5) $9mn^3\left(5m^3n^2+4n^3+9m\right)$
7) $(a+2)(a+1)$
9) $(x-9)(x+3)$

11) $(m+9)(m-6)$
13) $(a+3)(a-3)$
15) $(k+7)(k-7)$
17) $6(x+3)(x-3)$
19) $2(10+a)(10-a)$
21) $2(7+2k)(7-2k)$
23) $5(y+4)(y-4)$
25) $2(2y+7)(2y-7)$
27) $k(6+7k)(6-7k)$
29) PRIME
31) $(3x-4)(3x-2)$
33) $(2x+3)(x+2)$
35) $2(5y+3)(2y+1)$
37) $3(2a-3)(4a-1)$
39) $(10-y)(3+y)$
41) $(12+x)(2-x)$
43) $(14+x)(6-x)$
45) $3\left(2y^2+8y+5\right)$
47) $4a(5x+1)(x-2)$

1.3: Quadratic Equations

Quadratic equations are equations of the second degree. The solution of quadratic equations has a long history in mathematics going back several thousand years to the geometric solutions produced by the Babylonian culture. The Indian mathematician Brahmagupta used "rhetorical algebra" (algebra written out in words) in the 7 th century to produce solutions to quadratic equations and Arab mathematicians of 9 th and 10 th centuries followed simlar methods. Leonardo of Pisa (also known as Fibonacci) included information on the Arab approach to solving quadratic equations in his book *Liber Abaci*, published in 1202

The quadratic formula is generally used to solve quadratic equations in standard form: $ax^2+bx+c=0$. The solutions for this are:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{1.3.1}$$

Now, the question is - why does this formula give solutions to the standard quadratic equation? We can proceed as we normally do in solving linear equations - that is, by getting the x by itself. The only problem here is that instead of just x, there are also terms involving x^2. This is where the process of completing the square comes in handy.

We can begin with the quadratic equation in standard form:

$$ax^2+bx+c=0 \tag{1.3.2}$$

Just as it is easier to factor a quadratic trinomial if the leading coefficient is 1, this process of completing the square is also easier if the leading coefficient is 1. So next we will divide through on both sides of this equation by a.

$$\begin{gathered}\frac{ax^2+bx+c}{a}=\frac{0}{a}\\ \frac{ax^2}{a}+\frac{bx}{a}+\frac{c}{a}=\frac{0}{a}\\ x^2+\frac{b}{a}x+\frac{c}{a}=0\end{gathered} \tag{1.3.3}$$

Then, we will move the $\frac{c}{a}$ to the other side of the equation to clear out some room for completing the square:

$$\begin{gathered}x^2+\frac{b}{a}x+\frac{c}{a}=0\\ -\frac{c}{a}=-\frac{c}{a}\\ x^2+\frac{b}{a}x=-\frac{c}{a}\end{gathered} \tag{1.3.4}$$

Now we need to complete the square. If you are already familiar with this process, you may wish to skip the following explanation.

If we look at what happens when we square a binomial like $(x+3)^2$, we will begin to notice a pattern.

$$\begin{gathered}(x+3)^2=(x+3)(x+3)=x^2+6x+9\\ (x+4)^2=(x+4)(x+4)=x^2+8x+16\\ (x+5)^2=(x+5)(x+5)=x^2+10x+25\\ (x+6)^2=(x+6)(x+6)=x^2+12x+36\end{gathered} \tag{1.3.5}$$

Our goal in the derivation of the Quadratic Formula is to rewrite the expression $x^2+\frac{b}{a}x$ as a perfect square in the form $(x+\quad)^2$. The reason that we want to do this is that writing an expression as a binomial squared eliminates the problem of having both an x and an x^2, which was preventing us from getting the x by itself in the standard quadratic equation.

If we can figure out what should take the place of the blanks in the statement:

$$x^2+\frac{b}{a}x+\quad=(x+\quad)^2 \tag{1.3.6}$$

then we will be well on our way to deriving the quadratic formula.

If we re-examine the sample perfect binomial squares from the previous page, we note a useful pattern. This is that the blank in the parentheses $(x+\quad)^2$ is filled by a number that is one-half the value of the linear coefficient - or the coefficient of the x^1 term. Notice that in $x^2+6x+9=(x+3)^2, 3$ is half of 6, in $x^2+8x+16= \ (x+4)^2$, the 4 is half of 8, and so on. If we want to write $x^2+\frac{b}{a}x+\quad$ as a perfect square in the form $(x+\quad)^2$, the blank in the parentheses should be filled by:

$$\frac{1}{2}*\frac{b}{a}=\frac{b}{2a} \tag{1.3.7}$$

Now, it's not true that $x^2+\frac{b}{a}x+\quad=\left(x+\frac{b}{2a}\right)^2$

We're missing the constant term on the left. However, if we return to our perfect square examples, we can see that the constant term is always the square of the term inside inside the parentheses. So, we can restate our problem now as:

$$\begin{gathered} x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(x+\frac{b}{2a}\right)^2 \\ x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\left(x+\frac{b}{2a}\right)^2 \end{gathered} \tag{1.3.8}$$

So, if we return to our originial problem, we were saying that:

$$\begin{gathered} x^2+\frac{b}{a}x+\frac{c}{a}=0 \\ -\frac{c}{a}=-\frac{c}{a} \\ x^2+\frac{b}{a}x=-\frac{c}{a} \end{gathered} \tag{1.3.9}$$

We can add $\frac{b^2}{4a^2}$ to both sides of this equation and then restate the left hand side as a perfect square of a binomial:

$$\begin{gathered} x^2+\frac{b}{a}x \quad =-\frac{c}{a} \\ +\frac{b^2}{4a^2}=+\frac{b^2}{4a^2} \\ x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2} \\ \left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2} \end{gathered} \tag{1.3.10}$$

The last tricky bit of this derivation is adding the two fractions on the right hand side. The common denominator for these fractions is $4a^2$, so we'll need to multiply the $-\frac{c}{a}$ by $\frac{4a}{4a}$ to get $-\frac{4ac}{4a^2}$. Then the right hand side will be $\frac{b^2-4ac}{4a^2}$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2} \tag{1.3.11}$$

Then, we can take the square root of both sides and get the x by itself:

$$\begin{gathered} \sqrt{\left(x+\frac{b}{2a}\right)^2}=\pm\sqrt{\frac{b^2-4ac}{4a^2}} \\ x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{\sqrt{4a^2}} \\ x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a} \end{gathered} \tag{1.3.12}$$

Subtracting $\frac{b}{2a}$ from both sides is easy since we already have a common denominator:

$$\begin{gathered} x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a} \\ -\frac{b}{2a}=-\frac{b}{2a} \\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \end{gathered} \tag{1.3.13}$$

At the College Algebra level, it is often useful to program the quadratic formula onto a graphing calculator both for easy use and also to learn a little bit about programming. The following program is a simple example of this for the TI84 series of graphing calculators. Graphing calculators also often have built-in polynomial solver feature that can be used to solve quadratics.

Press the "prgm" key in the top middle of the calculator keypad. This will bring up a screen that shows EXEC EDIT NEW across the top. Arrow over across the top to "NEW," and then select 1: Create New.

This will bring up a screen asking you to name the program. You should see PROGRAM and then underneath it, "Name=". The alpha lock is on automatically, so any key you press will type the letter associated with it. Name your program and press ENTER. You should see PROGRAM: Name, with whatever name you've chosen for your program. Underneath this you will se a colon:
This is where you will enter the commands for the program.

First we need to enter the values for A, B and C from the quadratic equation into the calculator. To do this, press the "prgm" key again. Across the top of the screen you should see CTL I/O COLOR EXEC.

Arrow over to I/O. This is the "input/output" menu. Choose number 2:Prompt. This will return you to the program screen where you will see :Prompt under the name of the program. After :Prompt, type A, B, C. You'll need to use the "alpha" key to access the letters and the comma is right above the 7 key.

PROGRAM: Name (whatever name you've chosen should show here) :Prompt A, B, C
On the next line of the program, we will take the values of A, B and C and use them to calculate the values of the roots of the equation. Type in the following:
PROGRAM: Name :Prompt $\mathrm{A}, \mathrm{B}, \mathrm{C}$
: $(-B+\sqrt{(B^2-4AC)})/(2A) \rightarrow R$
: $(-B-\sqrt{(B^2-4AC)})/(2A) \rightarrow S$

In typing these two lines it's important that when you type $-B$, you use the negative key next to the decimal point, rather than the subtraction key. The calculator is very picky about this. When you're typing the B^2-4AC, you'll need to use the subtraction key on the far right of the keypad.

Also notice the double parentheses - one set for the numerator of the fraction and one set for the square root. If you don't type this in correctly it will produce Wrong answers. The arrow in the formula stores the values of the answer in the variables R and S, and the arrow is produced by the "sto $\rightarrow$" key just above the ON button in the lower left of the keyboard.

Now that we've given the calculator the values for A, B and C and then had the calculator find the roots of the eqatiuon, we need to display the answers. If you press the prgm key and arrow over to the I/O menu again, you can choose 3:Disp. This will Display the answers that we've stored as R and S.

PROGRAM: Name :Prompt $A, B, C : (-B+\sqrt{B^2-4AC}))/(2A) \rightarrow R$
: $(-B-\sqrt{B^2-4AC}))/(2A) \rightarrow S$
: Disp R, S
Now we can test the program with some simple equations. To run the program, press the program key and choose the program you've created either by selecting it and pressing enter, or by pressing the number for the program in the list. This should bring you back to the calculation screen, where you can run the program by pressing enter. The calculator should then ask you for the values of A, B and
C

Solve for $x: \quad 2x^2-x-1=0$
In this example the values for A, B and C are:
$A=2$
$B=-1$
$C=-1$
Again, it's important that you use the negative sign key next to the decimal point for the values of any negative coefficients and not the subtraction key. The calculator should return values of 1 and -0.5 as the solutions.

Solve for
$x: \quad x^2+x+1=0$
In this example the values for A, B and C are:

$A=1$
$B=1$
$C=1$
The calculator should return values of $-0.5 \pm 0.8660254038 i$ as solutions.

If you get an error message saying "NONREAL AnswerS," you'll need to adjust the calculator setting to allow for complex valued answers. You can do this by presing the "mode" key in the top left of the keypad and arrowing down to the line that reads "REAL a+bi re$^\wedge(\theta i)$." You can then arrow over to "a+bi" and press enter. This will allow the calculator to compute complex valued answers.
Something very important to remember about the quadratic formula is that the equation must be in standard form in order to identify the values of A, B and C to use in the formula. For example in the equation:

$$3 x^{2}-7=2 x \tag{1.3.14}$$

it is important to understand that the values of A, B and C come from the standard form of the equation and not the present form of the equation. There are several pitfalls to watch out for in this equation. First of all, the $2x$ is on the opposite side of the equation from the other terms. That means that the value of B IS $\text{NOT}+2$. Also, if we were to move the $2x$ to the other side to put the equation in standard form, it is not the order of the terms, but degree of the variable that determines whether a coefficient is identified as A, B or C

In moving the $2x$ to the other side of the equation, I have seen students put the term they're adding to that side as the last term. There is nothing wrong about this, but if you do that, you must be careful about identifying the values of A, B and C

$$\begin{aligned} 3 x^{2}-7 &=2 x \\ -2 x &=-2 x \\ 3 x^{2}-7-2 x &=0 \end{aligned}$$

There is nothing wrong about the way the equation above is written depsite the fact that it is not in "standard form." The important thing to remember is that "A" is not the coefficient of whichever term is listed first. It is the coefficient of the quradratic, or x^2 term. Likewise, "B" is not the coefficient of the second term, but rather the coefficient of the linear, or x^1 term. And "C" is not whichever number comes last, but rather the value of the constant term. So in the equation above, however it is written, the value of A is $+3$, B is -2 and C is -7

Exercises 1.3.1

Solve for x in each equation. Round any irrational values to the nearest 1000 th.

1) $x^{2}+7 x=2$
2) $5 x^{2}-3 x=4$
3) $\frac{3}{4} x^{2}=\frac{7}{8} x+\frac{1}{2}$
4) $\frac{2}{3} x^{2}-\frac{1}{3}=\frac{5}{9} x$
5) $2 x^{2}+(\sqrt{5}) x-3=0$
6) $3 x^{2}+x-\sqrt{2}=0$
7) $2.58 x^{2}-3.75 x-2.83=0$
8) $3.73 x^{2}+9.74 x+2.34=0$
9) $5.3 x^{2}+7.08 x+1.02=0$
10) $3.04 x^{2}+1.35 x+1.234=0$
11) $7 x(x+2)+5=3 x(x+1)$
12) $5 x(x-1)-7=4 x(x-2)$
13) $14(x-4)-(x+2)=(x+2)(x-4)$
14) $11(x-2)+(x-5)=(x+2)(x-6)$

Answer

1) $x \approx 0.275, -7.275$
3) $x \approx 1.587, -0.420$
5) $x \approx 0.787, -1.905$
7) $x \approx -0.548, 2.002$
9) $x \approx -0.164, -1.172$
11) $x \approx -0.575, -2.175$
13) $x = 10, 5$

1.4: Complex Numbers

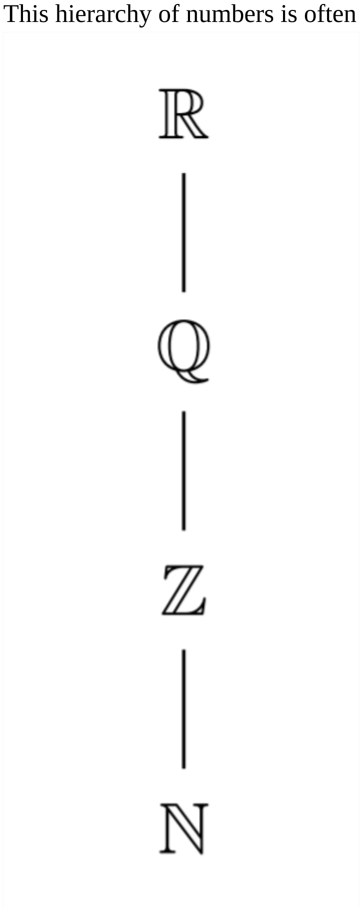
Our number system can be subdivided in many different ways. The most basic form of mathematics is counting and almost all human cultures have words to represent numbers (the Pirahã of South America are a notable exception). Thus the most basic set of numbers is the set of counting numbers represented by the double barred $\mathrm{N}: \mathbb{N}=\{1,2,3,4,5,6,7, \ldots\}$ (we will set aside the debate as to whether or not zero should be included in this set).

If we try to subtract a larger counting number from a smaller counting number we find that there are no members in the set of counting numbers to represent the answer in this situation. This extends the set of natural numbers to the set of integers: $\mathbb{Z}=\{\ldots,-3,-2,-1,0,1,2,3, \ldots\}$.The integers are represneted by the double barred Z, for the German word for numbers- "zahlen." In the earliest appearances of negative numbers in the Chinese and Indian mathematical systems, negative values were often used to represent debt. Because Greek mathematics was based on Geometry, they did not use negative numbers.

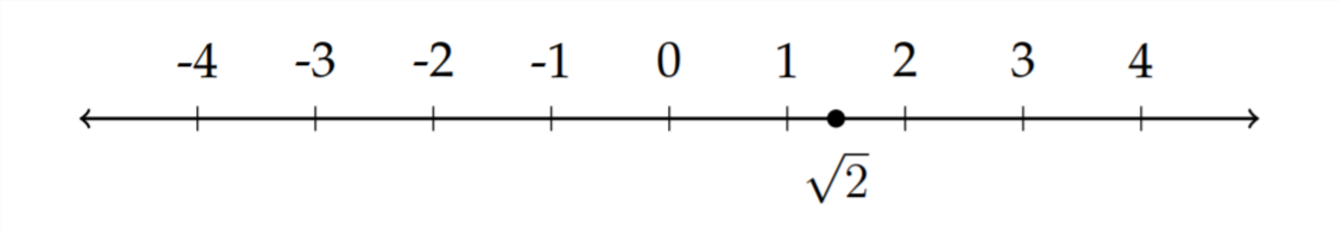
Moving to multiplication and division, if we question the value of $8 \div 2=4$ versus $8 \div 3=?$, we once again must expand our conception of numbers to allow for an answer to the second question $8 \div 3=?$. Understanding ratios of whole numbers or Rational numbers allows solutions to such problems. The set of Rational numbers is represented by the double barred Q, to represent a quotient:
$\mathbb{Q}=\left\{\frac{a}{b}: a, b \in \mathbb{Z}\right\}$

The Greek understanding of numbers mostly stopped here. They felt that all quantities could be represented as the ratio of whole numbers. The length of the diagonal of a square whose sides are of length 1 produced considerable consternation among the Pythagoreans as a result of this. Using the Pythagorean Theorem for the diagonal of a square whose sides are of length 1 shows that the diagonal would be $c^2=1^2+1^2=2$, thus $c=\sqrt{2}$. This number cannot be represented as a ratio of whole numbers. This new class of numbers adds the set
of irrational numbers to the existing set of rational numbers to create the Real numbers, represented with a double barred $\mathrm{R}: \mathbb{R}$.

This hierarchy of numbers is often represented in the following diagram:



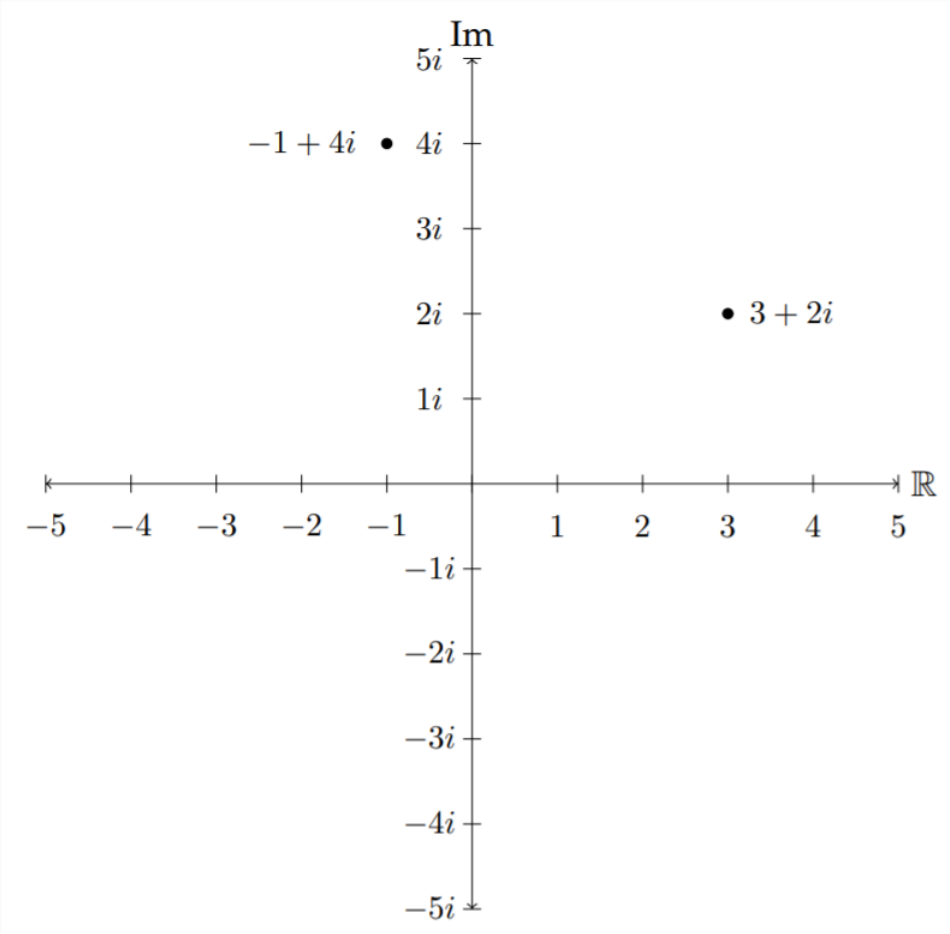
One of the best ways to conceptualize the Real number system is on the number line - every point on the number line corresponds to a unique Real number and every Real number corresponds to a unique position on the Real number line.



After the development of the printing press in the 15 th century, Fibonacci's Liber Abaci was translated into Italian from Latin and read throughout Italy. As a result, Italy became a thriving center of mathematics until the 17 th century, when the center of European mathematics moved north to France, Germany and England.

Throughout the 1500 's Italian mathematicians such as Girolamo Cardano, Raphael Bombelli and Niccolo Fontana Tartaglia worked to extend the ideas in Fibonacci's book. They produced formulas to solve cubic (x^3), and quartic (x^4) degree equations. In solving some of these equations they found that their formulas sometimes produced negative values under a square root. None of the known number systems could accomodate this possibility. In Cardano's book on algebra Ars Magna, he encounters a problem which involves the square root of a negative number. He says, "It is clear that this case is impossible. Nevertheless we will work thus..." and he proceeds to compute a valid complex solution to the problem. Mathematicians eventually defined the complex unit $\sqrt{-1}=i$ and then

devised a system in which all complex numbers are a combination of a Real part
(a) and an "imaginary" part (bi).
The complex numbers are a two dimensional number system represented by the double barred $C: \mathbb{C}=\{a+bi: a, b \in \mathbb{R}\},$ where i is the complex unit defined as $i=\sqrt{-1}.$ Throughout the late 1700 's and early 1800 's mathematicians gradually moved towards a geometrical interpretation of the two-dimensional complex numbers. What is today known as the "Argand diagram" represents the real valued portion of a complex number along a horizontal axis and the multiple of the complex unit along the vertical axis.



Expressing square roots of negative numbers

Square roots of negative quantities are generally expressed as a multiple of i

Example 1.4.1

$$\begin{gathered}\sqrt{-4}=2i\\ \sqrt{-25}=5i\\ \sqrt{-7}\approx 2.646i\end{gathered} \tag{1.4.1}$$

Adding, subtracting and multiplying with complex numbers

Calculating with complex numbers has many similarities to working with variables. The real part and imginary part are treated separately for addition and subtraction, but can be multiplied and divided.

Example 1.4.2

Compute the following:

$$\begin{gathered}(6-4i)+(-2+7i)=4+3i\\ (-9+2i)-(-4+6i)=-9+2i+4-6i=-5-4i\\ 3(10+i)=30+3i\\ -7i(-5+8i)=35i-56i^2\end{gathered} \tag{1.4.2}$$

Now we encounter an interesting fact about complex numbers and, in particular, the complex unit i. By definition, $i=\sqrt{-1}$. Therefore, if we square i we should get -1. In the last example problem above, we can replace the i^2 with -1 to finish the problem.

$$\begin{aligned}-7i(-5+8i) &= 35i-56i^2\\ &= 35i-56(-1)\\ &= 35i+56\\ &= 56+35i\end{aligned}$$

Example 1.4.3

Compute the following:

$$\begin{aligned}(8-5i)(1-4i) &= 8-32i-5i+20i^2\\ &= 8-37i+20(-1)\\ &= 8-37i-20\\ &= -12-37i\end{aligned}$$

$$\begin{aligned}(9+2i)^2 &= (9+2i)(9+2i)\\ &= 81+18i+18i+4i^2\\ &= 81+36i+4(-1)\\ &= 81+36i-4\\ &= 77+36i\end{aligned}$$

Powers of i

The powers of i follow an interesting pattern based on the definition that $i^2=-1$
We can see that $i^1=i$ and that $i^2=-1$, as a result, $i^3=i^2*i^1=-1*i=-i$
In a similar fashion, $i^4=i^2*i^2=(-1)(-1)=1$
This means that $i^5=i^4*i=1*i=i$
If we put all of this information together we get the following:
$i^1=i$
$i^2=-1$
$i^3=-i$
$i^4=1$
$i^5=i^1=i$
$i^6=i^2=-1$
$i^7=i^3=-i$
$i^8=i^4=1$
In other words, every power of i is equivalent to either $i,-1,-i$, or 1. To determine which of these values a power of i is equivalent to, we need to find the remainder of the exponent when it is divided by 4

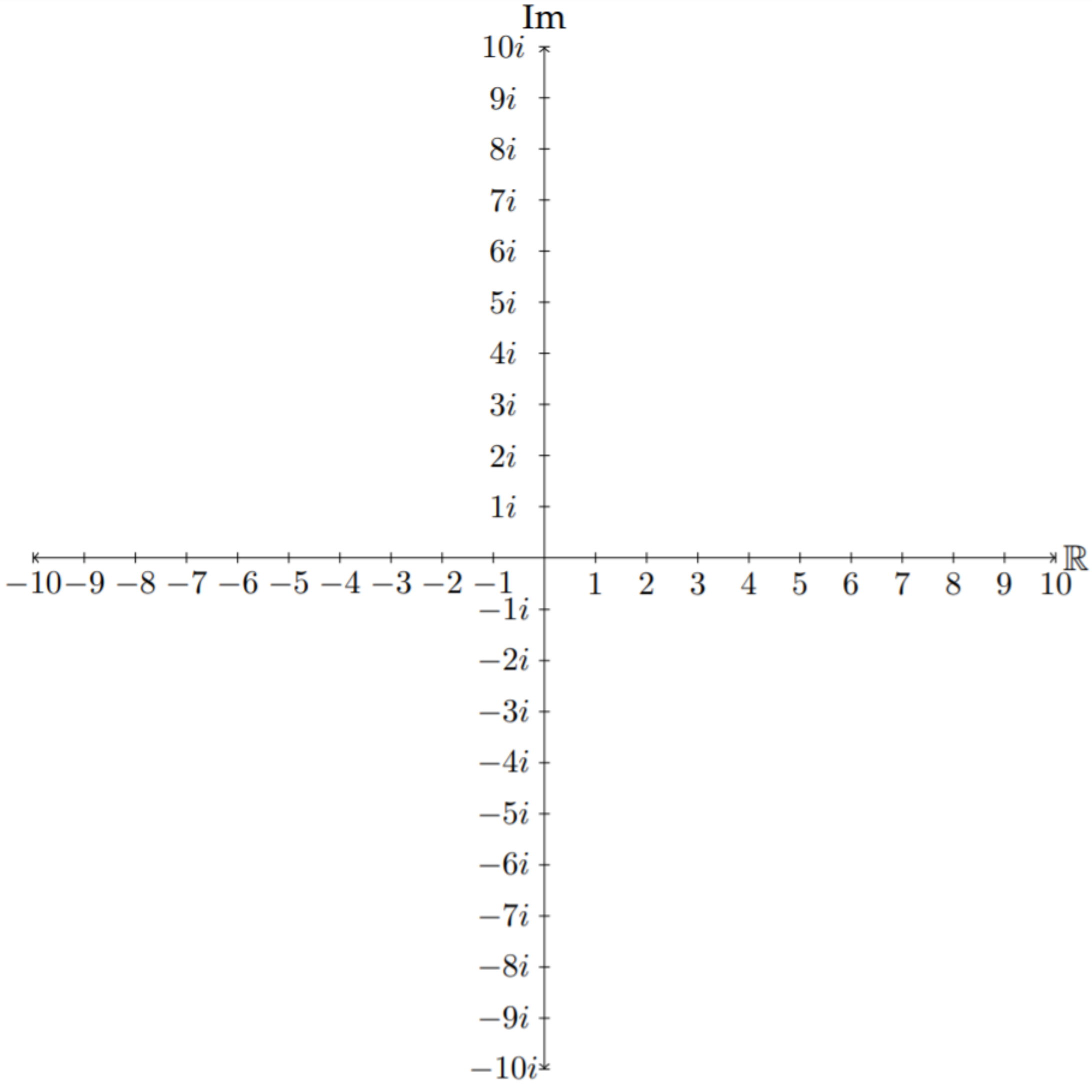
Example 1.4.4

Simplify i^{38}

Solution

since every $i^4=1$, then $i^{38}=i^{36}*i^2=\left(i^4\right)^9*i^2=1^9*i^2=i^2=-1$
since 38 is 2 more than a multiple of 4, then $i^{38}=i^2=-1$

Exercise 1.4.1



Graph the following complex numbers:

1) $2+5i$
2) $4-3i$
3) $-2+6i$
4) $-3-5i$
5) 4
6) $-2i$
7) $7-i$
8) $-1+i$
9) $-8+4i$
10) $8+3i$
11) $7i$
12) $-5-9i$

Express each quantity in terms of i. Round irrational values to the nearest 1000 th.

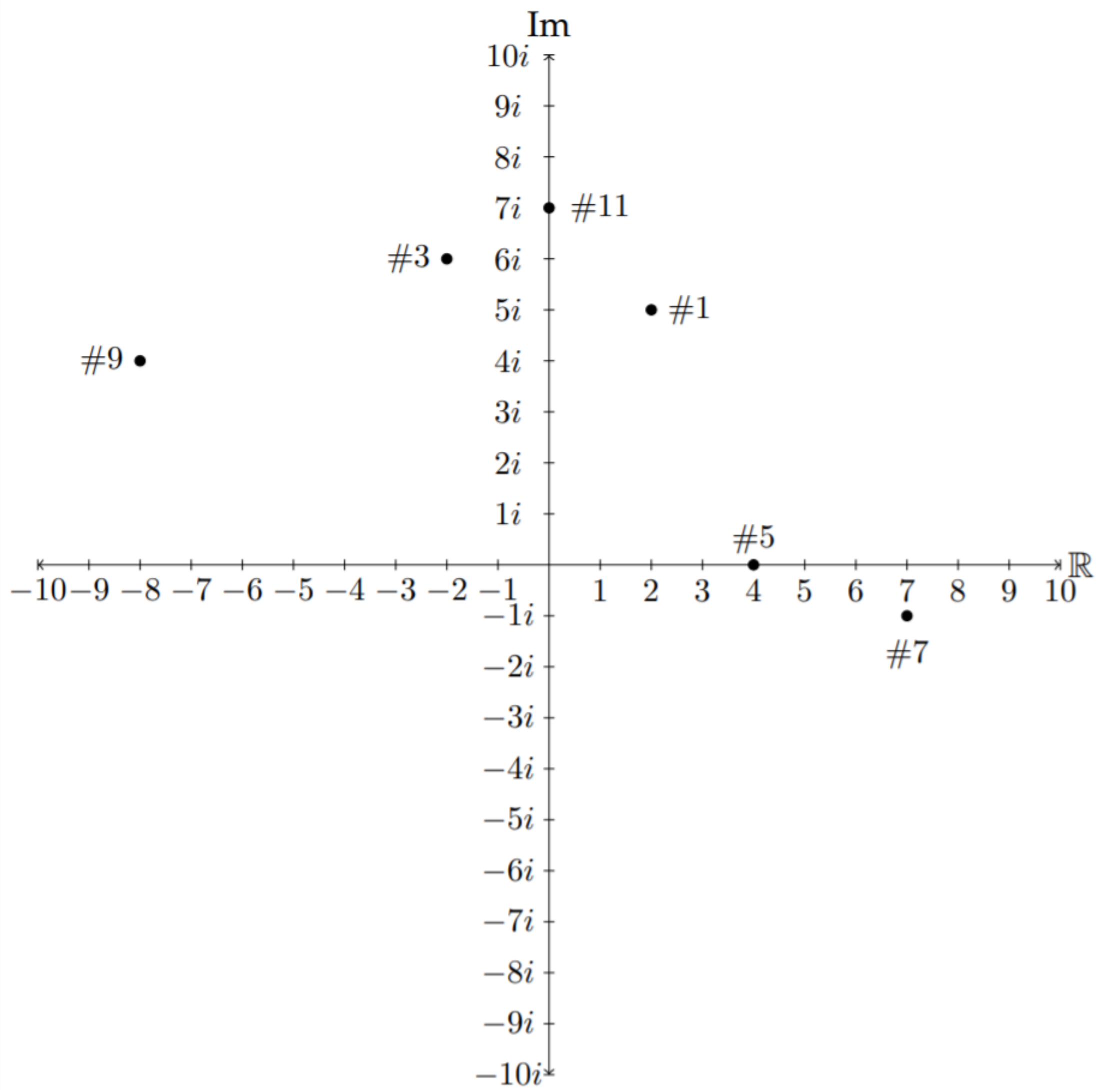
13) $\sqrt{-36}$
14) $\sqrt{-81}$
15) $\sqrt{-100}$
16) $\sqrt{-49}$
17) $\sqrt{-4}$

18) $\sqrt{-25}$
19) $\sqrt{-2}$
20) $\sqrt{-6}$
21) $\sqrt{-10}$
22) $\sqrt{-31}$
23) $\sqrt{-5}$
24) $\sqrt{-3}$

Perform the indicated operation and simplify
25) $(6+7i)+(5+3i)$
26) $(4-5i)+(3+9i)$
27) $(9+8i)-(1-2i)$
28) $(2+i)-(6-4i)$
29) $(7-4i)-(5-3i)$
30) $(8+i)-(4+3i)$
31) $(7i)(6i)$
32) $(4i)(-8i)$
33) $(-2i)(5i)$
34) $(12i)(3i)$
35) $(1+i)(3+2i)$
36) $(1+5i)(4+3i)$
37) $(6-5i)(2-3i)$
38) $(8-3i)(2+i)$
39) $(-3+4i)(-1-2i)$
40) $(-7-i)(3-5i)$
41) $(4-2i)^2$
42) $(-5+i)^2$
43) $(3+i)(3-i)$
44) $(2+6i)(2-6i)$
45) $(9-4i)(9+4i)$
46) $(5+2i)(5-2i)$

Express as either $i, -1, -i,$ or 1
47) i^3
48) i^7
49) i^{21}
50) i^{13}
51) i^{29}
52) i^{56}
53) i^{72}
54) i^{35}
55) i^{66}
56) i^{103}
57) i^{16}
58) i^{53}
59) i^{11}
60) i^{42}
61) i^{70}
62) i^9

Answer



13) $6i$
15) $10i$
17) $2i$
19) $1.414i$
21) $3.162i$
23) $2.236i$
25) $11+10i$
27) $8+10i$
29) $2-i$
31) -42
33) 10
35) $1+5i$
37) $-3-28i$
39) $11+2i$
41) $12-16i$
43) 10
45) 97
47) $-i$
49) i
51) i

53) 1
55) -1
57) 1
59) $-i$
61) -1

1.5: Quadratic Equations with Complex Roots

In Section 1.3, we considered the solution of quadratic equations that had two real-valued roots. This was due to the fact that in calculating the roots for each equation, the portion of the quadratic formula that is square rooted (b^2-4ac, often called the discriminant) was always a positive number.

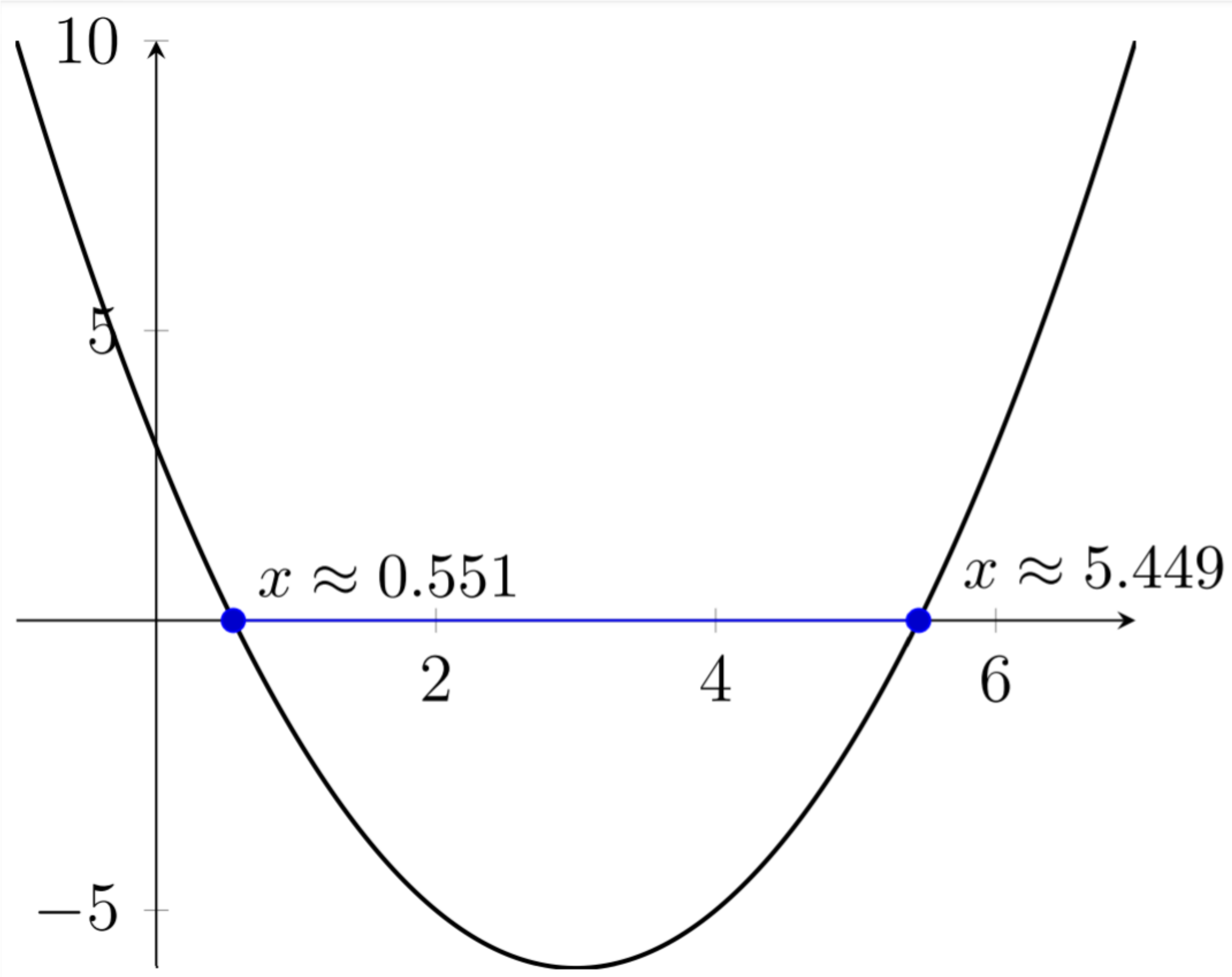
For example, in using the quadratic formula to calculate the the roots of the equation $x^2-6x+3=0,$ the discriminant is positive and we will end up with two real-valued roots:

$$\begin{gathered} x^2-6x+3=0 \\ a=1, b=-6, c=3 \\ =\frac{-(-6) \pm \sqrt{(-6)^2-4(1)(3)}}{2 * 1} \\ =\frac{6 \pm \sqrt{36-12}}{2} \\ =\frac{6 \pm \sqrt{24}}{2} \\ =\frac{6 \pm 4.899}{2} \end{gathered} \tag{1.5.1}$$

$$\begin{array}{ll} \approx \frac{6+4.899}{2} & \approx \frac{6-4.899}{2} \\ \approx \frac{10.899}{2} & \approx \frac{1.101}{2} \\ \approx 5.449 & \approx 0.551 \end{array} \tag{1.5.2}$$

When we added and subtracted the square root of 24 to 6 in the quadratic formula, this created two answers, and they were real-valued because the square root of 24 is real-valued.

Another way to see this is graphically. If we graph $y=x^2-6x+3$ and find the x values that make $y=0$, these will appear along the x -axis, and will be the same values that solve the equation $x^2-6x+3=0$



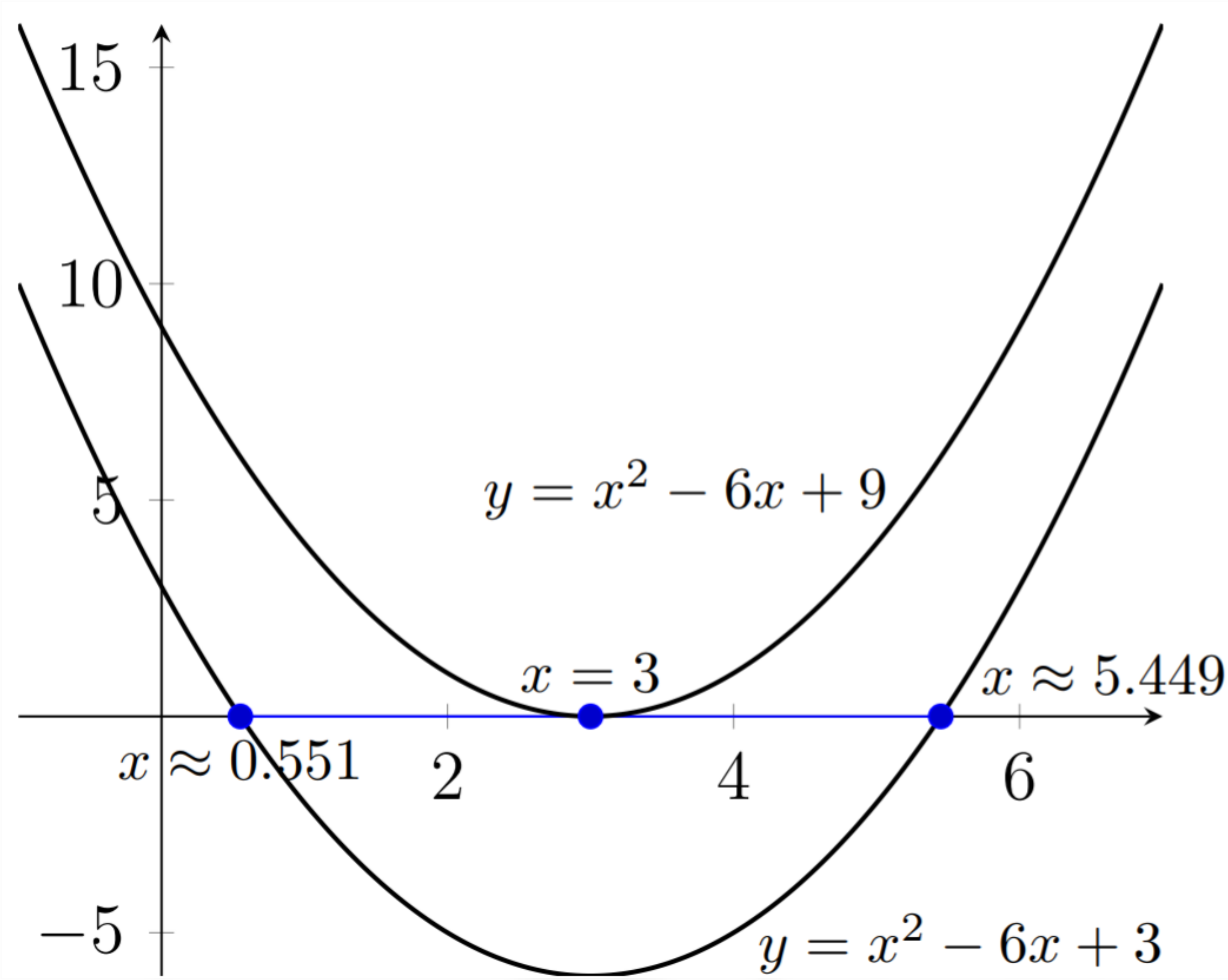
If we consider a related, but slightly different equation to start with, these relationships between the roots, the discriminant and the graphical intersections will be slightly different.

$$\begin{gathered}x^2-6x+9=0\\a=1,b=-6,c=9\end{gathered} \tag{1.5.3}$$

$$\begin{aligned}x&=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(9)}}{2*1}\\&=\frac{6\pm\sqrt{36-36}}{2}\\&=\frac{6\pm\sqrt{0}}{2}\\&=\frac{6}{2}=3\end{aligned}$$

Because the discriminant was 0 in this problem, we only get one real-valued answer.

Graphically, the additional 6 that was added to the original equation to change it from x^2-6x+3 to x^2-6x+9 shifts every y value on the graph up 6 units.

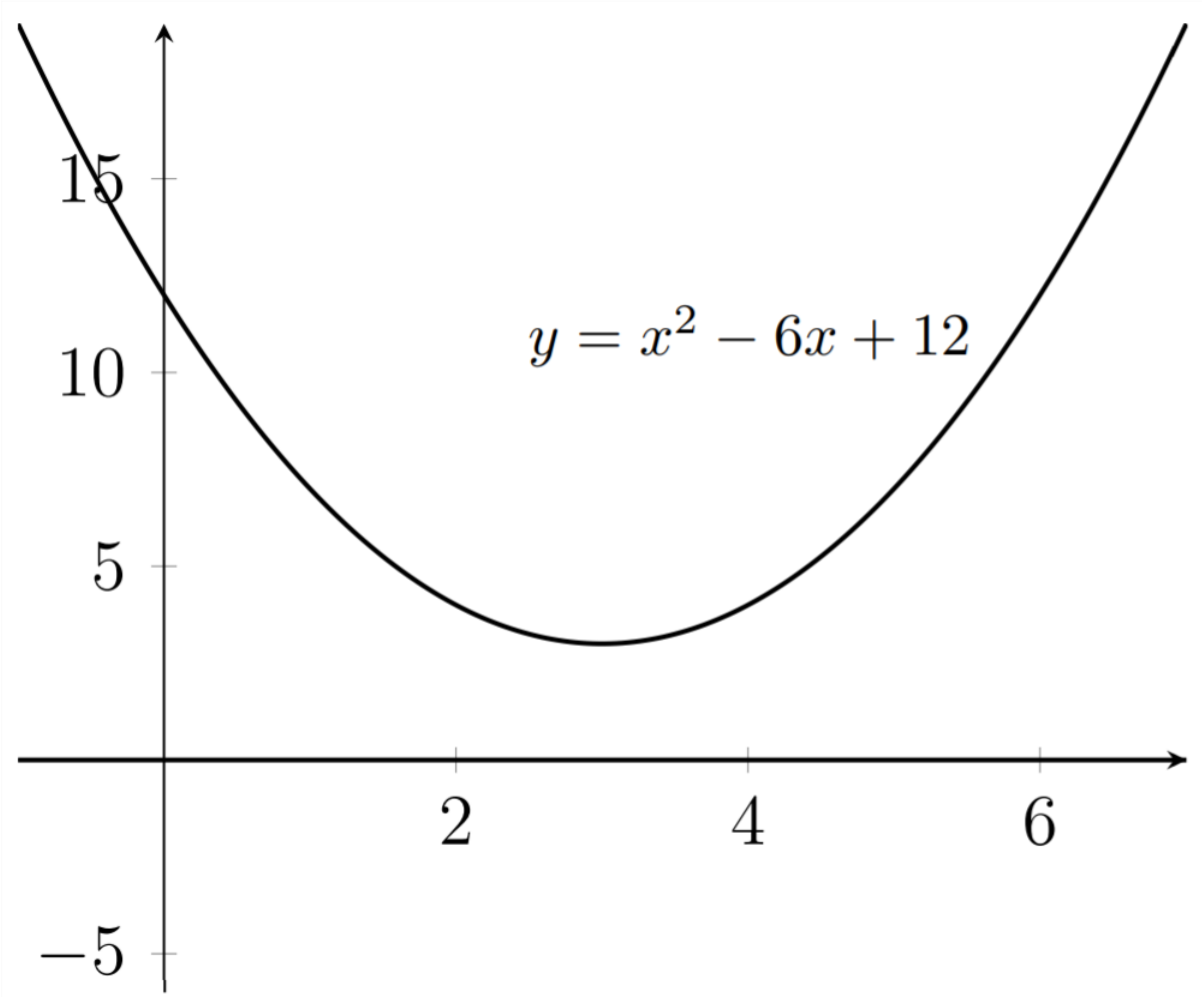


If we add an additional three units to the constant term of this quadratic equation, we encounter a third possibility.

$$\begin{gathered} x^2 - 6x + 12 = 0 \\ a = 1, b = -6, c = 12 \\ = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(12)}}{2*1} \\ = \frac{6 \pm \sqrt{36-48}}{2} \\ = \frac{6 \pm \sqrt{-12}}{2} \\ = \frac{6}{2} \pm \frac{6 \cdot 464 i}{2} \\ \approx 3 \pm 1.732 i \end{gathered} \tag{1.5.4}$$

Here the discriminant is negative, which leads to two complex-valued answers. If the equation has real-valued coefficients, the complex roots will always come in conjugate pairs. Complex conjugates share the same real-valued part and have opposite signs in their complex-valued (or imaginary) parts: $a \pm bi$

Graphically, the previous problem was one step away from not intersecting the x -axis at all and the additional three units that we added on to get $y = x^2 - 6x + 12$ moves the graph entirely away from the x -axis. Because the roots are complex-valued, we don't see any roots on the x -axis. The x -axis contains only real numbers.



since the calculator has been programmed for the quadratic formula, the focus of the problems in this section will be on putting them into standard form.

Example 1.5.1

Solve for x

$(2x+1)(x+5)-2x(x+7)=5(x+3)^2$

Solution

$$\begin{gathered}(2x+1)(x+5)-2x(x+7)=5(x+3)^2 \\ 2x^2+11x+5-2x^2-14x=5(x+3)(x+3) \\ -3x+5=5\left(x^2+6x+9\right) \\ -3x+5=5x^2+30x+45 \\ 0=5x^2+33x+40 \\ x=5, b=33, c=40 \\ x=-5,-1.6\end{gathered} \tag{1.5.5}$$

The fact that the roots of this equation were rational numbers means that the equation could have been solved by factoring.

$$\begin{aligned} 0 &= 5x^2 + 33x + 40 \\ 0 &= (5x+8)(x+5) \\ 5x &= -8 \qquad x+5=0 \\ 5x+8 &= 0 \qquad x=-5 \\ x &= -1.6 \end{aligned} \tag{1.5.6}$$

Example 1.5.1

Solve for x

(x-2)^{2}+3(4 x-1)(x+1) &=7(x+1)(x-1)

Solution

$$\begin{aligned} x^2-4x+4+3\left(4x^2+3x-1\right) &= 7\left(x^2-1\right) \\ x^2-4x+4+12x^2+9x-3 &= 7x^2-7 \\ 13x^2+5x+1 &= 7x^2-7 \\ 6x^2+5x+8 &= 0 \\ a=6, b=5, c=8 & \\ x \approx -0.41\overline{6} \pm 1.077i \approx -\frac{5}{12} \pm 1.077i & \end{aligned}$$

Exercise 1.5.1

Solve for x in each equation. Round any irrational values to the nearest 1000 th.

1) $3x^2-3x=4$
2) $4x^2-2x=7$
3) $5x^2=3-7x$
4) $3x^2=21-14x$
5) $6x^2+1=2x$
6) $5x-3x^2=17$
7) $(5x-1)(2x+3)=3x-20$
8) $(x+4)(3x-1)=9x-5$
9) $(x-2)^2=8x(x-1)+10$
10) $(2x-3)^2=2x-7x^2$
11) $(x+5)(x-6)=(2x-1)(x-4)$
12) $(3x-4)(x+2)=(2x-5)(x+5)$

Answer

1) $x \approx 1.758, -0.758$
3) $x \approx 0.344, -1.744$
5) $x \approx 0.1\overline{6} \pm 0.373i$
7) $x \approx -0.5 \pm 1.204i$
9) $x \approx 0.286 \pm 0.881i$
11) $x \approx 4 \pm 4.243i$

1.6: Multiplying and Dividing Rational Expressions

Reducing Rational Expressions

A rational expression is simply an algebraic fraction, and our first consideration will be to reduce these expressions to lowest terms in the same way that we reduce numerical fractions to lowest terms. When we reduce $\frac{6}{15}$ to $\frac{2}{5}$ by canceling the common factor of three, we are removing a redundant factor of 1 in the form of $\frac{3}{3}$

$$\begin{aligned}\frac{6}{15} &= \frac{3*2}{3*5} = \frac{3}{3}*\frac{2}{5} \\ &= 1*\frac{2}{5} \\ &= \frac{2}{5}\end{aligned}$$

Similarly, if there is a common factor that can be factored out of an algebraic fraction, this also can be canceled.

$$\begin{aligned}\frac{21x+14}{7x+7} &= \frac{7(3x+2)}{7(x+1)} \\ &= \frac{7}{7}*\frac{3x+2}{x+1} \\ &= 1*\frac{3x+2}{x+1} \\ &= \frac{3x+2}{x+1}\end{aligned}$$

It's important to remeber that only common factors can be canceled. This means that the first priority in each problem will be to identify the factors of the numerator and denominator to see if they share any common factors.

Example

Reduce to lowest terms.

$$\frac{x^2+4x-12}{x^2-4}$$

$$\begin{aligned}\frac{x^2+4x-12}{x^2-4} &= \frac{(x+6)(x-2)}{(x+2)(x-2)} \\ &= \frac{(x+6)\ \cancel{(x-2)}}{(x+2)\ \cancel{(x-2)}} \\ &= \frac{x+6}{x+2}\end{aligned}$$

Notice that we can't cancel the 6 and the 2 in the final answer because they aren't factors. The plus signs in the numerator and denominator prevent us from cancelling the 6 and the 2.

In the previous examples, we saw that cancelling out common factors in the numerator and denominator was actually a process of eliminating a redundant factor of 1. In the following example, we'll see a slightly different form of cancelling.

Example

Reduce to lowest terms.

$$\frac{16-x^2}{x^2+x-20}$$

$$\frac{16-x^2}{x^2+x-20} = \frac{(4+x)(4-x)}{(x+5)(x-4)}$$

In this problem, there are no common factors, but we can do some cancelling. We can see that $(4-x)$ and $(x-4)$ are not the same expression. In the first binomial, the 4 is positive and the x is negative, whereas in the second binomial, the 4 is negative and the x is positive. So, we know that $\frac{4-x}{x-4} \neq 1$. However, if we factor a(-1) out of the numerator, we will see an interesting phenomenon:

$$\begin{aligned}\frac{4-x}{x-4} &= \frac{-1(-4+x)}{x-4} \\ &= \frac{-1(x-4)}{x-4} \\ &= -1 * \frac{x-4}{x-4} \\ &= -1 * 1 = -1\end{aligned}$$

Therefore, although $\frac{4-x}{x-4} \neq 1$, we can say that $\frac{4-x}{x-4} = -1$. This will allow us to cancel $(4-x)$ and $(x-4)$ and replace them with (-1)

$$\begin{aligned}\frac{16-x^2}{x^2+x-20} &= \frac{(4+x)(4-x)}{(x+5)(x-4)} \\ &= \frac{(4+x)\ \cancel{(4-x)}}{(x+5)\ \cancel{(x-4)}(-1)}\end{aligned}$$

In the final answer, the (-1) can be placed in the denominator or the numerator, but not both. It can also be placed in front of the fraction.

$$\begin{aligned}\frac{4+x}{-1(x+5)} &= \frac{-1(4+x)}{x+5} \\ &= -\frac{4+x}{x+5}\end{aligned}$$

Multiplying and Dividing Rational Expressions

In multiplying and dividing rational expressions, it is often easier to identify and cancel out common factors before multiplying rather than afterwards. Multiplying rational expressions works the same way that multiplying numerical fractions does - multiply straight across the top and straight across the bottom. As a result, any factor in either numerator of the problem will end up in the numerator of the answer. Likewise, any factor in either denominator of the problem will end up in the denominator of the answer. Thus, any factor in either numerator can be cancelled with any factor in either denominator.

Example

Multiply. Express your answer in simplest form.

$$\begin{aligned}&\frac{x^2+5x+6}{25-x^2} * \frac{x^2-2x-15}{x^2+6x+9} \\ &\frac{x^2+5x+6}{25-x^2} * \frac{x^2-2x-15}{x^2+6x+9} = \frac{(x+2)(x+3)}{(5+x)(5-x)} * \frac{(x-5)(x+3)}{(x+3)(x+3)} \\ &= \frac{(x+2)\ \cancel{(x+3)}}{(5+x)\ \cancel{(5-x)}(-1)} * \frac{\cancel{(x-5)}\ \cancel{(x+3)}}{\cancel{(x+3)}\ \cancel{(x+3)}} \\ &= -\frac{x+2}{x+5}\end{aligned}$$

Dividing rational expressions works in much the same way that dividing numerical fractions does. We multiply by the reciprocal. There are several ways to demonstrate that this is a valid definition for dividing. First, it is important to understand that the fraction bar is the same as a "divided by" symbol:

$$\frac{8}{2} = 8 \div 2 = 4 \tag{1.6.1}$$

The same is true for dividing fractions:

$$\frac{1}{3} \div \frac{2}{5} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} \tag{1.6.2}$$

We can take the complex fraction $\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)}$ and multiply it by 1 without changing its value:

$$\begin{aligned} \frac{1}{3} \div \frac{2}{5} &= \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} * 1 \\ &= \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} \end{aligned}$$

We can multiply by any form of 1 we want to and not change the value of the result.

$$\begin{gathered} \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} * 1 = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} \\ \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} * \frac{9}{9} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} \\ \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} * \frac{12}{12} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} \end{gathered} \tag{1.6.3}$$

With a carefully chosen form of 1, we can transform the division problem into a multiplication problem.

$$\begin{aligned} \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{5}\right)} * \frac{\left(\frac{5}{2}\right)}{\left(\frac{5}{2}\right)} &= \frac{\frac{1}{3} * \frac{5}{2}}{\frac{2}{5} * \frac{5}{2}} \\ &= \frac{\frac{1}{3} * \frac{5}{2}}{1} \\ &= \frac{1}{3} * \frac{5}{2} \\ &= \frac{5}{6} \end{aligned}$$

In this way, we can redefine division as multiplication by a reciprocal.

Example

Divide the expressions. Express your answer in lowest form.

$$\begin{aligned} &\frac{2x^2-x-3}{x^2-x-12} \div \frac{x^2+5x+4}{16-x^2} \\ \frac{2x^2-x-3}{x^2-x-12} \div \frac{x^2+5x+4}{16-x^2} &= \frac{2x^2-x-3}{x^2-x-12} * \frac{16-x^2}{x^2+5x+4} \\ &= \frac{(2x-3)(x+1)}{(x-4)(x+3)} * \frac{(4+x)(4-x)}{(x+1)(x+4)} \\ &= \frac{(2x-3)\ \cancel{(x+1)}}{\cancel{(x-4)}(x+3)} * \frac{\cancel{(4+x)}\ \cancel{(4-x)}(-1)}{\cancel{(x+1)}\ \cancel{(x+4)}} \\ &= -\frac{2x-3}{x+3} \end{aligned}$$

Exercises 1.6

Reduce each expression to lowest terms.

1) $\frac{3x+9}{x^2-9}$

2) $\frac{4x^2+8x}{12x+24}$

3) $\frac{x^2-2x}{6-3x}$

4) $\frac{15x^2+24x}{3x^2}$

5) $\frac{24x^2}{12x^2-6x}$

6) $\frac{x^2+4x+4}{x^2-4}$

7) $\frac{25-y^2}{2y^2-8y-10}$

8) $\frac{3y^2-y-2}{3y^2+5y+2}$

9) $\frac{x^2+4x-5}{x^2-2x+1}$

10) $\frac{x-x^2}{x^2+x-2}$

11) $\frac{x^2+5x-14}{2-x}$

12) $\frac{2x^2+5x-3}{1-2x}$

Multiply or divide the expressions in each problem.
Express your answers in lowest terms.

13) $\frac{3x+6}{5x^2} * \frac{x}{x^2-4}$

14) $\frac{4x^2}{x^2-16} * \frac{7x-28}{6x}$

15) $\frac{2a^2-7a+6}{4a^2-9} * \frac{4a^2+12a+9}{a^2-a-2}$

16) $\frac{4a^2-4a-3}{8a+4a^2} * \frac{16a^2}{4a^2-6a}$

17) $\frac{x^2-y^2}{(x+y)^3} * \frac{(x+y)^2}{(x-y)^2}$

18) $\frac{2x^2+x-3}{x^2-1} * \frac{2x-2}{2x^2+5x+3}$

19) $\frac{6x}{x^2-4} \div \frac{3x-9}{2x+4}$

20) $\frac{12x}{5x+20} \div \frac{4x^2}{x^2-16}$

21) $\frac{9x^2+3x-2}{6x^2-2x} \div \frac{3x+2}{6x^2}$

22) $\frac{2a^2-5a-3}{4a^2+2a} \div \frac{2a+1}{4a}$

23) $\frac{x^2+7x+6}{x^2+x-6} \div \frac{x^2+5x-6}{x^2+5x+6}$

24) $\frac{x^2+7x+10}{x^2-x-30} \div \frac{3x^2+7x+2}{9x^2-1}$

25) $\frac{2x^2-x-28}{3x^2-x-2} \div \frac{4x^2+16x+7}{3x^2+11x+6}$

26) $\frac{9x^2+3x-2}{12x^2+5x-2} \div \frac{9x^2-6x+1}{8x^2-10x-3}$

1.7: Adding and Subtracting Rational Expressions

Just as we do with numerical fractions, we will need to have common denominators in order to add or subtract algebraic fractions. When we add $\frac{1}{2}+\frac{1}{3}$, we make a common denominator of 6 so that we can add them together.

$$\begin{aligned}\frac{1}{2}+\frac{1}{3} &= \frac{1}{2}*\frac{3}{3}+\frac{1}{3}*\frac{2}{2}\\ &= \frac{3}{6}+\frac{2}{6}\\ &= \frac{5}{6}\end{aligned}$$

Because the denominators, 2 and 3, are prime and don't share any common factors, the common denominator is simply $3*2=6$. We can see a similar result in adding algebraic fractions.

Example

Add the fractions. Express your answer in lowest terms.

$$\frac{2}{x}+\frac{x}{x-3}$$

$$\begin{aligned}\frac{2}{x}+\frac{x}{x-3} &= \frac{2}{x}*\frac{x-3}{x-3}+\frac{x}{x-3}*\frac{x}{x}\\ &= \frac{2(x-3)}{x(x-3)}+\frac{x*x}{x(x-3)}\\ &= \frac{2x-6+x^2}{x(x-3)}=\frac{x^2+2x-6}{x(x-3)}\end{aligned}$$

It's important to be aware that in subtraction, the negative sign representing subtraction must be distributed to all terms in the second numerator.

Example

Subtract the given expressions. Express your answer in lowest terms.

$$\frac{6}{x+1}-\frac{x+5}{x-2}$$

$$\begin{aligned}\frac{6}{x+1}-\frac{x+5}{x-2} &= \frac{6}{x+1}*\frac{x-2}{x-2}-\frac{x+5}{x-2}*\frac{x+1}{x+1}\\ &= \frac{6(x-2)-(x+5)(x+1)}{(x+1)(x-2)}\\ &= \frac{6x-12-\left(x^2+6x+5\right)}{(x+1)(x-2)}\\ &= \frac{6x-12-x^2-6x-5}{(x+1)(x-2)}\\ &= \frac{-x^2-17}{(x+1)(x-2)}\end{aligned}$$

In other situations, the denominators may share a common factor. In this case, we can turn one of the denominators into the other one:

Example

Add the given fractions.

$\frac{7}{x^2+8x+15}+\frac{2}{x+3}$

$$\frac{7}{x^2+8x+15}+\frac{2}{x+3}=\frac{7}{(x+3)(x+5)}+\frac{2}{x+3} \tag{1.7.1}$$

We can turn $(x+3)$ into $x^2+8x+15$ by multiplying by $(x+5)$

$$\begin{aligned}\frac{7}{(x+3)(x+5)}+\frac{2}{x+3} &= \frac{7}{(x+3)(x+5)}+\frac{2}{x+3}*\frac{x+5}{x+5}\\ &= \frac{7}{(x+3)(x+5)}+\frac{2(x+5)}{(x+3)(x+5)}\\ &= \frac{7+2x+10}{(x+3)(x+5)}\\ &= \frac{2x+17}{(x+3)(x+5)}\end{aligned}$$

Sometimes, the answer we end up with is not in lowest terms:

Example

Add the fractions.

$$\frac{x}{x+2}+\frac{8}{x^2+8x+12}$$

$$\begin{aligned}\frac{x}{x+2}+\frac{8}{x^2+8x+12} &= \frac{x}{x+2}+\frac{8}{(x+2)(x+6)}\\ &= \frac{x}{x+2}*\frac{x+6}{x+6}+\frac{8}{(x+2)(x+6)}\\ &= \frac{x(x+6)}{(x+2)(x+6)}+\frac{8}{(x+2)(x+6)}=\frac{x(x+6)+8}{(x+2)(x+6)}\end{aligned}$$

$$\frac{x(x+6)+8}{(x+2)(x+6)}=\frac{x^2+6x+8}{(x+2)(x+6)} \tag{1.7.2}$$

The numerator is factorable:

$$\begin{aligned}\frac{x^2+6x+8}{(x+2)(x+6)} &= \frac{(x+2)(x+4)}{(x+2)(x+6)}\\ &= \frac{\cancel{(x+2)}(x+4)}{\cancel{(x+2)}(x+6)}\\ &= \frac{x+4}{x+6}\end{aligned}$$

Add or subtract the given expressions.

1) $\frac{1}{x-1}-\frac{1}{x}$

2) $\frac{3}{y-6}-\frac{1}{y}$

3) $\frac{2}{x-3}+\frac{4}{x+3}$

4) $\frac{3}{x+4}-\frac{4}{x-2}$

5) $\frac{3}{k+2}-\frac{k-4}{k+5}$

6) $\frac{a+1}{a}-\frac{a}{a+1}$

7) $\frac{2y}{y^2-25}-\frac{y}{y-5}$

8) $\frac{x}{x^2-1}+\frac{4}{x+1}$

9) $\frac{1}{x-3}+\frac{x}{x+1}$

10) $\frac{9y}{y-4}-\frac{y+1}{y+5}$

Add or subtract the given expressions. Express your answers in lowest terms.

11) $\frac{b}{b+1}-\frac{b+1}{2b+2}$

12) $\frac{4x+1}{8x-12}+\frac{x-3}{2x-3}$

13) $\frac{2}{a^2+4a+3}+\frac{1}{a+3}$

14) $\frac{1}{y+6}-\frac{4}{y^2+8y+12}$

15) $\frac{x+1}{2x+4}-\frac{x^2}{2x^2-8}$

16) $\frac{x+1}{x+2}-\frac{x^2+1}{x^2-x-6}$

17) $\frac{2x}{x^2-3x+2}+\frac{2x}{x-1}-\frac{x}{x-2}$

18) $\frac{3x+3}{2x^2-x-1}+\frac{1}{2x+1}$

19) $\frac{4a}{a-2}-\frac{3a}{a-3}+\frac{4a}{a^2-5a+6}$

20) $\frac{2}{y-3}-\frac{8-4y}{y^2-8y+15}$

21) $\frac{2x}{x-1}+\frac{3x}{x+1}-\frac{x+3}{x^2-1}$

22) $\frac{a}{a-1}-\frac{2}{a+2}+\frac{3(a-2)}{a^2+a-2}$

23) $\frac{x}{x-1}+\frac{x+7}{x^2-1}-\frac{x-2}{x+1}$

24) $\frac{2y+5}{y^2-16}-\frac{y-9}{y^2-y-12}$

1.8: Complex Fractions

1.8 Complex Fractions

Complex fractions involve simplifying a rational expression which has a complicated numerator and/or denominator

Example

Simplify.

$$\frac{3+\frac{x}{x+2}}{1-\frac{x+3}{x-1}} \tag{1.8.1}$$

There are a variety of ways to approach this problem. One of the most straightforward ways to simplify the expression above is to create common denominators for the numerator and the denominator so that each one is a single fractional expression:

$$\begin{aligned}\frac{3+\frac{x}{x+2}}{1-\frac{x+3}{x-1}} &= \frac{\frac{3}{1}*\frac{x+2}{x+2}+\frac{x}{x+2}}{\frac{1}{1}*\frac{x-1}{x-1}-\frac{x+3}{x-1}} \\ &= \frac{\left(\frac{3x+6+x}{x+2}\right)}{\left(\frac{x-1-(x+3)}{x-1}\right)}\end{aligned}$$

$=\frac{\left(\frac{4x+6}{x+2}\right)}{\left(\frac{-4}{x-1}\right)}$ (Now this is a division problem)

$$\begin{aligned}&=\frac{4x+6}{x+2}*\frac{x-1}{-4}=\frac{2(2x+3)}{x+2}*\frac{x-1}{-4} \\ &=\frac{\cancel{2}(2x+3)}{x+2}*\frac{x-1}{\cancel{4}(-2)}=\frac{(2x+3)(x-1)}{-2(x+2)}\end{aligned} \tag{1.8.2}$$

Simplifying complex fractions uses all of the previous concepts about rational expressions which we've covered in this chapter.

Example

Simplify.

$$\frac{x-\frac{x}{x+3}}{1+\frac{2}{x}} \tag{1.8.3}$$

$\frac{x-\frac{x}{x+3}}{1+\frac{2}{x}}=\frac{\frac{x}{1}*\frac{x+3}{x+3}-\frac{x}{x+3}}{\frac{1}{1}*\frac{x}{x}+\frac{2}{x}}$ creating common denominators

$$\begin{aligned}&=\frac{\left(\frac{x(x+3)-x}{x+3}\right)}{\left(\frac{x+2}{x}\right)} \\ &=\frac{\left(\frac{x^2+3x-x}{x+3}\right)}{\left(\frac{x+2}{x}\right)}=\frac{\left(\frac{x^2+2x}{x+3}\right)}{\left(\frac{x+2}{x}\right)} \text{ dividing fractions} \\ &=\frac{x^2+2x}{x+3}*\frac{x}{x+2}=\frac{x(x+2)}{x+3}*\frac{x}{x+2}\end{aligned} \tag{1.8.4}$$

$=\frac{x\,\cancel{(x+2)}}{x+3}*\frac{x}{\cancel{x+2}}$ factor and cancel to reduce to lowest terms

$$=\frac{x^2}{x+3} \tag{1.8.5}$$

Exercises 1.8

Simplify each complex fraction. Express your answer in lowest terms.

1) $\frac{1}{\left(x+\frac{y}{2}\right)}$

2) $\frac{\left(\frac{1}{x}+\frac{1}{y}\right)}{\left(\frac{y}{x}-\frac{x}{y}\right)}$

3) $\frac{\left(1+\frac{m}{n}\right)}{\left(1-\frac{n^2}{m^2}\right)}$

4) $\frac{\left(\frac{1}{x}-\frac{1}{y}\right)}{\left(\frac{1}{x^2}-\frac{1}{y^2}\right)}$

5) $\frac{\left(\frac{x}{y}-\frac{x-y}{x+y}\right)}{\left(\frac{y}{x}+\frac{x-y}{x+y}\right)}$

6) $\frac{\left(\frac{7}{a+1}-\frac{3}{a}\right)}{\left(\frac{3}{a}+\frac{1}{a-1}\right)}$

7) $\frac{\left(x-\frac{1}{2x+1}\right)}{\left(1-\frac{2}{2x+1}\right)}$

8) $\frac{\left(\frac{1}{2x-2}-\frac{1}{x}\right)}{\left(\frac{2}{x}-\frac{1}{x-1}\right)}$

9) $\frac{\left(x+\frac{4}{x+4}\right)}{\left(x-\frac{4x+4}{x+4}\right)}$

10) $\frac{\left(x-\frac{x+6}{x+2}\right)}{\left(x-\frac{4x+15}{x+2}\right)}$

11) $\frac{\left(\frac{1}{x+2}-\frac{1}{x-3}\right)}{\left(1+\frac{1}{x^2-x-6}\right)}$

$\frac{\left(1-\frac{1}{x+1}\right)}{\left(1+\frac{1}{x-1}\right)}$

13) $\frac{\left(\frac{1}{a-b}-\frac{3}{a+b}\right)}{\left(\frac{2}{b-a}+\frac{4}{b+a}\right)}$

14) $\frac{\left(\frac{3}{y^2-4}\right)}{\left(\frac{1}{y+2}-\frac{1}{y-2}\right)}$

$\frac{\left(n+2-\frac{5}{n-2}\right)}{\left(1-\frac{1}{(n-2)^2}\right)}$

16) $\frac{\left(4+\frac{1}{x+1}\right)}{\left(16-\frac{1}{(x+1)^2}\right)}$

17) $\frac{\left(2+\frac{x-2}{1-x^2}\right)}{\left(2-\frac{3}{x+1}\right)}$

18) $\frac{\left(\frac{1}{2x-1}-\frac{1}{2x+1}\right)}{\left(4-\frac{1}{x^2}\right)}$

1.9: Rational Equations

1.9 Rational Equations

We can also use the skills we have covered in previous sections to solve equations involving rational expressions. There are three main methods of solution that will be explored in this section - multiplying on both sides to clear a denominator, cross-multiplying, and making common denominators. Each of these techniques is actually the same process, but approached from a slightly different perspective.

Clearing a denominator

Often, if the denominator is a single variable, it can be easy and striaghtforward to multiply on both sides by the denominator to cancel it out.

Example

Solve for x

$$x+\frac{4}{x}=7 \tag{1.9.1}$$

If we multiply on both sides by x, it will clear the variable from the denominator:

$$\begin{array}{c} x+\frac{4}{x}=7 \\ x*\left(x+\frac{4}{x}\right)=(7)*x \quad \text{Multiply on both sides by } x \end{array} \tag{1.9.2}$$

$$x*x+x*\frac{4}{x}=7x \quad \text{Distribute the } x \tag{1.9.3}$$

$$\begin{array}{r} x^2+4=7x \\ x^2-7x+4=0 \end{array} \quad \text{Standard form} \tag{1.9.4}$$

$$x\approx 0.628, 6.372 \tag{1.9.5}$$

If a problem is stated simply as the equality of two fractions, cross-multiplying can be a useful method of solution.

Example

Solve for x

$$\frac{x}{2x+3}=\frac{7}{x-4}$$

$$\begin{aligned} \frac{x}{2x+3} &= \frac{7}{x-4} \\ x(x-4) &= 7(2x+3) \\ x^2-4x &= 14x+21 \\ x^2-18x-21 &= 0 \\ x &\approx 19.100, -1.100 \end{aligned}$$

Cross-multiplying is really just a short-cut method of clearing out the denominators by multiplying on both sides by both denominators:

$$\begin{aligned} \frac{x}{2x+3} &= \frac{7}{x-4} \\ (x-4)(2x+3) * \frac{x}{2x+3} &= \frac{7}{x-4} * (x-4)(2x+3) \\ (x-4)\,\cancel{(2x+3)} * \frac{x}{\cancel{2x+3}} &= \frac{7}{\cancel{x-4}} * \cancel{(x-4)}(2x+3) \\ x(x-4) &= 7(2x+3) \end{aligned}$$

Then the equation is ready to be solved as shown above - but by just cross-mulitiplying, we skip directly to the solution portion of the problem.

Sometimes, it is helpful to create a common denominator in order to set up a situation where cross-multiplying can be used.

Example
Solve for x

$$\begin{aligned} \frac{1}{x+6} + \frac{4}{x-2} &= \frac{3}{x+1} \\ \frac{1}{x+6} + \frac{4}{x-2} &= \frac{3}{x+1} \\ \frac{1}{x+6} * \frac{x-2}{x-2} + \frac{4}{x-2} * \frac{x+6}{x+6} &= \frac{3}{x+1} \\ \frac{1(x-2)+4(x+6)}{(x+6)(x-2)} &= \frac{3}{x+1} \\ \frac{x-2+4x+24}{(x+6)(x-2)} &= \frac{3}{x+1} \\ \frac{5x+22}{(x+6)(x-2)} &= \frac{3}{x+1} \\ (5x+22)(x+1) = 3(x+6)(x-2) &= 3\left(x^2+4x-12\right) \\ 5x^2+27x+22 &= 3x^2+12x-36 \\ 2x^2+15x+58 &= 0 \\ x &\approx -3.75 \pm 3.865i \end{aligned}$$

Example
Solve for x

$$\begin{aligned} \tfrac{2}{x-2} + \tfrac{x}{2x-1} &= 4 \\ \tfrac{2}{x-2} + \tfrac{x}{2x-1} &= 4 \\ \tfrac{2}{x-2} * \tfrac{2x-1}{2x-1} + \tfrac{x}{2x-1} * \tfrac{x-2}{x-2} &= 4 \\ \tfrac{2(2x-1)+x(x-2)}{(x-2)(2x-1)} &= 4 \\ \tfrac{4x-2+x^2-2x}{(x-2)(2x-1)} &= 4 \\ \tfrac{x^2+2x-2}{(x-2)(2x-1)} &= \tfrac{4}{1} \\ 1\left(x^2+2x-2\right) = 4(x-2)(2x-1) &= 4\left(2x^2-5x+2\right) \\ x^2+2x-2 &= 8x^2-20x+8 \\ \frac{2}{x} &= 7x^2-22x+10 \\ x &= 2.592, 0.551 \end{aligned}$$

In some situations, we can create a single common denominator for every fraction in the problem and then clear them all at once.

Example
Solve for x

$$\frac{2}{x+3}-\frac{4}{3x-1}=\frac{x}{3x^2+8x-3}$$

$$\begin{aligned}\frac{2}{x+3}-\frac{4}{3x-1}&=\frac{x}{3x^2+8x-3}\\ \frac{2}{x+3}*\frac{3x-1}{3x-1}-\frac{4}{3x-1}*\frac{x+3}{x+3}&=\frac{x}{3x^2+8x-3}\\ \frac{2(3x-1)-4(x+3)}{(x+3)(3x-1)}&=\frac{x}{(x+3)(3x-1)}\\ 2(3x-1)-4(x+3)&=x\end{aligned}$$

The missing step above is the clearing of both denominators:

$$(x+3)(3x-1)*\frac{2(3x-1)-4(x+3)}{(x+3)(3x-1)}=\frac{x}{(x+3)(3x-1)}*(x+3)(3x-1) \tag{1.9.6}$$

$$\cancel{(x+3)}\ \cancel{(3x-1)}*\frac{2(3x-1)-4(x+3)}{\cancel{(x+3)}\ \cancel{(3x-1)}}=\frac{x}{\cancel{(x+3)}\ \cancel{(3x-1)}}*\ \cancel{(x+3)}\ \cancel{(3x-1)}$$

$$2(3x-1)-4(x+3)=x$$

As is true in the process of cross-multiplying, it isn't necessary to actually cancel out the denominators in completing the problem.

$$\begin{aligned}\frac{2}{x+3}-\frac{4}{3x-1}&=\frac{x}{3x^2+8x-3}\\ \frac{2}{x+3}*\frac{3x-1}{3x-1}-\frac{4}{3x-1}*\frac{x+3}{x+3}&=\frac{x}{3x^2+8x-3}\\ \frac{2(3x-1)-4(x+3)}{(x+3)(3x-1)}&=\frac{x}{(x+3)(3x-1)}\\ 2(3x-1)-4(x+3)&=x\\ 6x-2-4x-12&=x\\ 2x-14&=x\\ x&=14\end{aligned}$$

Exercises 1.9

1) $x+\frac{5}{x}=-6$
2) $x+\frac{6}{x}=-7$
3) $y-\frac{5}{y}=2$
4) $\frac{7}{a}+1=a$
5) $\frac{9}{2y+4}=\frac{3}{y}$
6) $\frac{4}{3n+7}=\frac{1}{2}$
7) $\frac{x}{x+3}=\frac{8}{x+6}$
8) $\frac{y-2}{2}=\frac{5}{y-5}$
9) $\frac{2}{n}=\frac{n}{5n+12}$
10) $\frac{x}{4-x}=\frac{2}{x}$
11) $\frac{5x}{14x+3}=\frac{1}{x}$
12) $\frac{a}{8a+3}=\frac{1}{3a}$
13) $\frac{9}{x-1}-\frac{2}{x+4}=\frac{1}{x+2}$
14) $\frac{1}{x-2}+\frac{4}{x+5}=\frac{1}{x-3}$
15) $\frac{5}{3x+2}+\frac{1}{x-1}=\frac{3}{x+2}$
16) $\frac{1}{y-2}-\frac{4}{2y+5}=\frac{6}{y-1}$
17) $\frac{5}{x+1}+\frac{1}{x+2}=3$

18) $\frac{1}{2x-1}-\frac{2}{x+7}=1$
19) $\frac{6}{y-4}-\frac{1}{y+2}=3$
20) $\frac{10}{a+1}+\frac{3}{a-2}=2$
21) $\frac{3a}{a^2-2a-15}-\frac{a}{a+3}=\frac{2a}{a-5}$
22) $\frac{u^2+2}{u^2+u-2}-\frac{3u}{u+2}=\frac{-2u-1}{u-1}$
23) $\frac{4}{2x-1}+\frac{2}{x+3}=\frac{5}{2x^2+5x-3}$
24) $\frac{5}{x+5}-\frac{2}{x^2+2x-15}=\frac{2}{x-3}$
25) $\frac{5}{y-2}-\frac{3}{2y-1}=\frac{4}{2y^2-5y+2}$
26) $\frac{x+2}{x-1}+\frac{x+4}{x}=\frac{2x+1}{x^2-x}$
27) $\frac{x}{x+2}+\frac{x+1}{x^2-7x-18}=\frac{5}{x-9}$
28) $\frac{2a}{a+7}-\frac{a}{a+3}=\frac{5}{a^2+10a+21}$
29) $\frac{x-1}{2x+1}-\frac{2x-3}{x+3}=\frac{3}{2x^2+7x+3}$
30) $\frac{y}{y+4}+\frac{6}{y+1}=\frac{y^2+4}{y^2+5y+4}$

Addendum - Word Problems

Following are a selection of word problems - some from ancient times, some from the Renaissance and Enlightenment, and some from the 19 th and 20 th century.

1) A teacher agreed to work 9 months for $\$562.50$ and board. At the end of the term, on account of two months absence caused by illness, he received only $\$409.50$ for his seven months work. If the teacher used all nine months of his board during the term, what was his board per month? (American 1892)

2) A servant is promised $\$100$ plus a cloak as wages for a year. After seven months, he leaves and receives $\$20$ plus the cloak. How much is the cloak worth? (Clavius, German 1608)

3) The sales tax on garments is $\frac{1}{20}$ of their value. A certain man buys 42 garments, paying in copper coins. Two garments and 10 copper coins are paid as tax. What is the price of a garment, O learned one? (Ancient India)

4) Two wine merchants enter Paris, one of them with 64 casks of wine, the other with 20 casks (all of the same value). since they do not have enough money to pay the customs duties, the first pays 5 casks of wine and 40 francs, and the second pays 2 casks of wine and receives 40 francs change. What is the price of each cask of wine and what is the duty on each cask? (Chuquet, French 1484)

5) One of two men had 12 fish and the other had 13 fish, and all of the fish were of the same price. From the first man, a customs agent took away 1 fish and
12 denarii for payment. From the other man he took 2 fish and gave him back 7 denarii as change. Find the customs fee and the price of each fish. (Fibonacci, Italian 1202)

6) Two traders transporting sheepskins approach their country's border. The first trader has 100 sheepskins and the border guard takes 10 sheepskins plus $\$25$ as a tariff. The second trader has 42 sheepskins and for a tariff, the border guard takes 7 sheepskins but returns $\$14$ change. What is the tariff per sheepskin and what is the value of each sheepskin?

7) Two people have a certain amount of money. The first says to the second, "If you give me 5 denarii, I will have 7 times what you have left." The second says to the first, "If you give me 7 denarii, I will have 5 times what you have left." How much money does each have? Round to the nearest 10 th. (Leonardo, Italian c. 1500)

8) Two different scenarios from Ancient Greece:
Two friends were walking. One said to the other, "If I had 10 more coins, I would have 3 times as much money as you." The other said, "If I had 10 more coins, I would have five times as much as you." How many coins does each have?

Two friends were walking. One said to the other, "If you give me 10 of your coins, I would have 3 times as much money as you." The other said, "If you give me 10 of your coins, I would have five times as much as you." How many coins does each have?

9) Andy and Betty together have $\$6$ less than Christine. If Betty gives $\$5$ to Andy, then Andy will have half as much as Christine. If, instead, Andy gives $\$5$ to Betty, then Andy will have one-third as much as Betty. How much does each person have to begin with?

10) Three friends $(\mathrm{A}, \mathrm{B} \text{ and } \mathrm{C})$ each have a certain amount of money. A says, "I have as much as B plus one-third as much as C." B says, "I have as much as C plus one-third as much as A." Csays, "I have 10 more than one-third of B." How much does each person have? (Ancient Greece)

11) On a test, 39 more pupils passed than failed. On the next test, 7 who passed the first test failed and one-third of those who failed the first test passed the second. As a result, 31 more passed the second test than failed it. What was the record of passing and failing on the first test?

12) At two stations, A and B, six miles apart on the railway, the prices of coal are $\$20$ per ton and $\$24$ per ton respectively. The rates of cartage of coal are $\$2.00$ per ton per mile from A and $\$3.00$ per ton per mile from B. At a certain customer's home, on the railroad from A to B, the cost of coal is the same whether delivered from A or B. Find the distance to this home from A.

13) There were three-fourths as many women as there were men on the train. At the next station six men and eight women got off the train, and twelve men and five women got on. There were then three-fifths as many women as men on the train. How many men and how many women were originally on the train?

14) If a theater could put 5 more seats in a row, it would need 20 rows less, but if each row had 3 fewer seats, it would take 20 rows more to seat the same number. How many people will it seat?

15) An audience of 450 people is seated in rows, with the same number of people in each row. It would take 5 rows less if 3 more people were seated in each row. In how many rows are the people seated?

16) If evergreens are planted 4 feet closer together it will take 44 more trees for a certain piece of road, but if they are planted 6 feet farther apart, it will take 44 fewer trees for the same length of road. How many miles is the piece of road? (use $5280 \text{ feet } = 1 \text{ mile }$)

17) A movie theater owner found that by raising the price of each ticket by $\$1.00, 200$fewer people attened and she broke even, but that if she lowered the price by $\$1.00$ per person, 550 people attended and she increased her receipts by $\$1000.$What is the usual rate per person?

18) The brine pipes in an 84 -foot width artificial hockey rink are equally spaced. If the space between each pair of pipes were increased by 1 inch, then 84 fewer lengths of pipe would be needed. What is the distance between the pipes now?

19) A living room shelf is 36 inches long and contains a certain number of books of uniform width. If each book were one-half inch narrower, the shelf would hold six more books. How many books of the wider variety does it hold?

20) I am a brazen lion; my spouts are my two eyes, my mouth and flat of my right foot. My right eye fills a jar in two days, my left eye in three and my foot in four. My mouth is capable of filling it in six hours. Tell me how long all four together will take to fill it. (Ancient Greece)

21) A man wishes to have 500 rubii of grain ground. He goes to a mill that has five stones. The first stone grinds 7 rubii of grain in an hour, the second grinds 5 rubii in an hour, the third 4 rubii in an hour, the fourth grinds 3 rubii per hour and the fifth grinds 1 rubii per hour. In how long will the grain be ground and how much is done by each stone? (Clavius, German 1583)

22) If two men and three boys can plow an acre in one-sixth of a day, how long would it require three men and two boys to plow it? (Edward Brooks, American 1873)

23) A cobbler can cut leather for ten pairs of shoes in one day. He can finish 5 pairs of shoes in one day. How many pairs of shoes can he both cut and finish in one day? (Ancient Egypt)

24) Four waterspouts are filling a tank. Of the four spouts, one can fill the tank in one day, the second takes two days, the third takes three days and the fourth takes four days. How long will it take all four spouts working together to fill the tank? (Ancient Greece)

25) If, in one day, a person can make 30 arrows or fletch [put feathers on] 20 arrows, how many arrows can this person both make and fletch in one day? (Ancient China)

26) One military horse and one ordinary horse can pull a load of 40 dan. Two ordinary horses and one inferior horse can pull the same load of 40 dan as can three inferior horses and one military horse. How much can each horse pull individually? (Ancient China)

27) A barrel of water has several holes in it. The first hole empties the full barrel in 3 days. The second hole empties the full barrel in 5 days. The third hole empties the full barrel in 20 hours and the last hole empties the full barrel in 12 hours. With all the holes open, how long will it take to empty the barrel? (Levi ben Gershon, French 1321)

28) A certain lion could eat a sheep in 4 hours; a leopard could do so in 5 hours; and a bear in 6 hours. How many hours would it take for all three animals to devour a sheep if it were thrown in among them? (Fibonacci, Italian 1202)

29) Two ships are some distance apart, which journey the first can complete in 5 days and the other in 7 days - it is sought in how many days they will meet if they begin the journey at the same time. (Fibonacci, Italian 1202)

30) Sarah, alone, can paint the garage in 24 hours, her sister Jenny, alone, can paint the same garage in 12 hours. With the aid of their mother, the three together can paint the garage in 4 hours. How long would it take their mother, working alone, to paint the garage?

31) It required 75 workers 38 days to build an embankment to be used for flood control. Had 18 workers been removed to another job at the very start of operations, how much longer would it have taken to build the embankment?

32) Mark, alone, requires 6 hours to paint a fence; however, his younger brother, who alone could do it in 9 hours, helps him. If they start work at 8: 30 am, at what time should they finish the work?

33) A group decides to build a cabin together. The job can be done by 3 skilled workers in 4 days or by 5 amateurs in 6 days. How long would it take if they all work together?

34) If it requires 18 workers 50 days to build a piece of road, how many days sooner would it be done if 7 more workers were hired at the beginning of operations?

35) A contractor estimated that a certain piece of work would be done by 9 carpenters in 8 hours or by 16 amateurs in 9 hours. The contractor wishes to get the job done as quickly as possible and uses both professional carpenters and amateurs on the same job. Four carpenters and 4 amateurs begin work at 6 am. Allowing 45 minuntes for lunch, at what time should they finish the job?

36) Mrs. Ellis alone can do a piece of work in 6 days. Her oldest daughter takes 2 days longer; her youngest daughter takes twice as long as her mother. How long will it take to complete the job if all three work together?

37) If 5 men and 2 boys work together, a piece of work can be completed in one day and if 3 men and 6 boys work together, it can be completed in one day. How long would it take a boy to do the work alone?

38) A coal company can fill a certain order from one mine in 3 weeks and from a second mine in 5 weeks. How many weeks would be required to fill the order if both mines were used?

39) If 25 skilled workers work for 8 days, they can complete the construction of a concrete dam; 12 skilled workers and 15 untrained workers together can complete the dam in 10 days. How long would it take an untrained worker alone to complete the work on the dam?

40) A and B working together can complete a piece of work in 30 days. After they have both worked 18 days, however, A leaves and B finishes the work alone in 20 more days. Find the time in which each can do the work alone.

41) A dump cart can haul enough gravel to fill a pit in 6 days. A truck can do the same work in 2 days. How long would it take two dump carts and one truck working together to fill the pit?

CHAPTER OVERVIEW

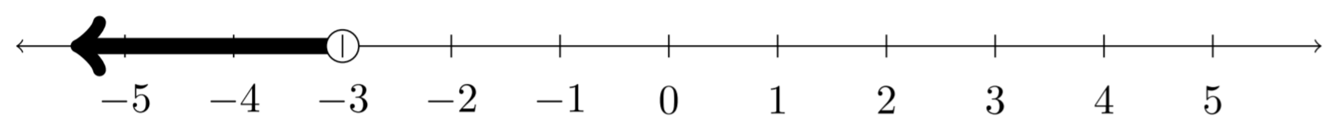
2: Polynomial and Rational Functions

This chapter will explore the solution of equations and inequalities involving both polynomial and rational functions, primarily through the examination of their graphical representations. We will also explore the use of polynomial long division and synthetic division in breaking down polynomials into their prime factors and the relationship between factors and roots.

2.1: Representing Intervals

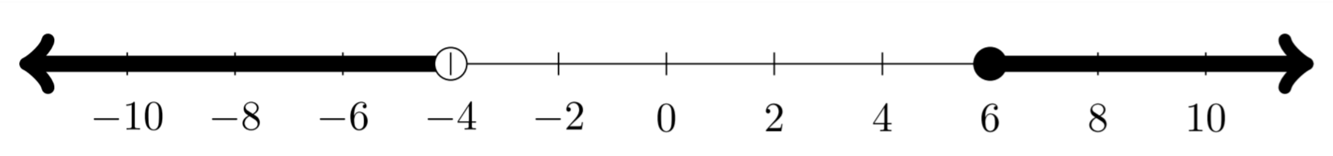
Many of the problems in this chapter will have solutions that must be expressed as an interval. This means a range of x values that will satisfy the original problem. In this section, we will introduce the translation of graphical intervals into algebraic notation.

For example, in the diagram below, we would represent the interval shown on the graph as $x < -3$



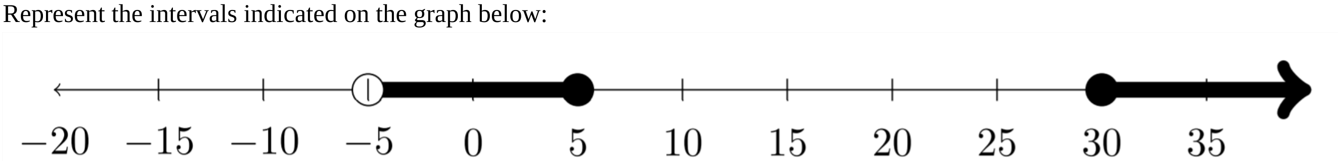
In this diagram, we would represent the interval shown on the graph as $1 \leq x < 7$.

In the next diagram, we have two separate intervals to represent-and we will need two separate statements to represent them.



These intervals would be represented as $x < -4$ OR $x \geq 6$
Sometimes students try to represent the intervals above as $6 \leq x < -4$, however, this expression would represent a single interval where x is both less than -4 and, at the same time, greater than or equal to $+6$. This is simply not possible, and would result in the empty set, which is the reason that the OR portion is needed in the correct answer.

Example
Represent the intervals indicated on the graph below:



On this graph, there is one interval beginning at -5 and ending at +5 and another beginning at 30 and continuing to infinity. Thus, these intervals would be represented as $-5 < x \leq 5$ OR $x \geq 30$

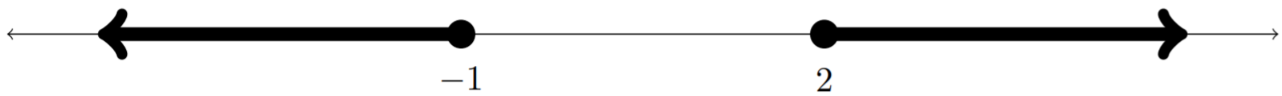
Students familiar with another form of interval notation may wish to represent this interval as $(-5, 5] \cup [30, \infty)$. Both forms of notation accomplish the same goal.

Exercises 2.1
In each problem below, represent the intervals indicated on the graph.

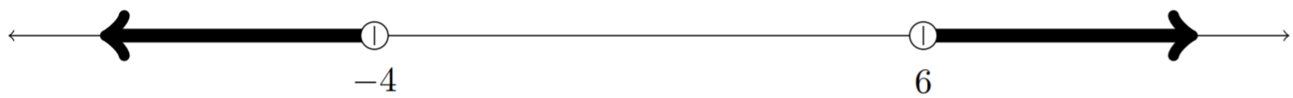
1)

2)

3)



4)



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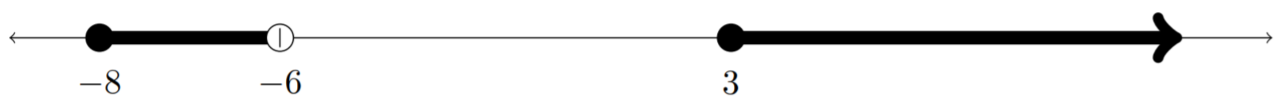
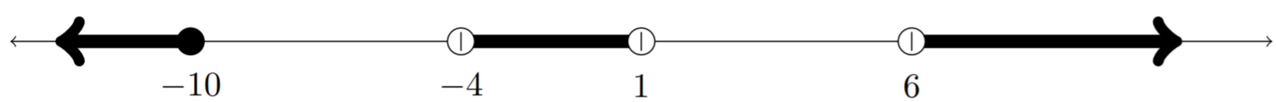


6)

7)



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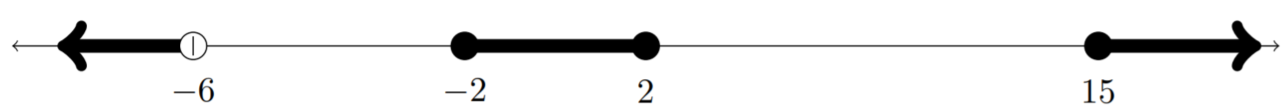
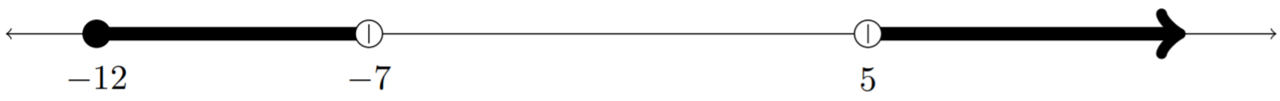
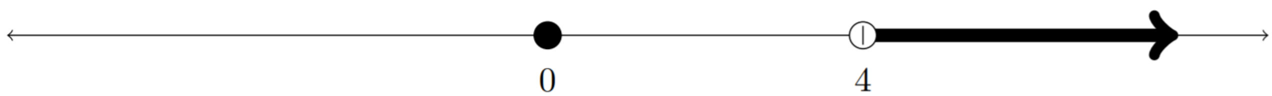


9)

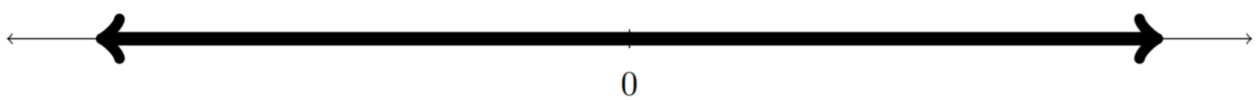


10)

11)



12)



13)

14)

2.2: Solution by Graphing

In previous courses the solution of linear equations is covered, generally by separating the variables and constants on opposite sides of the equation to isolate the variable. In the previous chapter we examined the solution of quadratic equations, in which the variable is isolated using the technique of completing the square. The major difference between these methods of solution is that, in the solving of quadratic equations, we must contend with several different powers of the variable which makes it considerably more difficult to isolate the variable. There are formulas like the quadratic formula available to solve cubic $\left(x^{3}\right)$ and quartic $\left(x^{4}\right)$ equations, however these formulas are somewhat cumbersome and archaic. The primary method for solving eqautions of degree higher than 2 is solution by graphing or by algorithm.

Solution by algorithm is a very interesting process as there are many different algorithms available. Which algorithm is most appropriate often depends on the types of equations being solved and the technology available to solve them. Two major types of algorithms that rely on the graphincal representation of an equation are called "Double False Position" and "The Newton-Raphson Method." Many commonly available pieces of techonolgy use one of these methods. since we have graphing calculators available to us, we will focus on solution by graphing.

Whether using a TI (Texas Instruments) or Casio graphing calculator, or a software based graphing ustility such as Graph or Desmos, the solution of these equations focuses on finding the x -intercepts of the graph since this is where the y value is 0. The specific processes for solving equations using each of these different tools will be covered in class.

Example

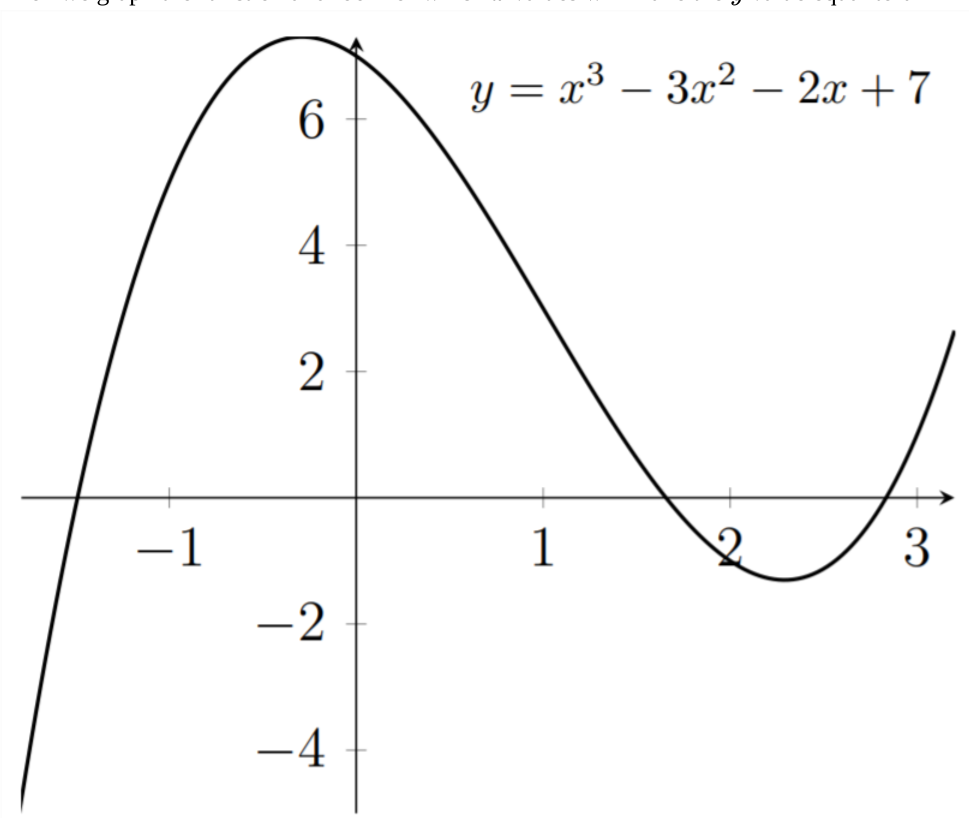
Solve for x

$x^{3}-3 x^{2}=2 x-7$

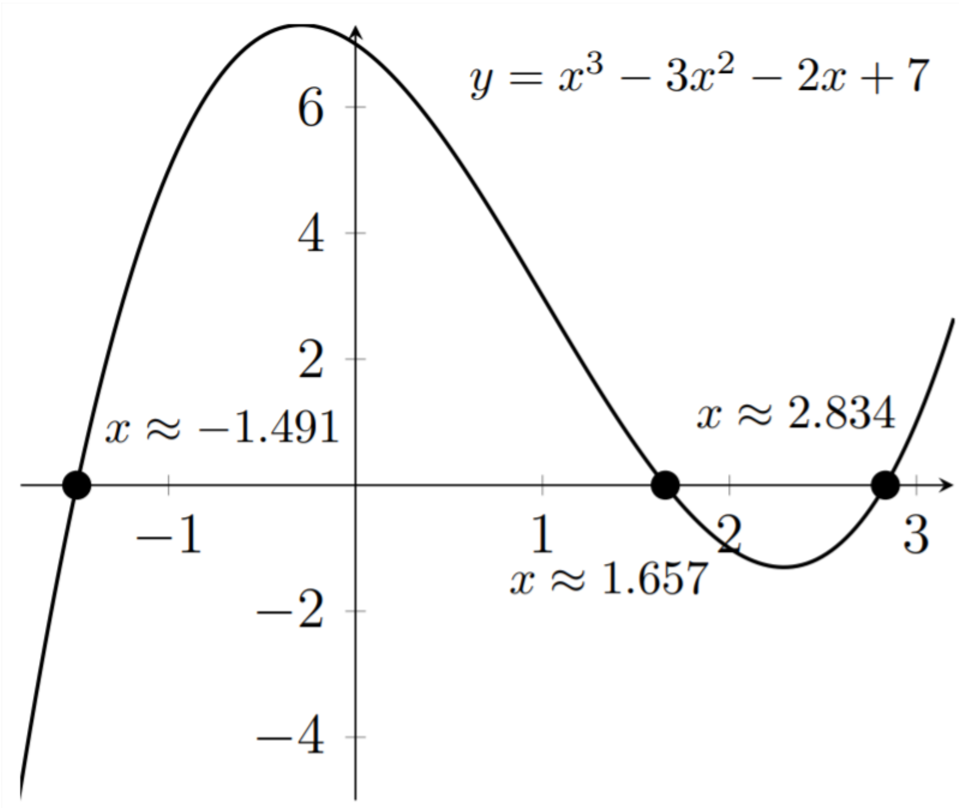
The first step is to move all terms to one side of the equation and set them equal to zero.

$x^{3}-3 x^{2}-2 x+7=0$

Then we graph the function and look for which x values will make the y value equal to 0



In this graph, we can see three roots, or x -intercepts, where the graph crosses the x -axis. These are the x -values that make the y -value equal to 0. We can use the available technology to find these x -values.



So, we can see that the solutions to the eqaution $x^3 - 3x^2 - 2x + 7 = 0$ are $x \approx -1.491, 1.657, 2.834$.
We can check these answers by plugging them back into the original equation.

$$\begin{aligned}(-1.491)^3 - 3 * (-1.491)^2 - 2 * (-1.491) + 7 &= -3.3146 - 3 * 2.223 + 2.982 + 7 \\ &= -3.3146 - 6.669 + 2.982 + 7 \\ &= -0.0016\end{aligned}$$

The result is not exactly 0 since our answers have been rounded off to the 1000 th place. If we wanted greater accuracy, then we should include greater accuracy in the value of our answers.

$$\begin{aligned}(1.657)^3 - 3 * (1.657)^2 - 2 * (1.657) + 7 &= 4.5495 - 3 * 2.7456 - 3.314 + 7 \\ &= 4.5495 - 8.2368 - 3.314 + 7 \\ &= -0.0013\end{aligned}$$

and

$$\begin{aligned}(2.834)^3 - 3 * (2.834)^2 - 2 * (2.834) + 7 &= 22.7614 - 3 * 8.03155 - 5.668 + 7 \\ &= 22.7614 - 24.09465 - 5.668 + 7 \\ &= -0.00125\end{aligned}$$

Exercises 2.2

Solve for x

1) $3x^3 - 7x^2 - x + 1 = 0$
2) $24x^4 + 5x^2 - 13x + 3 = 0$
3) $2x^3 - 2x^2 - 28x + 51 = 0$
4) $2x^3 + 5x^2 - 15x + 7 = 0$
5) $x^4 - 4x^3 + x^2 + 6x + 1 = 0$
6) $x^4 + 2x^3 + x^2 - x - 6 = 0$
7) $x^4 - 5x^3 - 3x^2 + 17x - 9 = 0$

8) $2x^3-5x-3=0$
9) $x^4-4x^3-7x^2-36x-18=0$
10) $6x^3-25x^2+21x+10=0$
11) $2x^4-5x^3+x^2+4x-2=0$
12) $x^4+x^3-5x^2+x-4=0$

2.3: Solution of Polynomial Inequalities by Graphing

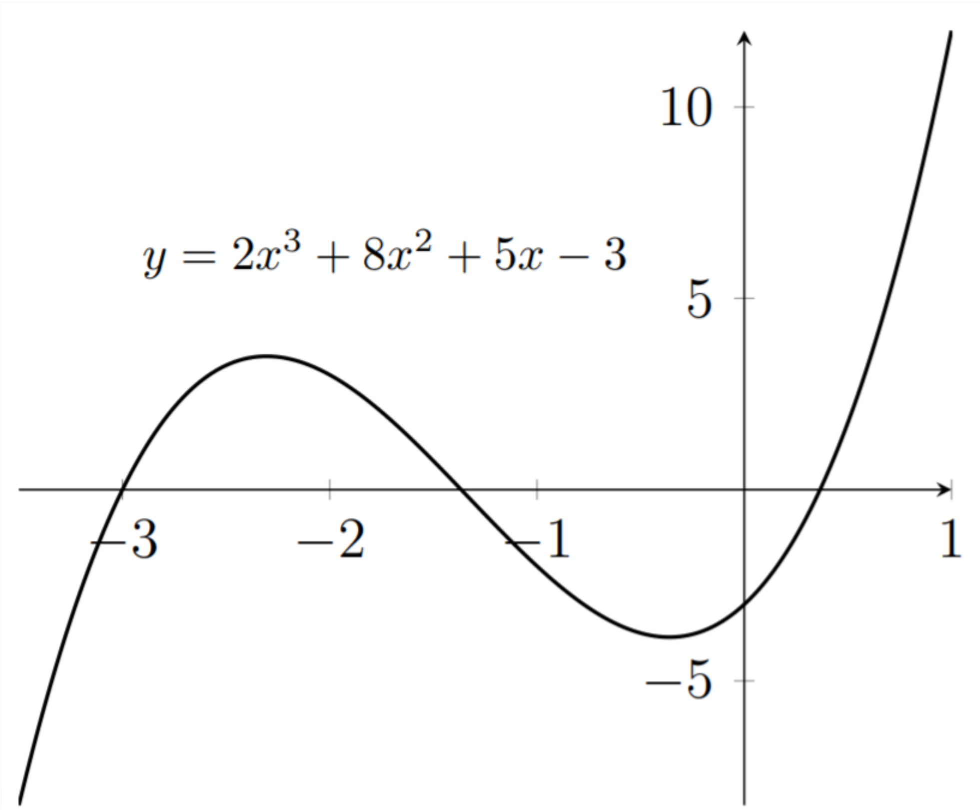
In this section, we will combine the concepts of the previous two sections to solve polynomial inequalities. In Section 2.2, we solved equations by graphing and finding the x -values which made $y=0$. In solving an inequality, we will be concerned with finding the range of x values that make y either greater than or less than 0, depending on the given problem.

Example

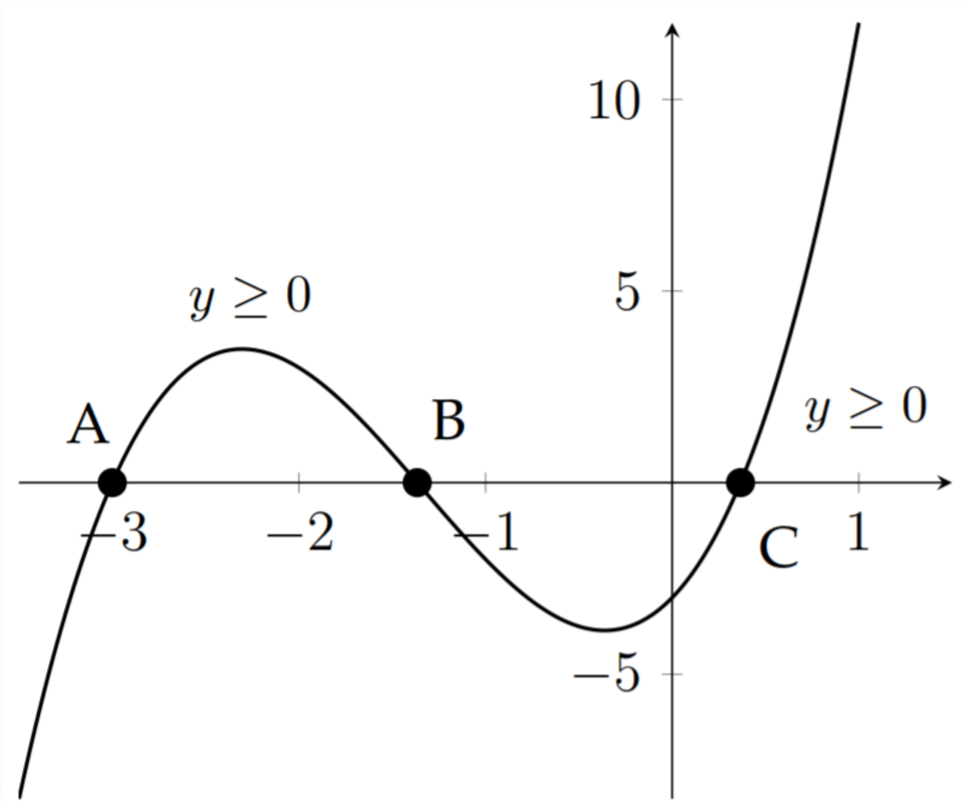
Solve the given inequality.

$2x^3+8x^2+5x-3\geq 0$

First, we graph the function:



Then we identify the intervals of x -values that make the y value greater than or equal to zero, as indicated in the problem.



The indicated roots of the function $(A, B \text{ and } C)$ are the x -values that make y equal to zero. These points divide the graph between the regions where y is greater than zero and the regions where y is less than zero. The solution to the given inequlaity $2x^3+8x^2+5x-3 \geq 0$ are $\mathrm{A} \leq x \leq \mathrm{B}$ OR $x \geq \mathrm{C}$

When we find the values of A, B and $\mathrm{C}: \mathrm{A}=-3, \mathrm{B} \approx-1.366$ and $\mathrm{C} \approx 0.366$, we can complete the solution to the problem.

$2x^3+8x^2+5x-3 \geq 0$

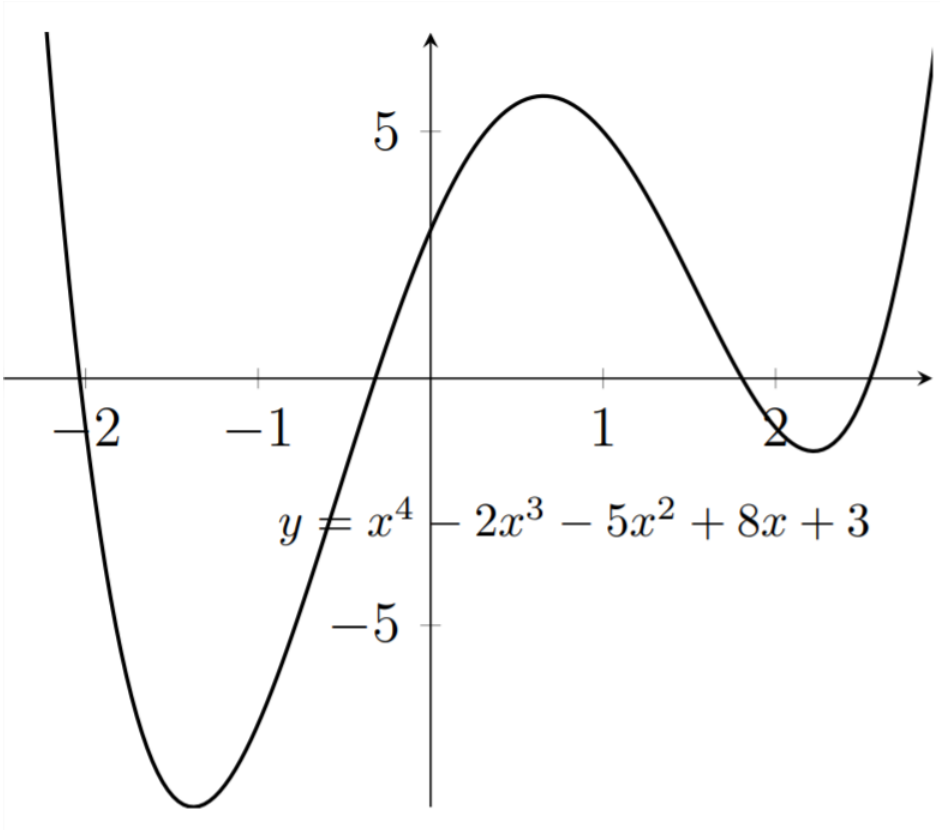
$-3 \leq x \leq -1.366$ OR $x \geq 0.366$

Example

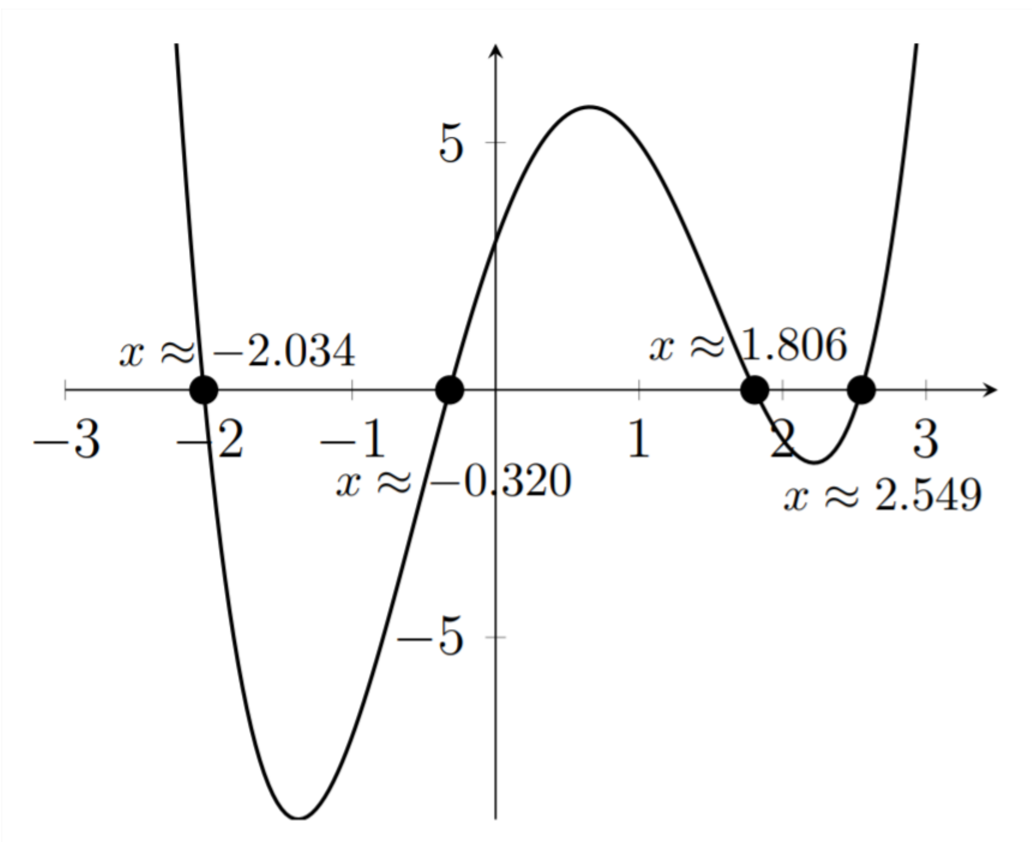
Solve the given inequality.

$x^4-2x^3-5x^2+8x+3 \leq 0$

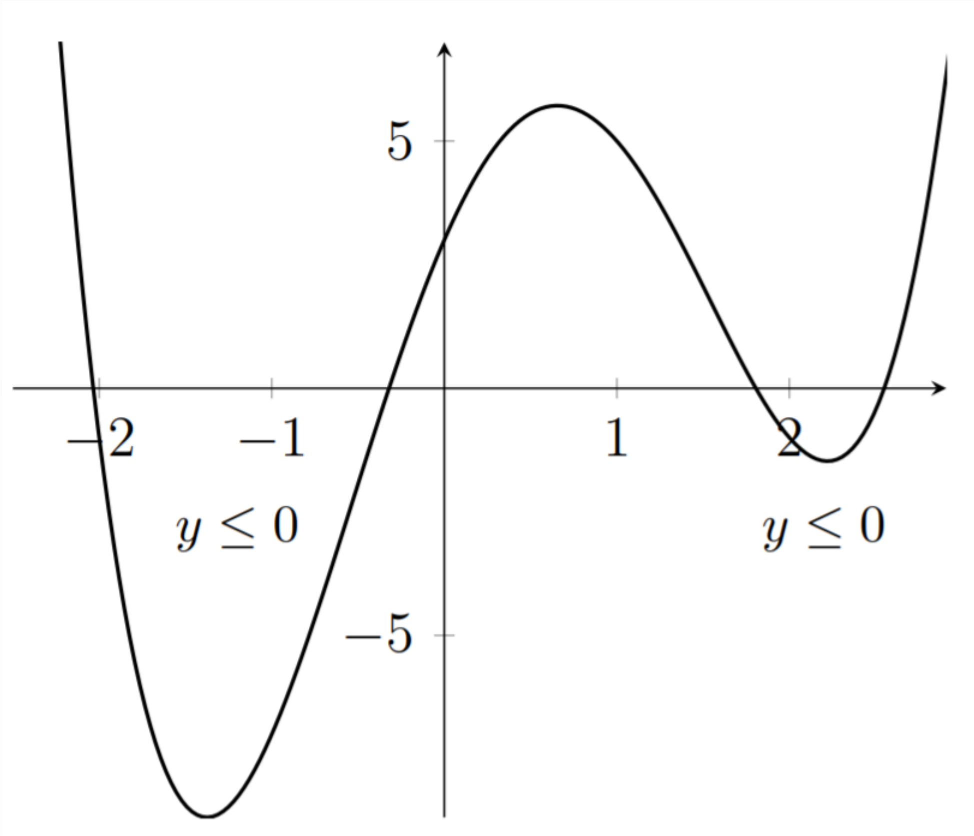
First, we graph the function:



In this problem, we're looking for the intervals of x values that make y less than or equal to zero. First, we identify the roots of the function:



Next, we'll identify the intervals where the y values are less than zero:



So, the solution to the original inequality is:

$x^4 - 2x^3 - 5x^2 + 8x + 3 \leq 0$

$-2.034 \leq x \leq -0.320$ OR $1.806 \leq x \leq 2.549$

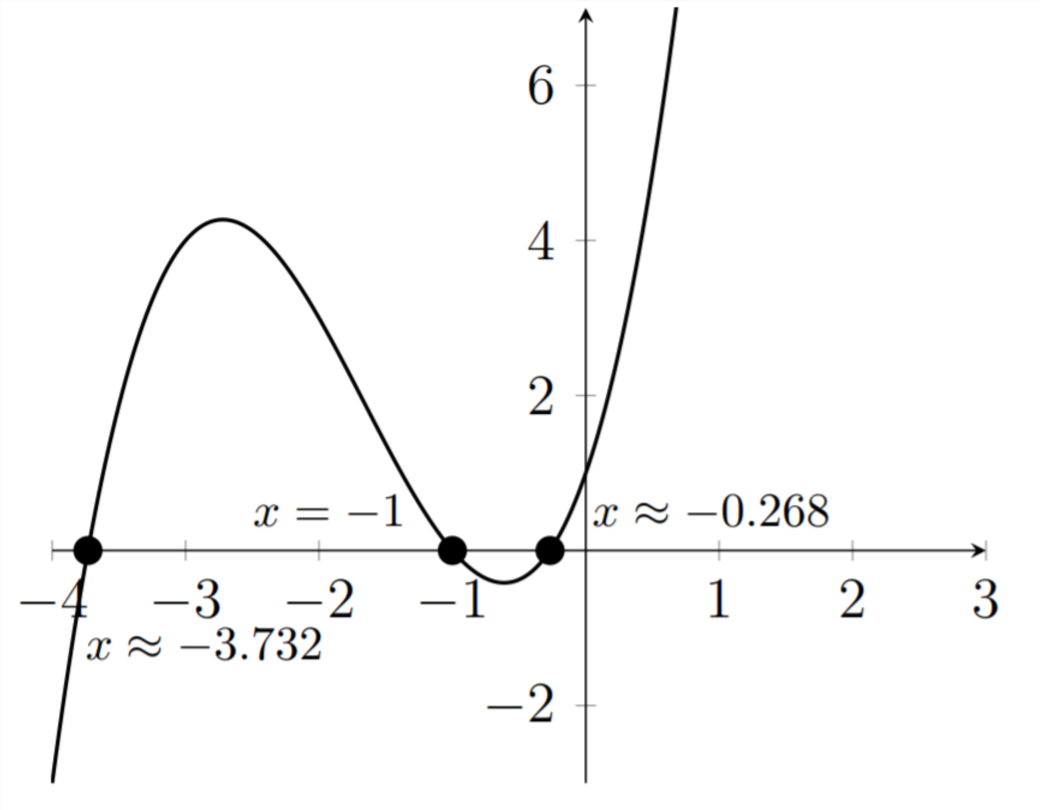
In the next example we'll be looking to identify both the intervals where y is greater than zero, and the intervals where y is less than zero.

Example

Determine the interval(s) for which $x^3 + 5x^2 + 5x + 1 \geq 0$

Determine the interval(s) for which $x^3 + 5x^2 + 5x + 1 < 0$

Once again, we'll start by graphing the function to find the roots:



Now that we've indentified the roots, we can determine where the y values are greater than zero and where they're less than zero.

For $y \geq 0$, we can see that this corresponds to: $-3.732 \leq x \leq -1$ OR $x \geq -0.268$
For $y < 0$, we can see that this corresponds to: $x < -3.732 \text{OR} -1 < x < -0.268$

Exercises 2.3

1) Determine the interval(s) for which $x^3 - 4x^2 + 2x + 3 \geq 0$
Determine the interval(s) for which $x^3 - 4x^2 + 2x + 3 < 0$
2) Determine the interval(s) for which $4x^3 - 4x^2 - 19x + 10 \geq 0$
Determine the interval(s) for which $4x^3 - 4x^2 - 19x + 10 < 0$
3) Determine the interval(s) for which $x^3 - 2.5x^2 - 7x - 1.5 \geq 0$
Determine the interval(s) for which $x^3 - 2.5x^2 - 7x - 1.5 < 0$
4) Determine the interval(s) for which $x^3 - 3.5x^2 + 0.5x + 5 \geq 0$
Determine the interval(s) for which $x^3 - 3.5x^2 + 0.5x + 5 < 0$
5) Determine the interval(s) for which $6x^4 - 13x^3 + 2x^2 - 4x + 15 \geq 0$
Determine the interval(s) for which $6x^4 - 13x^3 + 2x^2 - 4x + 15 < 0$
6) Determine the interval(s) for which $x^4 - x^3 - x^2 + 3x - 5 \geq 0$
Determine the interval(s) for which $x^4 - x^3 - x^2 + 3x - 5 < 0$
7) Determine the interval(s) for which $3x^4 + 3x^3 - 14x^2 - x + 3 \geq 0$
Determine the interval(s) for which $3x^4 + 3x^3 - 14x^2 - x + 3 < 0$
8) Determine the interval(s) for which $4x^4 - 4x^3 - 7x^2 + 4x + 3 \geq 0$
Determine the interval(s) for which $4x^4 - 4x^3 - 7x^2 + 4x + 3 < 0$

Determine the interval(s) that satisfy each inequality.
9) $x^3 + x^2 - 5x + 3 \leq 0$
10) $x^3 - 7x + 6 > 0$
11) $x^3 - 13x + 12 > 0$
12) $x^4 - 10x^2 + 9 < 0$

13) $6x^4-9x^2-4x+12\geq 0$
14) $x^4-5x^3+20x-16>0$
15) $x^3-2x^2-7x+6\leq 0$
16) $x^4-6x^3+2x^2-5x+2\leq 0$
17) $2x^4+3x^3-2x^2-4x+2>0$
18) $x^5+5x^4-4x^3+3x^2-2\leq 0$

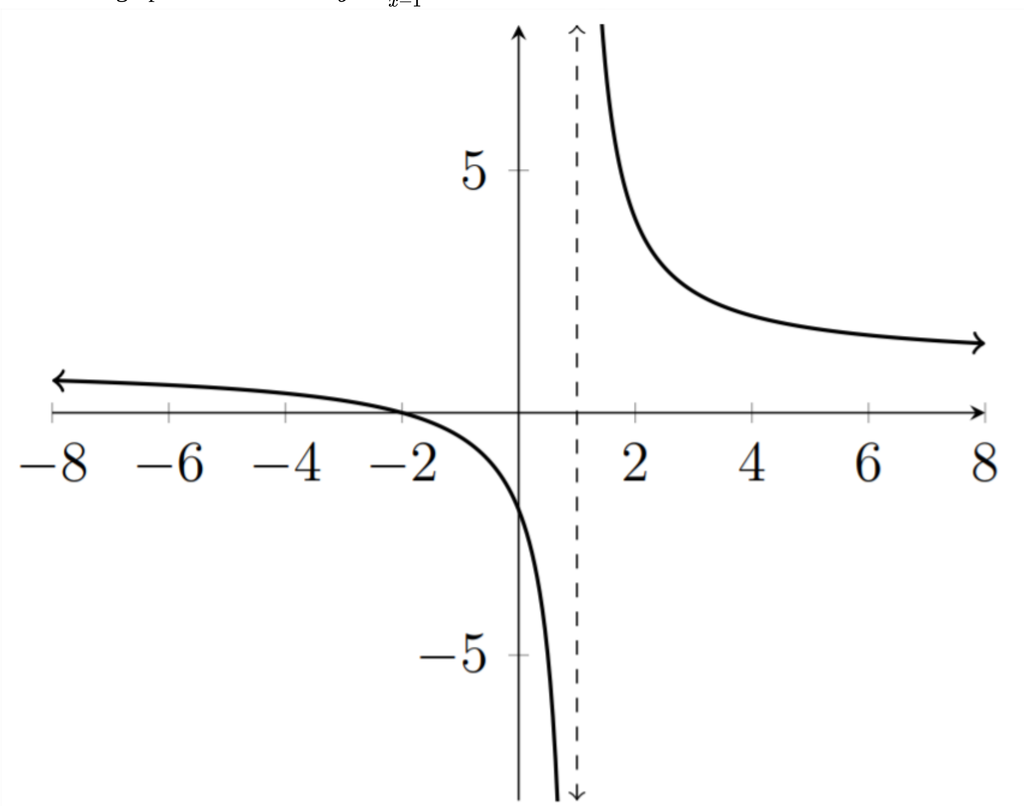
2.4: Solution of Rational Inequalities by Graphing

In the previous section, we saw how to solve polynomial inequalities by graphing. In this section, we will use similar methods to solve rational inequalities. Rational inequalities involve ratios of polynomials or fractions. Because these types of problems involve fractions, the graphs of the functions that we work with will have what are known as asymptotes. This word comes from a Greek root having to do with two lines that come very close to each other but never meet.

The vertical asymptotes of a graph will appear at places where the original expression has a zero denominator. This means that the function is not defined at those x values and so, rather than having a y value at that point, the graph has an asymptote.

Example

Below is a graph of the function $y=\frac{x+2}{x-1}$



Rather than having a y value at the point where $x=1$, the dotted line indicates the asymptote where the function is not defined. In the previous section, we were interested in finding the roots of the function because these are the places where $y=0$, and can be the dividing points between where the y values are greater than zero $(y>0)$ and the where the y values are less than zero $(y<0)$

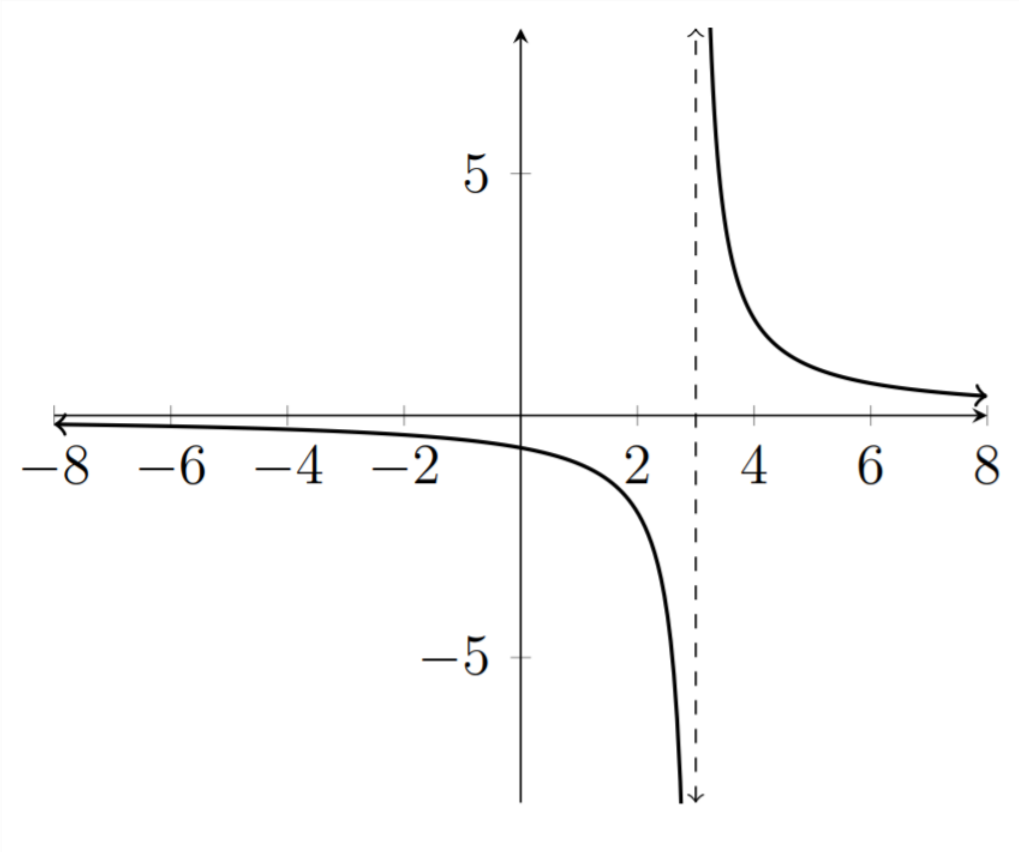
The importance of the asymptotes in analyzing rational functions is that, like the roots, these represent x values that can be the dividing points between where $y>0$ and where $y<0$

Example

Solve the given inequality.

$\frac{2}{x-3}>0$

First we examine the graph:



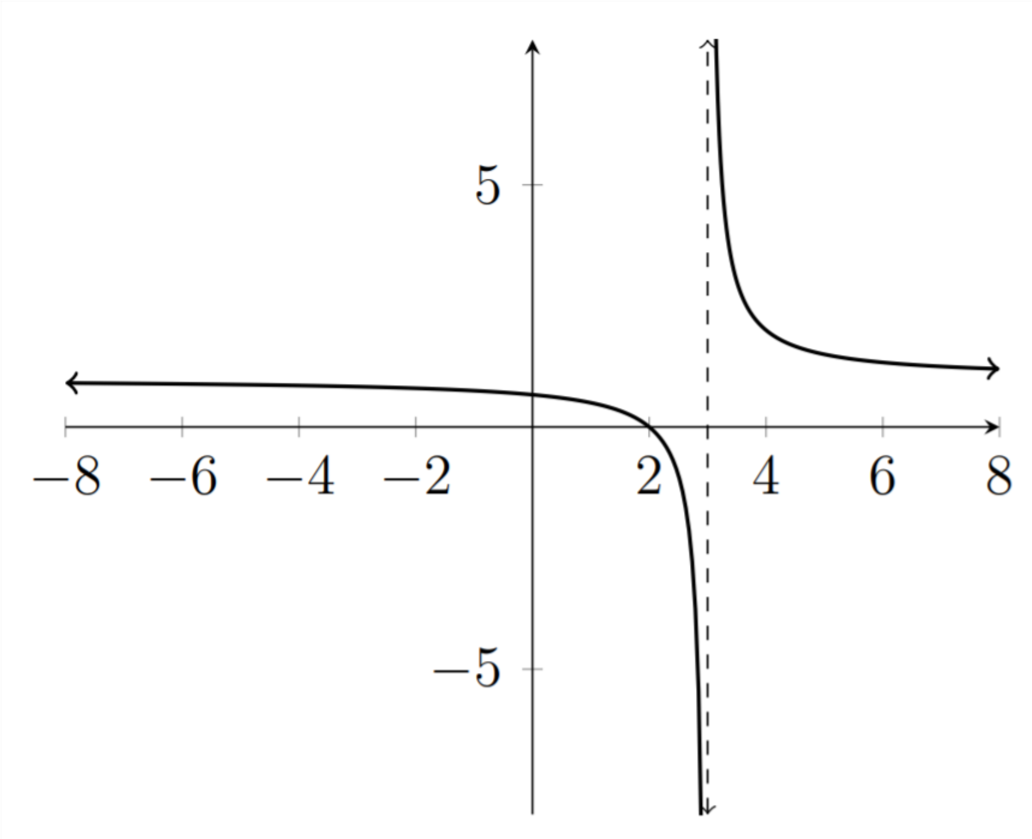
Notice that the asymptote for this graph occurs at the value $x = 3$, because this is the x value that creates a zero denominator. Also notice that the y values switch from being negative to being positive across the asymptote.

There are no roots for this function because there are no x values that make $y = 0$ For a fraction to be zero, the numerator must equal zero. In this example the numerator is 2 and no value of x will make it equal zero. Therefore the only possible dividing point on the graph is $x = 3$, and the solution to the inequality is $x > 3$

Example

Solve the given inequality.

$\frac{x-2}{x-3} > 0$



In this inequality, there is again an asymptote at $x = 3$, but there is also a root at the value $x = 2$, because when $x = 2$. $y = \frac{2-2}{2-3} = \frac{0}{-1} = 0$. So we have two dividing points to consider, $x = 2$ and $x = 3$. We can see from the graph that $y > 0$ for $x < 2$ or $x > 3$, so that is the solution to the given inequality.

Example

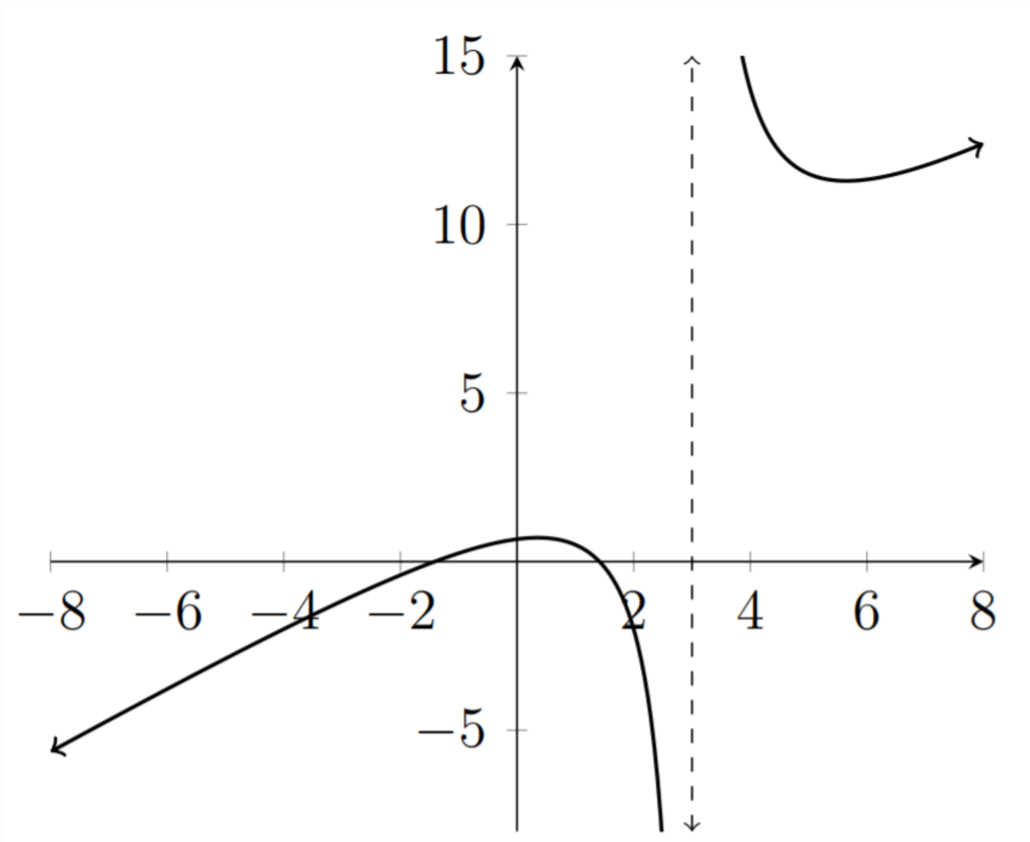
Solve the given inequality.

$\frac{x^2-2}{x-3} > 0$

In this problem, we have the same asymptote as the previous two problems: $x =$ 3. However, in this inequality, there are two roots, because there are two x values that make the numerator equal zero.

$x^2 - 2 = 0$ means that $x^2 = 2$ and $x = \pm\sqrt{2} \approx \pm 1.414$

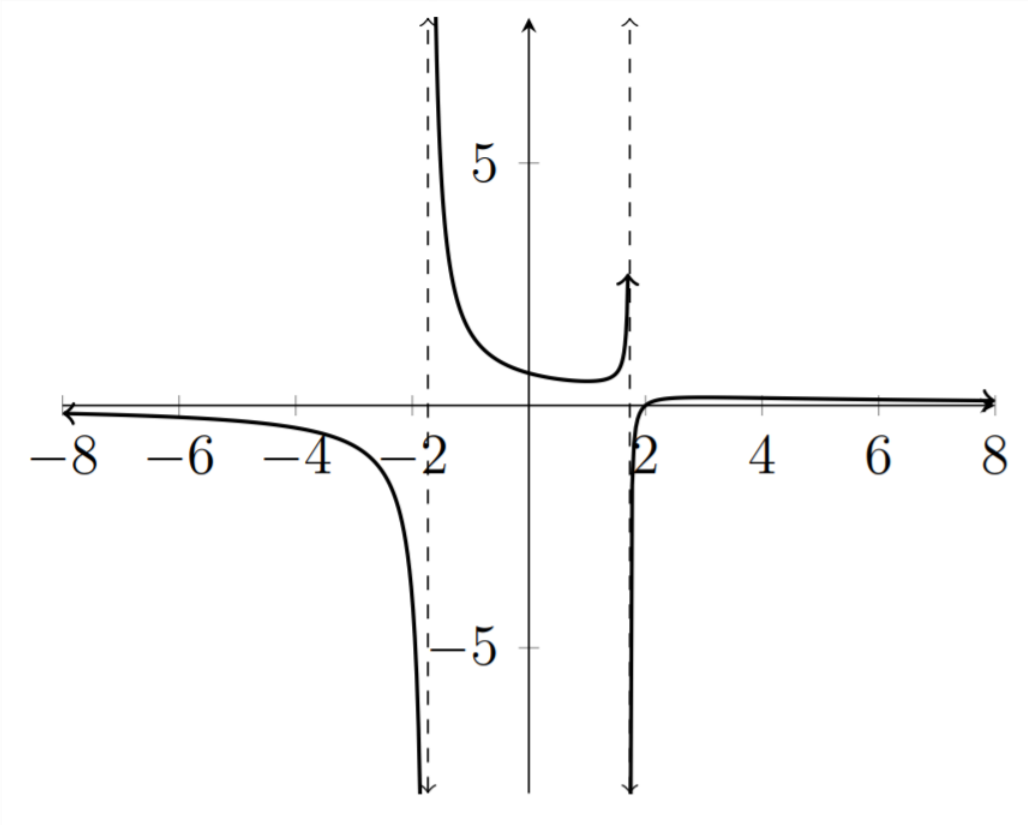
We can see these roots on the graph.



In the graph above, we can see the asymptote at $x = 3$ and the two roots at $x \approx 1.414, -1.414$
The x values that make $y > 0$ are $-1.414 < x < 1.414$ OR $x > 3$

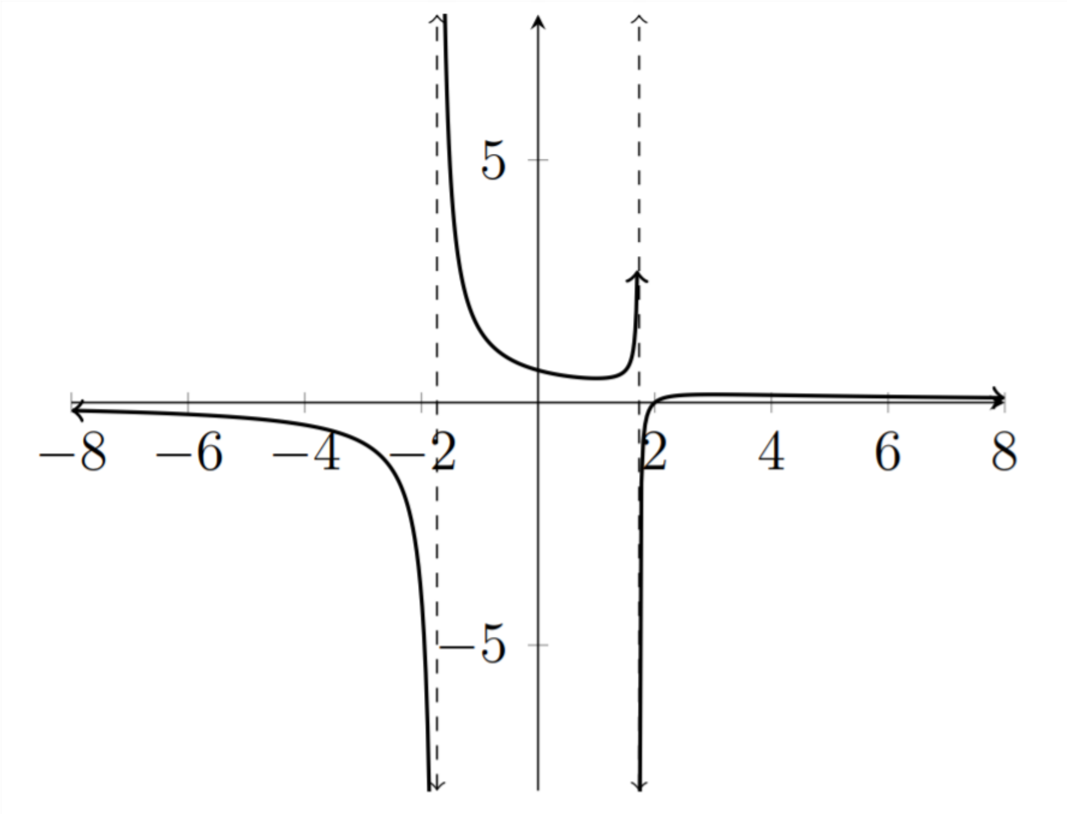
Example

Solve the given inequality.
$\frac{x-2}{x^2-3} > 0$



The roots for this function are the x values that make the numerator equal zero:
$x-2=0,$ therefore $x=2$, and we can see this root on the graph.

The asymptotes for the function are the x values that make the denominator equal zero:
$x^2-3=0$ means that $x^2=3$ and $x=\pm\sqrt{3}\approx\pm1.732$



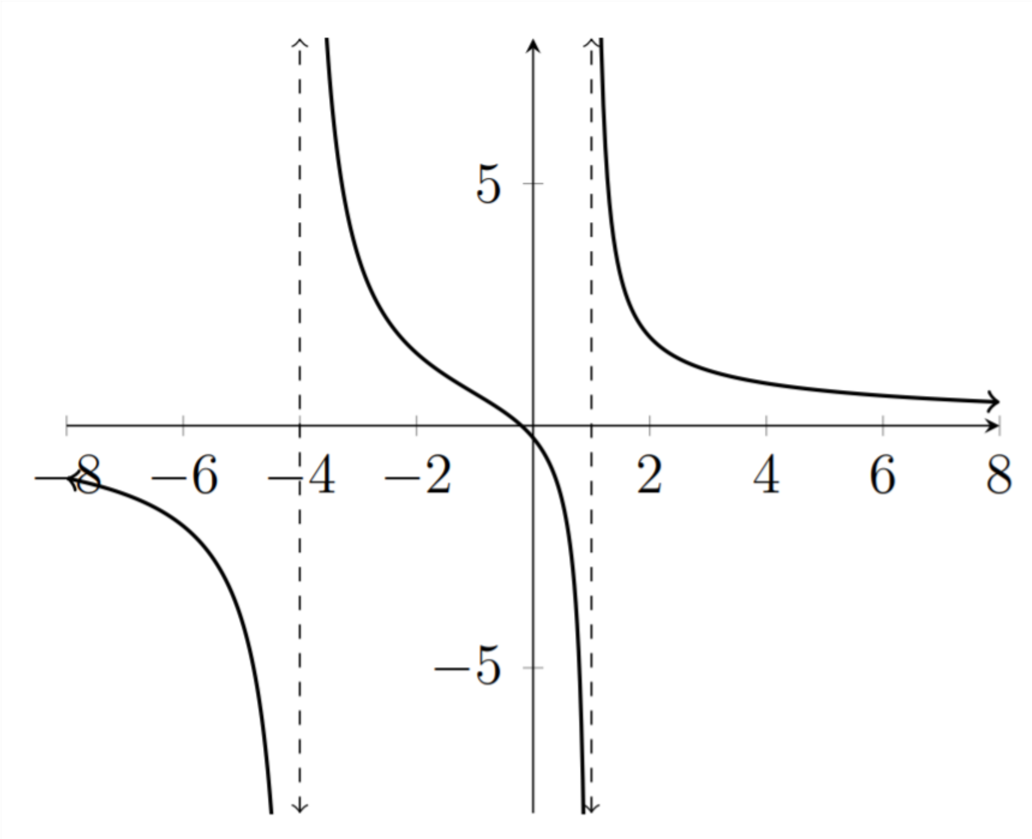
Therefore the solution for the given inequality is:

$-1.732 < x < 1.732$ OR $x > 2$

Example

Solve the given inequality

$\frac{5x+1}{x^2+3x-4} < 0$



Roots

$5x+1=0$

$5x=-1$

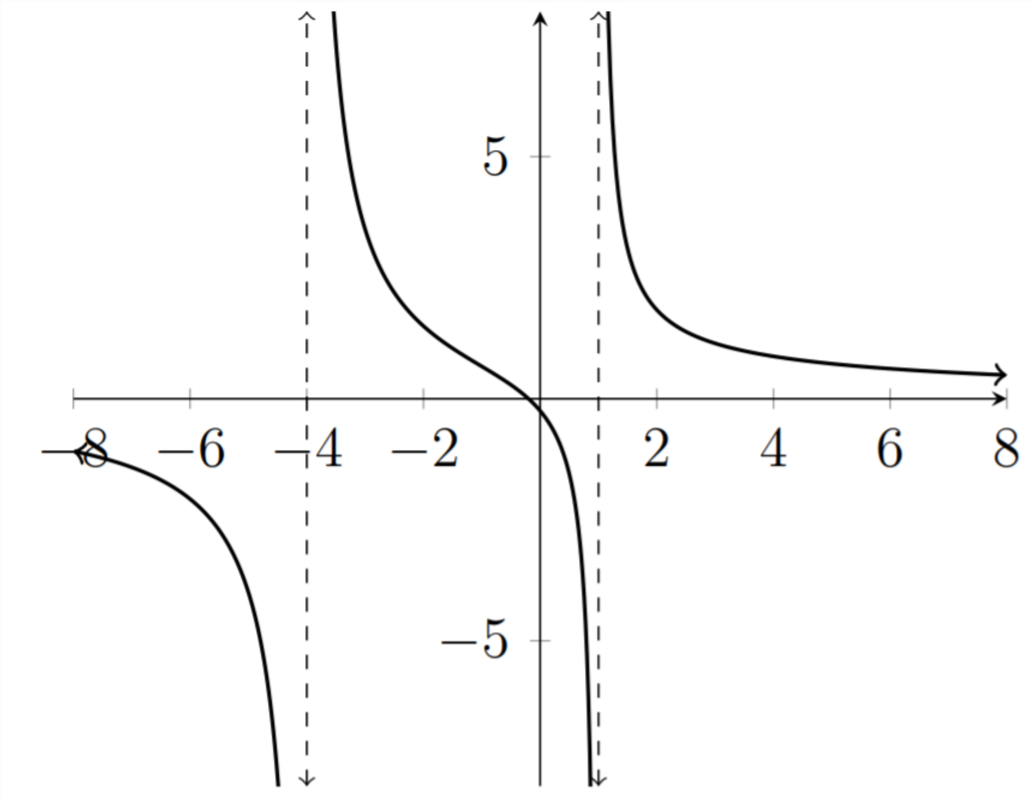
$x=-0.2=-\frac{1}{5}$

Asymptotes

$x^2+3x-4=0$

$(x+4)(x-1)=0$

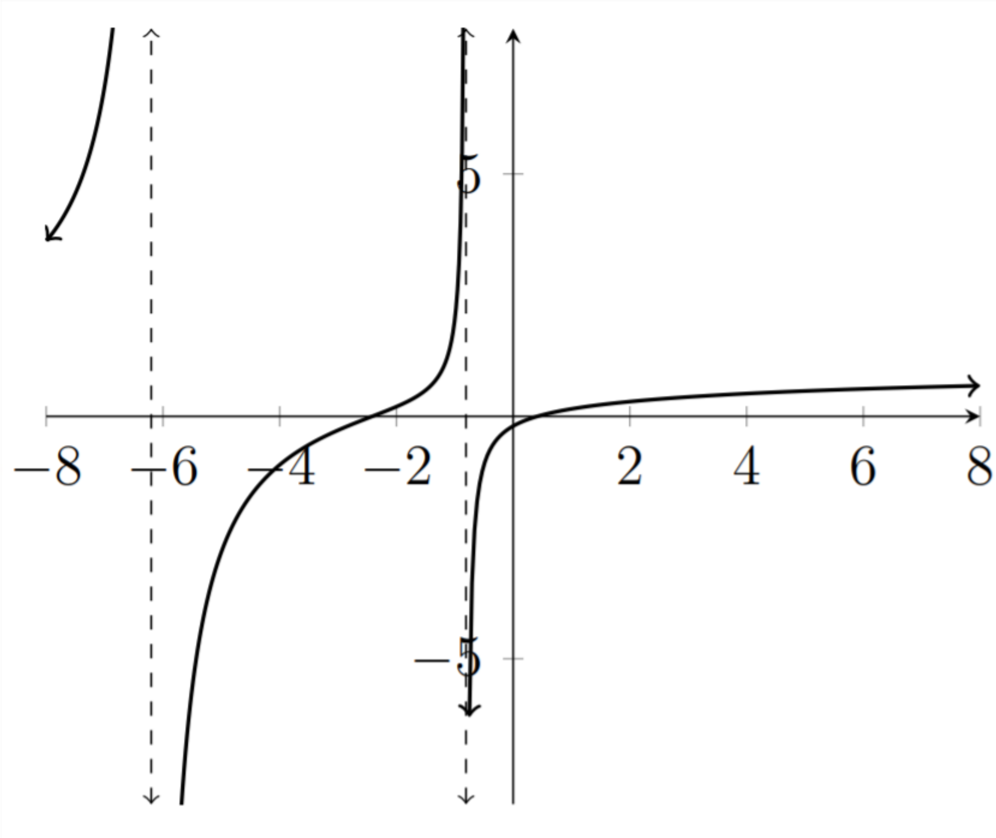
$x=-4,1$



If we combine the algebraic analysis above with what we see in the graph, then we know that the dividing points important to the solution of this inequality are at $x=-4,-0.2,1$. The intervals where the y values are less than zero are $x<-4$ OR $-0.2<x<1$

Example

$\frac{x^{2}+2x-1}{x^{2}+7x+5} \leq 0$



Roots

$$\begin{gathered} x^2+2x-1=0 \\ x\approx-2.414, 0.414 \end{gathered} \tag{2.4.1}$$

Asymptotes
$x^2+7x+5=0$

$$x\approx-6.193, -0.807 \tag{2.4.2}$$

We can see that the dividing points important to the solution of the inequality are $x\approx-6.193, -2.414, -0.807, 0.414$.The intervals where the y values are less than or equal to zero are $-6.193\leq x\leq-2.414$ OR $-0.807\leq x\leq 0.414$

Exercises 2.4

Solve each inequality.

1) $\frac{x+4}{x^2-8x+12}>0$
2) $\frac{2x+3}{x^2-2x-35}<0$
3) $\frac{x^2-5x-14}{x^2+3x-10}<0$
4) $\frac{2x^2-x-3}{x^2+10x+16}>0$
5) $\frac{3x+2}{x^2+x-5}<0$
6) $\frac{x^2+2x+5}{x^2-3x-7}>0$
7) $\frac{x^3+9}{x^2+x-1}>0$
8) $\frac{x^3+9}{x^2+x+1}>0$

Solve each inequality.

9) $\frac{x^2-2x-9}{3x+11}>0$

10) $\frac{x^2+4x+3}{2x+1} < 0$

11) $\frac{x^2+x-5}{x^2-x-6} > 0$

12) $\frac{x^3+2}{x^2-2} > 0$

13) $\frac{x^2+2x-7}{x^2+3x-6} < 0$

14) $\frac{2x-x^2}{x^2-4x+6} < 0$

15) $\frac{x^2-7}{x^2+5x-1} < 0$

16) $\frac{x-5}{3x^2-2x-3} > 0$

2.5: Finding Factors from Roots

One method of solving equations involves finding the factors of the polynomial expression in the equation and then setting each factor equal to zero.

$$\begin{array}{c} x^2+8x+15=0 \\ (x+5)(x+3)=0 \\ x+5=0 \quad x+3=0 \\ x=-5 \quad x=-3 \end{array} \tag{2.5.1}$$

In this process, the reasoning is that if $(x+5)$ times $(x+3)$ equals zero, then one of those expressions must be equal to zero. In setting them equal to zero, we find the solutions of $x=-5,-3$. Plugging them back into the factored expression we see the following:

$$(-5+5)(-5+3)=0*-2=0 \tag{2.5.2}$$

and

$$(-3+5)(-3+3)=2*0=0 \tag{2.5.3}$$

This process works in reverse as well. In other words, if we know a root of the function, we can find factors for the expression.
Example
Find a quadratic equation that has roots of -2 and +3

$$\begin{array}{cc} x=-2 & x=3 \\ x+2=0 & x-3=0 \\ (x+2)(x-3)=0 & \\ x^2-x-6=0 & \end{array} \tag{2.5.4}$$

Roots that are fractions are a little trickier, but really no more difficult:
Example
Find a quadratic equation that has roots of -5 and $\frac{2}{3}$

$$\begin{array}{cc} x=-5 & x=\frac{2}{3} \\ x+5=0 & 3x=2 \\ x+5=0 & 3x-2=0 \\ (x+5)(3x-2)=0 & \\ 3x^2+13x-10=0 & \end{array} \tag{2.5.5}$$

Exercises 2.5
Find a quadratic equation that has the indicated roots.
1) $4,-1$
2) $-2,7$
3) $\frac{3}{2},1$
4) $-\frac{1}{5},\frac{2}{3}$
5) $\frac{1}{3},3$
6) $-4,\frac{2}{5}$
7) $\frac{1}{2},-\frac{7}{2}$
8) $-1,\frac{3}{5}$
9) $-\frac{2}{3},-3$
10) $-\frac{2}{3},-\frac{3}{4}$

11) $-\frac{5}{2}, 3$

12) $-6, -2$

2.6: Polynomial Long Division

Polynomial long division has many similarities to numerical long division, so it is important that we understand how and why numerical long division works the way it does before discussing polynomial long division. First the HOW?

Given the numerical problem $87,462 \div 38$, the first step is to determine the highest place value in the answer.

$$\begin{array}{r@{}l} & 2 \\ 38\big) & \overline{87,462} \end{array}$$

Often the first step in numerical long division is to say "Does 38 divide into $8?$" "No." "Does 38 divide into $87?$" "Yes." This tells us that the first digit in the answer will be over the 7 in $87,462,$ and consequently will be in the thousands place. Once we know that the first digit in the answer will be in the thousands place, the next question is "How many thousands?" We can determine that $2 * 38 = 76$ but $3 * 38 = 114$ (too big), so we know that the first digit in the answer will be $2.$ Then we subtract, include the 4 and examine what is left over to continue.

$$\begin{array}{r@{}l} & 2 \\ 38\big) & \overline{87,462} \\ & \underline{-76} \\ & 114 \end{array}$$

Here, we see that $114 \div 38 = 3$, so we know that the next digit in the answer will be 3

$$\begin{array}{r} 2\;3 \\ 38\overline{)87,462} \\ -76 \\ \overline{114} \\ -114 \\ \overline{0006} \end{array}$$

After including the 6, we can see that 38 does not divide evenly into 6, so we put a zero as the next digit in our answer and proceed:

$$\begin{array}{r} 2\;\;30 \\ 38\overline{)87,462} \\ -76 \\ \overline{114} \\ -114 \\ \overline{0006} \\ 0 \\ \overline{00062} \end{array}$$

Now that we have included all the digits from our original number $87,462,$ the last step is to divide 38 into $62.$ This goes one time with 24 left over.

So, now we have the solution to the original problem $87,462 \div 38 = 2,301\text{R}24\text{or}$
$2,301\frac{24}{38}$
The WHY? of the long division algorithm is somewhat hidden by the HOW? In the first step, we are determining which place value will hold the first digit of our answer. When we determine that 38 does divide into 87, this indicates that the first digit in our answer will be the thousands place. Dividing 38 into 87 tells us how many thousands there will be. Then we subtract:

$$\begin{array}{r} 87,462 \\ \underline{-76,000} \\ 11,462 \end{array} \tag{2.6.1}$$

Now we need to determine how many times 38 will divide into $11,462$. We decided on $300 * 38 = 11,400$, then we subtract to see how much is left over:

$$\begin{array}{r} 11,462 \\ \underline{-11,400} \\ 00,062 \end{array} \tag{2.6.2}$$

We can see that we won't need any tens in our answer, and that 38 divides into 62 one time with 24 left over, thus the answer is 2 thousands, 3 hundreds, no tens, 1 and a remainder of 24. To check the answer, we multiply $38 * 2301$ and add 24:

$$\begin{array}{r} 2,301 \\ \underline{\times 38} \\ 18408 \\ \underline{6903} \\ 87438 \\ \underline{+24} \\ 87462 \end{array} \tag{2.6.3}$$

Polynomial long division works in much the same way that numerical long division does. Given a problem $A \div B$, the goal is to find a quotient Q and remainder R so that $A = B * Q + R$
Let's look at this with the example $2x^4 + 7x^3 + 4x^2 - 2x - 1 \div x^2 + 3x + 1$ or:

$$\frac{2x^4 + 7x^3 + 4x^2 - 2x - 1}{x^2 + 3x + 1} \tag{2.6.4}$$

So, we are looking to answer the question:

$$\begin{array}{ccccc} A & = & B & * & Q & +R \\ 2x^4 + 7x^3 + 4x^2 - 2x - 1 & = & \left(x^2 + 3x + 1\right) & * & (?+?+?) & +? \end{array} \tag{2.6.5}$$

If we want to mulitply $x^2 + 3x + 1$ times something and end up with $2x^4 + 7x^3 +\ 4x^2 - 2x - 1,$ then what we multiply by is going to have to start with $2x^2$, because

$$x^2 * 2x^2 = 2x^4 \tag{2.6.6}$$

Now we're working with this:

$$\begin{array}{cccccc} A & = & B & * & Q & +R \\ 2x^4 + 7x^3 + 4x^2 - 2x - 1 & = & \left(x^2 + 3x + 1\right) & * & \left(2x^2 +?+?\right) & +? \end{array} \tag{2.6.7}$$

But the $2x^2$ doesn't just get multiplied by the x^2, it will also get multiplied by the $3x$ and the 1. So now we have:

$$\begin{array}{rcccc} A & = & B & * & Q & +R \\ 2x^4+7x^3+4x^2-2x-1 & = & (x^2+3x+1) & * & (2x^2+?+?) & +? \\ & & =2x^4+6x^3+2x^2+????? & & & \end{array} \tag{2.6.8}$$

The issue this raises is that the next multiplication $(? * x^2 + ? * 3x + ? * 1)$ needs to add only $1x^3$ to our answer, because we need $7x^3$ and we already have $6x^3$ from the previous multiplication. That means we're going to want to multiply next by $1x$:

$$\begin{aligned} A = \quad & B \quad * \quad Q \quad +R \\ 2x^4+7x^3+4x^2-2x-1 \quad &= (x^2+3x+1)*(2x^2+1x+?)+? \\ &= 2x^4+6x^3+2x^2 \\ &= \quad 1x^3+3x^2+x \\ &= 2x^4+7x^3+5x^2+1x+??? \end{aligned}$$

In the next round of mutiplication, we're going to want to bring the $5x^2$ down to $4x^2$, so we'll need to multiply by -1

$$\begin{aligned} A = \quad & B \quad * \quad Q \quad +R \\ 2x^4+7x^3+4x^2-2x-1 \quad &= (x^2+3x+1)*(2x^2+1x-1)+0 \\ &= 2x^4+6x^3+2x^2 \\ &= \quad 1x^3+3x^2+x \\ &= \qquad -1x^2-3x-1 \\ &= 2x^4+7x^3+5x^2-2x-1 \end{aligned}$$

Now we also know that the remainder is zero, because x^2+3x+1 divides evenly into $2x^4+7x^3+4x^2-2x-1$ and so:

$$2x^4+7x^3+4x^2-2x-1 = (x^2+3x+1)*(2x^2+1x-1) \tag{2.6.9}$$

This method makes the reasoning behind dividing polynomials somewhat more apparent than the long division process, but it is more cumbersome. The way that polynomial long division is usually approached is as follows:

$$x^2+3x+1\overline{)\;2x^4+7x^3+4x^2-2x-1}$$

Then, just as we did in the other method, we question "What should we multiply x^2 by to get $2x^4?''$ Answer: $''2x^{2''}$ This is the first term in our answer:

$$\begin{array}{r} 2x^2 \\ x^2+3x+1\overline{)\;2x^4+7x^3+4x^2-2x-1} \end{array}$$

Then we multiply $2x^2\,(x^2+3x+1)$ and change all the signs to see what we'll be left with:

$$\begin{array}{r}2x^2\\ x^2+3x+1\overline{)\;2x^4+7x^3+4x^2-2x-1}\\ \underline{-2x^4-6x^3-2x^2}\end{array}$$

This indicates that we'll have the $2x^4$ we'll need in our answer, as well six of the seven x^3 's and two of the four x^2 's. We'll now need $1x^3$ next:

$$\begin{array}{r}2x^2\\ x^2+3x+1\overline{)\;2x^4+7x^3+4x^2-2x-1}\\ \underline{-2x^4-6x^3-2x^2}\\ x^3+2x^2-2x\end{array}$$

This means we'll need to multiply by $1x$:

$$\begin{array}{r}2x^2\quad+x\\ x^2+3x+1\overline{)\;2x^4+7x^3+4x^2-2x-1}\\ \underline{-2x^4-6x^3-2x^2}\\ x^3+2x^2-2x\\ \underline{-x^3-3x^2\quad-x}\\ -x^2-3x-1\end{array}$$

Here, we still need to pick up a $-1x^2$, which means that our next multiplication will be with -1:

$$\begin{array}{r} 2x^2 \quad +x-1 \\ x^2+3x+1 \overline{\smash{\big)}\; 2x^4+7x^3+4x^2-2x-1} \\ \underline{-2x^4-6x^3-2x^2} \qquad\qquad \\ x^3+2x^2-2x \quad \\ \underline{-x^3-3x^2 \quad -x} \quad \\ -x^2-3x-1 \\ \underline{x^2+3x+1} \\ 0 \end{array}$$

Because x^2+3x+1 divides evenly into $2x^4+7x^3+4x^2-2x-1$ we have a zero remainder. In the next example there will be a non-zero remainder:

Example

Divide:

$\frac{3x^4-8x^3+19x^2-15x+10}{x^2-x+4}$

First, we set up the problem:

$$x^2-x+4 \overline{\smash{\big)}\; 3x^4-8x^3+19x^2-15x+10}$$

Then, we question: "What do we need to multiply x^2 by to get $3x^4$? " Answer:

$$\begin{array}{r} 3x^2 \qquad\qquad\qquad\qquad \\ x^2-x+4 \overline{\smash{\big)}\; 3x^4-8x^3+19x^2-15x+10} \end{array}$$

Then, we multiply, change signs (subtract) and combine like terms:

$$\begin{array}{r} 3x^2 \\ x^2-x+4\overline{\smash{\big)}\; 3x^4-8x^3+19x^2-15x+10} \\ -3x^4+3x^3-12x^2 \\ \hline -5x^3+7x^2-15x \end{array}$$

Now we'll need to multiply by $-5x$, change signs and combine like terms:

$$\begin{array}{r} 3x^2-5x \\ x^2-x+4\overline{\smash{\big)}\; 3x^4-8x^3+19x^2-15x+10} \\ -3x^4+3x^3-12x^2 \\ \hline -5x^3+7x^2-15x \\ 5x^3-5x^2+20x \\ \hline 2x^2+5x+10 \end{array}$$

We need $2x^2$ so we'll need to multiply by 2 , change signs and combine like terms:

$$\begin{array}{r l}
 & 3x^2 \quad -5x \quad +2 \\
\hline
x^2-x+4) & 3x^4-8x^3+19x^2-15x+10 \\
 & -3x^4+3x^3-12x^2 \\
\hline
 & -5x^3 \quad +7x^2-15x \\
 & 5x^3 \quad -5x^2+20x \\
\hline
 & 2x^2 \quad +5x+10 \\
 & -2x^2 \quad +2x \quad -8 \\
\hline
 & 7x \quad +2
\end{array}$$

Because there is no positive power of x that we can multiply x^2 by to get $7x$, then this is our remainder: $7x+2$
So:

$$\begin{array}{ccccc} A & = & B & * & Q \quad + \quad R \\ 3x^4-8x^3+19x^2-15x+10 & =\left(x^2-x+4\right) & *\left(3x^2-5x+2\right) & +(7x+2) & \end{array} \tag{2.6.10}$$

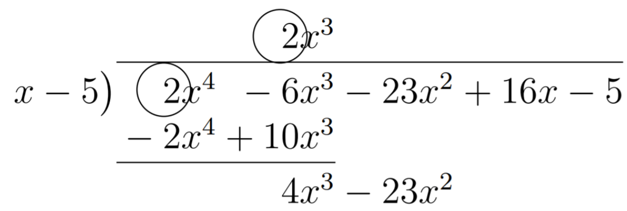
Exercises 2.6

Find the quotient in each problem.

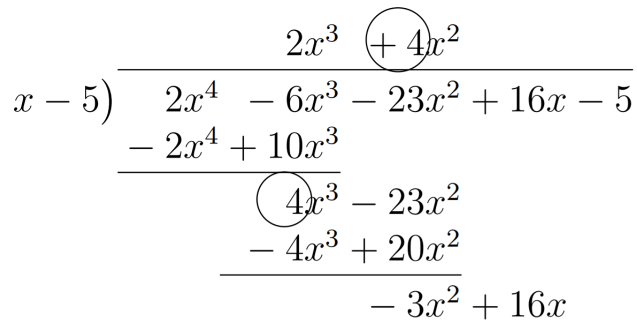
1) $\frac{y^3-4y^2+6y-4}{y-2}$
2) $\frac{x^3-5x^2+x+15}{x-3}$
3) $\frac{x^3-4x^2-3x-10}{x^2+x+2}$
4) $\frac{2x^3-3x^2+7x-3}{x^2-x+3}$
5) $\frac{x^4+2x^3-x^2+x+6}{x+2}$
6) $\frac{x^4+x^3+5x^2+3x+6}{x^2+x-1}$
7) $\frac{2z^3+5z+8}{z+1}$
8) $\frac{x^5+3x+2}{x^3+2x+1}$
9) $\frac{x^4+2x^3+4x^2+3x+2}{x^2+x+2}$
10) $\frac{2x^4+3x^3+3x^2-5x-3}{2x^2-x-1}$
11) $\frac{2y^5-3y^4-y^2+y+4}{y^2+1}$
12) $\frac{3x^5-4x^3+3x^2+12x-10}{x^2+2x-1}$
13) $\frac{5x^4-3x^2+2}{x^2-3x+5}$
14) $\frac{3y^3-4y^2-3}{y^2+5y+2}$

2.7: Synthetic Division

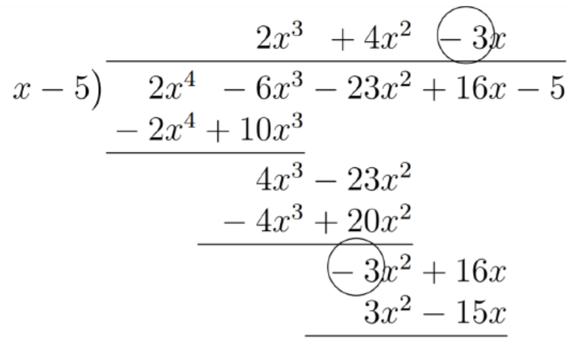
The process for polynomial long division (like the process for numerical long division) has been separated somewhat from its logical underpinnings for a more efficient method to arrive at an answer. For particular types of polynomial long division, we can even take this abstraction one step further. Synthetic Division is a handy shortcut for polynomial long division problems in which we are dividing by a linear polynomial. This means that the highest power of x we are dividing by needs to be x^1. This limits the usefulness of Synthetic Division, but it will serve us well for certain purposes. Let's examine where the coefficients in our answer come from when we divide by a linear polynomial:



Notice that the first coefficient in the answer is the same as the first coefficient in the polynomial we're dividing into. This is because we're dividing by a polynomial in the form $1x^1 - a$. This also makes the power of the first term in the answer one less than the power of the polynomial we are dividing into. Let's see where the subsequent coefficients in the answer come from:



The next coefficient in the answer (4) comes from the combination of the -6 and the $+10$. The +10 came from multiplying the 2 in the answer by the 5 in the divisor $x-5$. The next coefficient in the answer will be -3, which comes from multiplying the 4 (in the answer) by the 5 (in the divisor) and combining it with the -23 in the polynomial we're dividing into:



The last part of our answer will come from multiplying the -3 in the answer times the 5 in the divisor (making -15) and combining this with the +16 in the polynomial we're dividing into:

$$\begin{array}{r|l} & \quad 2x^3 \quad + 4x^2 \quad - 3x \; \textcircled{+1} \\ \hline x-5 & \; 2x^4 \quad - 6x^3 - 23x^2 + 16x - 5 \\ & -2x^4 + 10x^3 \\ & \quad\quad 4x^3 - 23x^2 \\ & \quad\quad -4x^3 + 20x^2 \\ & \quad\quad\quad\quad -3x^2 + 16x \\ & \quad\quad\quad\quad 3x^2 - 15x \\ & \quad\quad\quad\quad\quad\quad \textcircled{1x} - 5 \\ & \quad\quad\quad\quad\quad\quad -x + 5 \\ & \quad\quad\quad\quad\quad\quad\quad\quad 0 \end{array}$$

The process of Synthetic Division uses these relationships as a shortcut to finding the answer. The set-up for a Synthetic Division problem is shown below:

$$\begin{array}{c|ccccc} 5 & 2 & -6 & -23 & 16 & -5 \\ & & & & & \\ \hline & & & & & \end{array}$$

This set-up allows us to complete the division problem $\frac{2x^4-6x^3-23x^2+16x-5}{x-5}$.

$$\begin{array}{c|ccccc} 5 & 2 & -6 & -23 & 16 & -5 \\ & & & & & \\ \hline & & & & & \end{array}$$

The first coefficient in the answer is the same as the first coefficient in the polynomial we're dividing into:

$$\begin{array}{c|ccccc} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & & & & \\ \hline & 2 & & & & \end{array}$$

To get the next coefficient, we multiply the 2 by the 5 to get +10 and flll this in under the -6:

$$\begin{array}{c|ccccc} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & +10 & & & \\ \hline & 2 & & & & \end{array}$$

Then, $-6+10=+4,$ which is the next coefficient in the answer:

$$\begin{array}{c|ccccc} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & +10 & & & \\ \hline & 2 & 4 & & & \end{array}$$

Then, we continue this process, multiplying the 4 by the 5 to get 20 and combining this with the $-23: -23+20=-3:$

$$\begin{array}{c|ccccc} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & +10 & +20 & & \\ \hline & 2 & 4 & -3 & & \end{array}$$

Next, multiply the -3 by the 5 and combine the resulting -15 with the 16:

$$\begin{array}{c|ccccc} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & +10 & +20 & -15 & \\ \hline & 2 & 4 & -3 & 1 & \end{array}$$

In the last step, multiply the 1 times the 5 and combine the result with the -5 in the problem to get zero:

$$\begin{array}{c|cccc|c} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & +10 & +20 & -15 & 5 \\ \hline & 2 & 4 & -3 & 1 & 0 \end{array}$$

This last coefficient represents the remainder - in this case 0. The other numerals in the answer represent the coefficients for the powers of x in the answer. On the far right is the remainder, then the constant $\left(x^0\right)$ term, then the linear $\left(x^1\right)$ term and so on:

$$\begin{array}{c|cccc|c} 5 & 2 & -6 & -23 & 16 & -5 \\ & \downarrow & +10 & +20 & -15 & 5 \\ \hline & 2x^3 & 4x^2 & -3x^1 & 1x^0 & 0 \end{array}$$

$\frac{2x^4-6x^3-23x^2+16x-5}{x-5}=2x^3+4x^2-3x+1$

Let's look at another example:

Example

Use Synthetic Division to divide:

$\frac{3x^3+5x^2-9x+9}{x+3}$

since Synthetic Division is set up to divide by $x-a$, if we're dividing by $x+3$ we'll need to use a -3 in the Synthetic Division:

-3	3	5	-9	9
	$\downarrow$			
	3			

Then, $3 * -3 = -9$

-3	3	5	-9	9
	$\downarrow$	-9		
	3	-4		

Next, $-4 * -3 = +12$

-3	3	5	-9	9
	$\downarrow$	-9	12	
	3	-4	3	

And this example also has a zero remainder:

-3	3	5	-9	9
	$\downarrow$	-9	12	-9
	3	-4	3	0

The answer here is $3x^2-4x+3$

$$\frac{3x^3+5x^2-9x+9}{x+3}=3x^2-4x+3 \tag{2.7.1}$$

and

$$3x^3+5x^2-9x+9=(x+3)\left(3x^2-4x+3\right) \tag{2.7.2}$$

Let's look at an example that is a little bit different.

Example

Use Synthetic Division to divide:

$\frac{6x^4+x^3+9x^2+x-2}{2x+1}$

Synthetic Division is set up to handle problems in which we are dividing by $1x-a$. Clearly, this is not the case in this example, however we can work around this. Another way of looking at setting up the synthetic division is that we use the number that is the solution to $x-a=0$. When we divided by $x-5$, we used
+5. When we were dividing by $x+3$, we used -3. So, if we're going to divide by $2x+1$, we'll use $-\frac{1}{2}$ in the Synthetic Division:

$-\frac{1}{2}$	6	1	9	1	-2
	$\downarrow$				
	6				

Then, we proceed as usual:

$-\frac{1}{2}$	6	1	9	1	-2
	$\downarrow$	-3	1	-5	2
	6	-2	10	-4	0

Notice that, again, we have a zero remainder. Also, notice that each coefficient in our answer has a common factor of 2, which was the coefficient of the x in $2x+1$, which we originally were going to divide by. What we've done here is not division by $2x+1$, but division by $x+\frac{1}{2}$

So, in the end, our work tells us that:

$$\frac{6x^4+x^3+9x^2+x-2}{x+\frac{1}{2}}=6x^3-2x^2+10x-4 \tag{2.7.3}$$

and

$$6x^4+x^3+9x^2+x-2=\left(x+\frac{1}{2}\right)\left(6x^3-2x^2+10x-4\right) \tag{2.7.4}$$

Notice that if we factor out the common factor of 2 from our answer, we can multiply it back into the $x+\frac{1}{2}$ and get an answer for our original problem:

$$\begin{aligned} 6x^4+x^3+9x^2+x-2 &= \left(x+\frac{1}{2}\right)2\left(3x^3-x^2+5x-2\right) \\ &= (2x+1)\left(3x^3-x^2+5x-2\right) \end{aligned}$$

This means that:

$$\frac{6x^4+x^3+9x^2+x-2}{2x+1}=3x^3-x^2+5x-2 \tag{2.7.5}$$

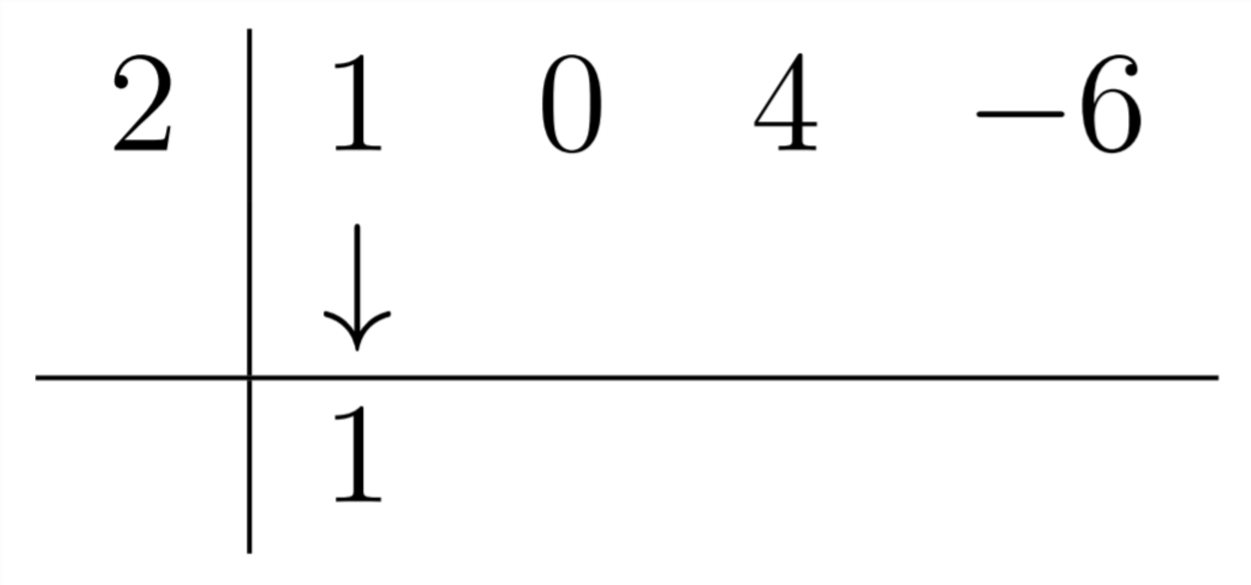
Another thing to understand about Synthetic Division is that if there is a missing power of x, then you should include a zero as the coefficient of that power

Example

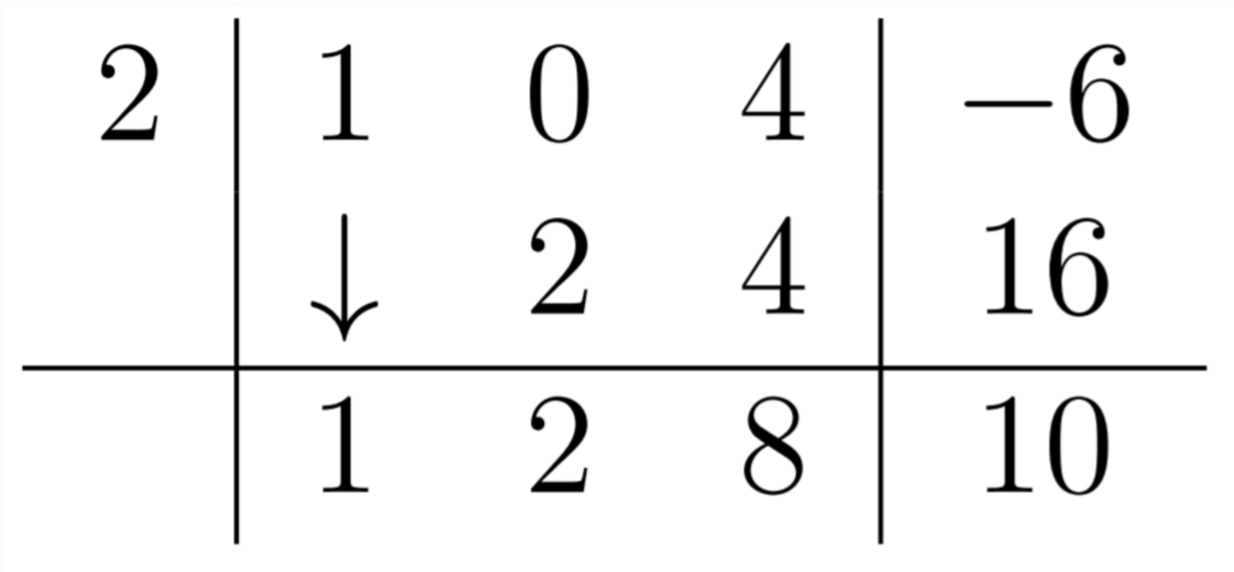
Use Synthetic Division to divide:

$\frac{x^3+4x-6}{x-2}$

since there is no x^2 term in the polynomial we're divding into, we'll enter a zero as the coefficient for that term:



And then proceed as usual:



So the answer for this problem is x^2+2x+8

Exercises 2.7

Use synthetic division to find the quotient in each problem.

1) $\frac{x^3-8x^2+5x+50}{x-5}$
2) $\frac{x^3+5x^2-x-14}{x+2}$
3) $\frac{x^3+12x^2+34x-7}{x+7}$
4) $\frac{x^3-10x^2+23x-6}{x-3}$
5) $\frac{x^4-15x^2+10x+24}{x+4}$
6) $\frac{x^4-3x^3+4x^2-36}{x-3}$
7) $\frac{x^4-2x^3-x+10}{x-2}$
8) $\frac{x^4-16x^2-5x-24}{x+4}$
9) $\frac{2x^4-x^3+2x-1}{2x-1}$
10) $\frac{3x^4+x^3-3x+1}{3x+1}$

11) $\frac{3x^4-8x^3+9x^2-2x-2}{3x+1}$

12) $\frac{6x^4-7x^3+5x^2-17x+10}{3x-2}$

13) $\frac{2x^3+7x^2+6x-5}{2x-1}$

14) $\frac{3x^4-x^3-21x^2-11x+6}{3x-1}$

2.8: Roots and Factorization of Polynomials

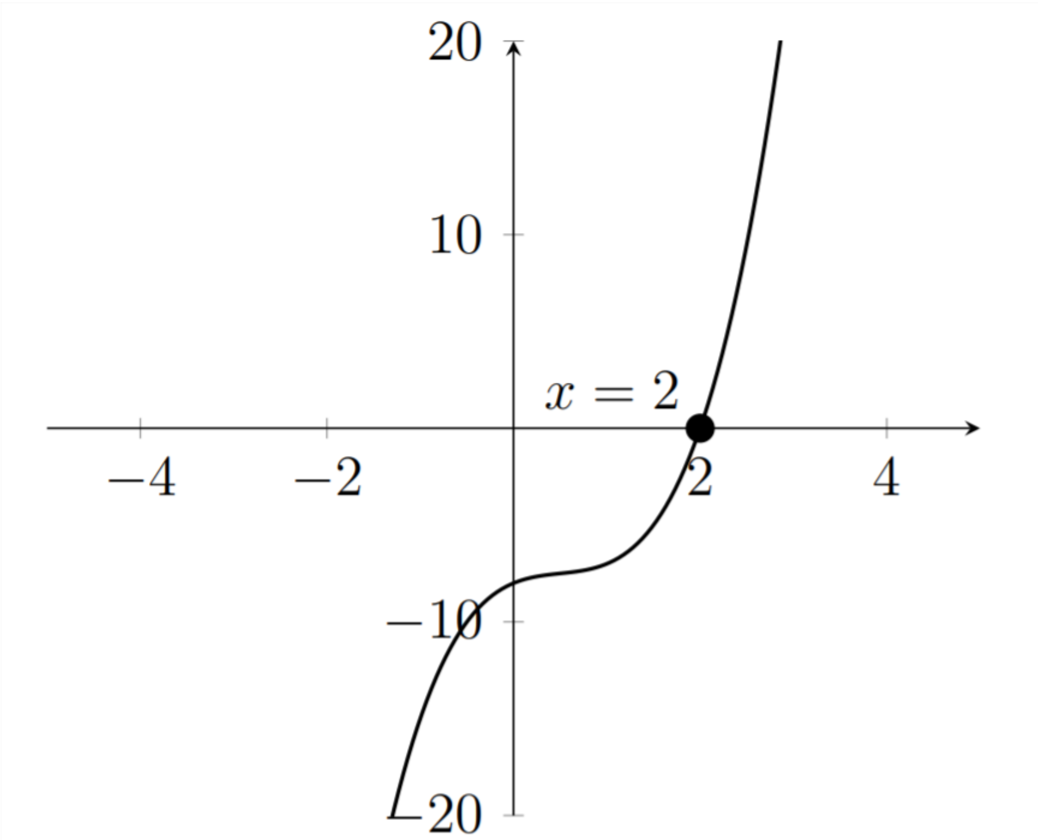
In this section we will use some of the skills we have seen in previous sections in order to find all the roots of a polynomial function (both real and complex) and also factor the polynomial as the product of prime factors with integer coefficients.

Example

Find all real and complex roots for the given equation. Express the given polynomial as the product of prime factors with integer coefficients.

$2x^3-3x^2+2x-8=0$

First we'll graph the polynomial to see if we can find any real roots from the graph:



We can see that there is a root at $x=2$. This means that the polynomial will have a factor of $(x-2)$. We can use Synthetic Division to find any other factors. Because $x=2$ is a root, we should get a zero remainder:

2	2	-3	2	-8
	$\downarrow$	4	2	8
	2	1	4	0

So, now we know that $2x^3-3x^2+2x-8=(x-2)\left(2x^2+x+4\right)$. To finish the problem, we can set each factor equal to zero and find the roots:

$$\begin{gathered}2x^3-3^2+2x-8=0\\(x-2)\left(2x^2+x+4\right)=0\\x-2=0\quad 2x^2+x+4=0\\x=2\quad x\approx-0.25\pm1.392i\end{gathered}\tag{2.8.1}$$

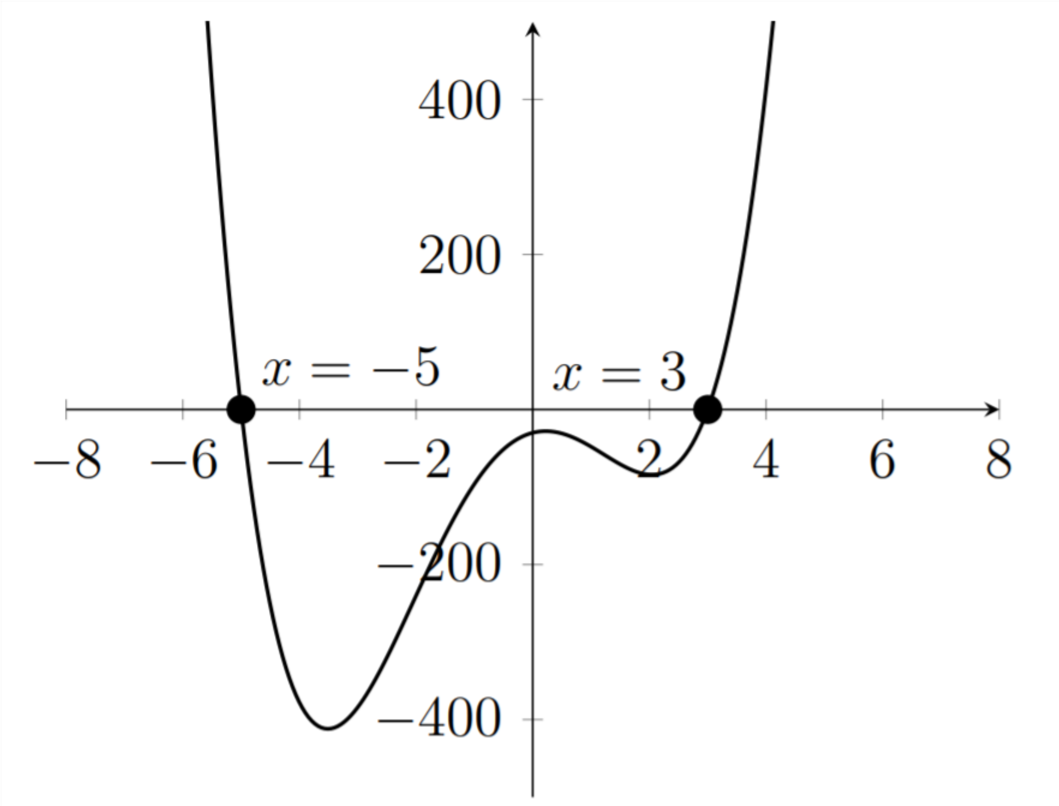
Let's look at an example that has more than one real root:

Example

Find all real and complex roots for the given equation. Express the given polynomial as the product of prime factors with integer coefficients.

$$3x^4+5x^3-45x^2+19x-30=0\tag{2.8.2}$$

First we'll graph the polynomial to see if we can find any real roots from the graph:



We can see roots at $x=-5,3$, which means that $(x+5)$ and $(x-3)$ are both factors of this polynomial. We'll need to divide by both of these factors to break down the polynomial. First, we divide by $(x-3)$:

3	3	5	−45	19	−30
	↓	9	42	−9	30
	3	14	−3	10	0

And then by $(x+5)$:

3	3	5	-45	19	-30
	$\downarrow$	9	42	-9	30
-5	3	14	-3	10	0
	$\downarrow$	-15	5	-10	
	3	-1	2	0	

Now we know that $3x^4+5x^3-45x^2+19x-30=(x+5)(x-3)\left(3x^2-x+2\right)$ and so, to finish the problem:

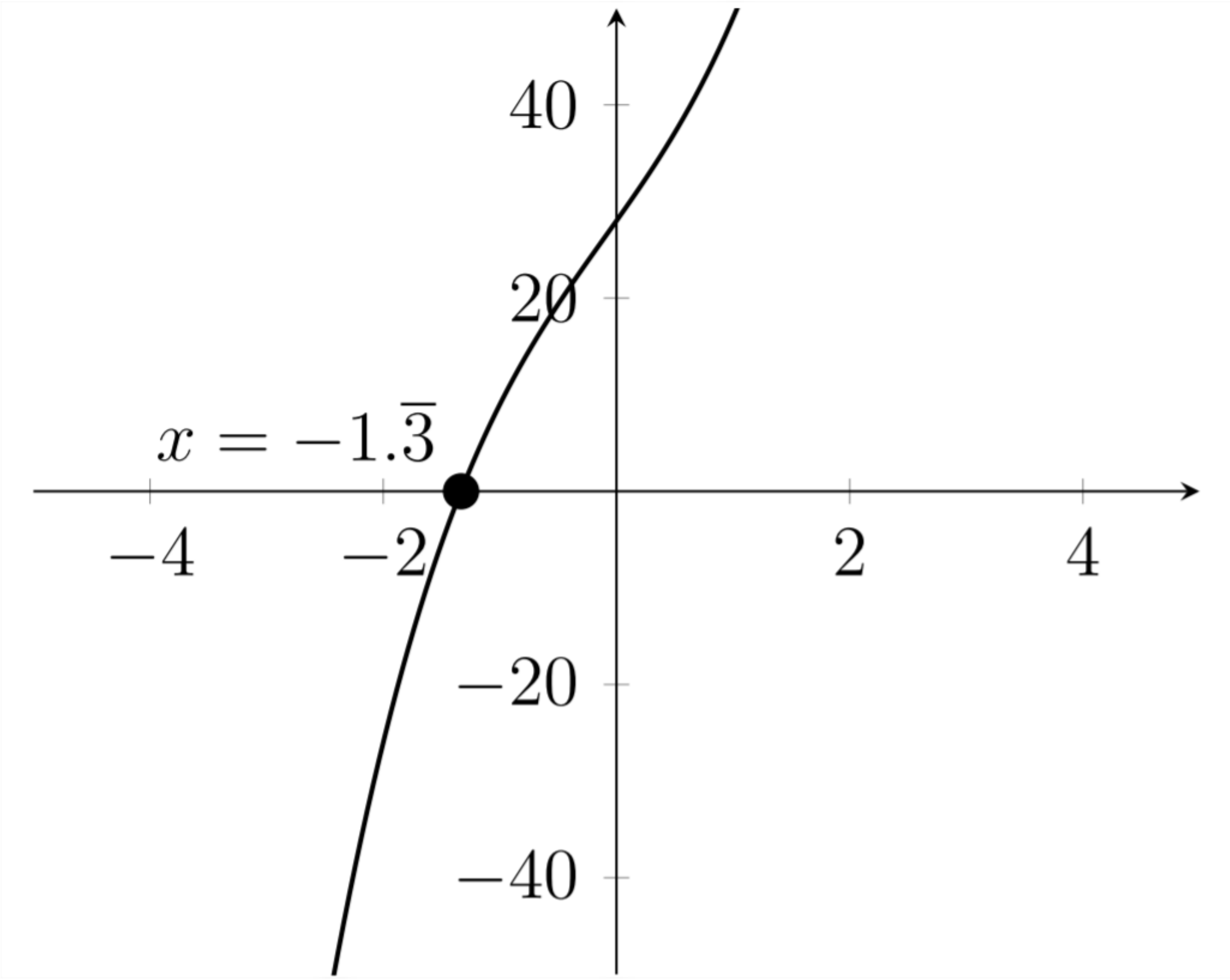
$$\begin{gathered}3x^4+5x^3-45x^2+19x-30=0\\(x+5)(x-3)\left(3x^2-x+2\right)=0\\x+5=0\quad x-3=0\quad 3x^2-x+2=0\\x=-5\quad x=3\quad x\approx\frac{1}{6}\pm 0.799i\end{gathered}\tag{2.8.3}$$

Next, let's look at an example where there is a root that is not a whole number:

Example

Find all real and complex roots for the given equation. Express the given polynomial as the product of prime factors with integer coefficients.
$3x^3+x^2+17x+28=0$

First we'll graph the polynomial to see if we can find any real roots from the graph:



We can see in the graph that this polynomial has a root at $x = -\frac{4}{3}$. That means that the polynomial must have a factor of $3x + 4$. We can use Synthetic Division to find the other factor for this polynomial. Because we know that $x = -\frac{4}{3}$ is a root, we should get a zero remainder:

$-\frac{4}{3}$	3	1	17	28
	$\downarrow$	−4	4	−28
	3	−3	21	0

Notice that, because the root we used was a fraction, there is a common factor of 3 in the answer to our Synthetic Division. We should factor this out to obtain the
answer:

$\left(x+\frac{4}{3}\right)\left(3x^2-3x+21\right)=(3x+4)\left(x^2-x+7\right)$
So, this means that:

$$\begin{gathered}3x^3+x^2+17x+28=0 \\ (3x+4)\left(x^2-x+7\right)=0 \\ 3x+4=0 \quad x^2-x+7=0 \\ x=-\frac{4}{3} \quad x\approx 0.5\pm 2.598i\end{gathered} \tag{2.8.4}$$

Exercises 2.8

Find all real and complex roots for the given equation. Express the given polynomial as the product of prime factors with integer coefficients.

Set #1

1) $x^4-3x^3+5x^2-x-10=0$
2) $3x^3-5x^2+2x-8=0$
3) $2x^4-5x^3+x^2+4x-4=0$
4) $x^4+x^3-3x^2-17x-30=0$
5) $x^4-9x^3+21x^2+21x-130=0$
6) $x^4-7x^3+14x^2-38x-60=0$
7) $x^5-9x^4+31x^3-49x^2+36x-10=0$
8) $x^4+4x^3+2x^2+12x+45=0$
9) $x^4-6x^3+12x^2-10x+3=0$
10) $x^4-6x^3+13x^2-24x+36=0$
11) $x^5-3x^4+12x^3-28x^2+27x-9=0$
12) $x^5+2x^4-3x^3-3x^2+2x+1=0$

Set #2

13) $15x^3-7x^2+13x+3=0$
14) $x^4-5x^3+3x^2-11x-20=0$
15) $6x^3+13x^2+12x+4=0$
16) $6x^3-5x^2+5x-2=0$
17) $4x^4+20x^3+29x^2+10x-15=0$
18) $3x^4-4x^3+10x^2+12x-5=0$
19) $2x^4-3x^3-6x^2-8x-3=0$
20) $12x^4-53x^3-31x^2-19x-5=0$
21) $12x^4+4x^3+x^2-3x-2=0$
22) $3x^4+13x^3-26x-40=0$

CHAPTER OVERVIEW

3: Exponents and Logarithms

In this chapter, we will examine concepts that are related to exponential, logarithmic and logistic relationships. In the first section, we will look at how to approach these problems from a graphical perspective. In the subsequent sections, we will examine the methods necessary to work with these problems algebraically.

3.1: Exponential and Logistic Applications

There are a variety of different types of mathematical relationships. The simplest mathematical relationship is the additive relationship. This is a situation in which the value of one quantity is always a certain amount more (or less) than another quantity. A good example of an additive relationship is an age relationship. In an age relationship, the age of the older person is always the same amount more than the age of the younger person. If the older person is five years older, then the age of the older person (y) will always be equal to the age of the younger person (x) plus five: $y=x+5$

Another type of additive relationship is seen where two quantities add up to a constant value. Let's say there is a board whose length is 20 inches. If we cut it into two pieces, with one piece being 6 inches, then the other piece will be 14 inches. If one piece is 9 inches, then the other will be 11 inches. If one piece is x inches, then the other piece (y) will be $20-x: y=20-x$ or $x+y=20$

The next type of mathematical relationshp is a multiplicative relationship. This represents a situation in which one quantity is always a multiple of the other quantity. This is commonly seen in proportional relationships. If a recipe for a cake calls for 2 cups of flour, then, if we want to make 3 cakes, we'll need 6 cups of flour. The amount of flour (y) is always two times the number of cakes we want to make $(x): y=2x$

If a recipe for a batch of cookies (with 20 cookies per batch) calls for 1.5 cups of sugar, then three batches would require 4.5 cups of sugar. The amount of sugar required (y) is always the number of batches (x) times $1.5: y=1.5x$. If we wanted to represent this relationship based on the number of cookies instead of the number of batches, we would need to adjust the formula. Given that there are 20 cookies per batch, we could adjust our formula so that we first calculate the number of batches from the number of cookies and then multiply by 1.5. If the number of cookies is x and the amount of sugar is y, then $y=1.5*\frac{x}{20}$ or $y=\frac{3}{40}x$
The next type of mathematical relationship is the polynomial relationship. In this type of relationship, one quantity is related to a power of another quantity. A good example of this type of relationship involves gravity. As Galileo discovered in the 16 th century, the distance that an object falls after it is dropped is not proportional to the time that it has been falling. Rather, it is proportional to the square of the time. The table below shows this type of relationship.

t	d
1	16
2	64
3	144
4	256

After one second, it looks like the distance will always be sixteen times the time the object has been falling. However, after two seconds, we can see that this relationship no longer is true. That's because this relationship is a polynomial relationship in which the distance an object has fallen (d) is proportional to the square of the time it has been falling $(t): d=16t^2$

Exponential Relationships

The next type of relationship is the focus of this chapter - the exponential relationship. In this situation, the rate of change of a quantity is proportional to the size of that quantity. This relationship can be explored in more depth in an integral calculus course, but we will discuss the basics here.

In a linear or proportional relationship, the slope, or rate of change, is constant. For example, in the equation $y=3x+1$, the slope is always three, no matter what the values of x and y are. In an exponential relationship, the rate of change (also called " y prime" or y') is proportional to the value of y. In this case, we say that

$$y'=k*y \tag{3.1.1}$$

This is what is known as a differential equation. This is an equation in which the variable and its rate of change are related. Through the processes of differential and integral calculus, we can solve the equation above $y'=k*y$ as:

$$y=Ae^{kt} \tag{3.1.2}$$

In the equation above, A is the value of y at time $t=0, k$ is a constant that determines how fast the quantity y increases or decreases and t plays the role of the independent variable (as x often does) and represents the time that has passed. If k is positive, then the quantity y is growing because its rate of change is positive. If k is negative, then the quantity y is decreasing because the rate of change is negative.

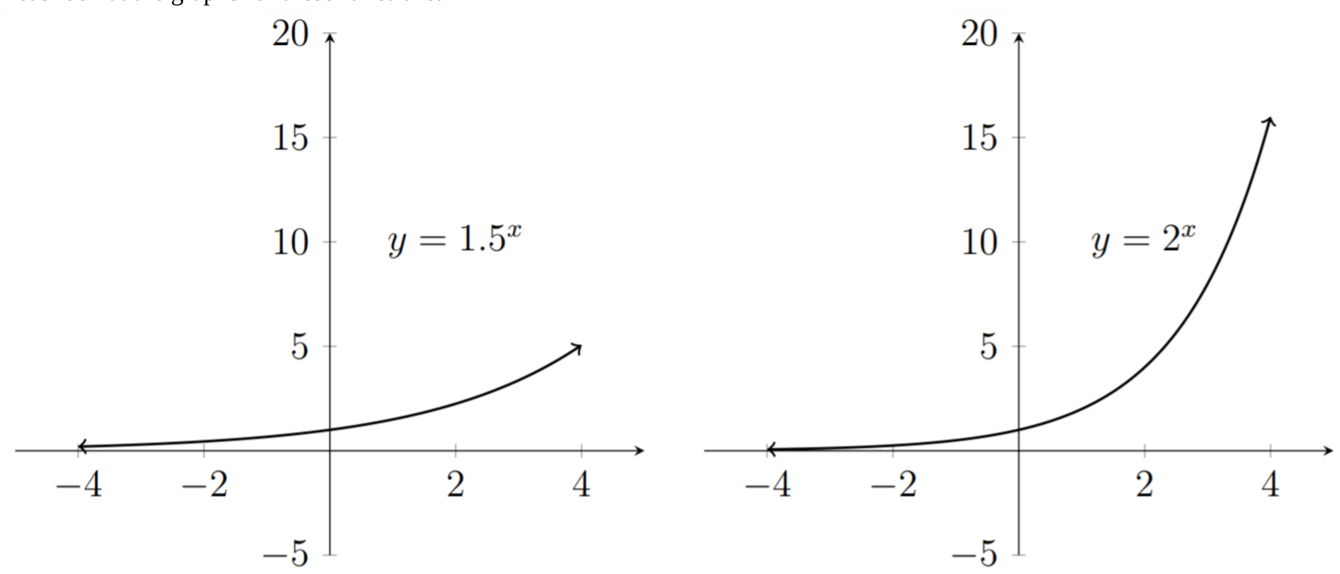
The quantity represented by e in the above equation is a mathematical constant (like π) that is often used to represent exponential relationships. The best way to understand the value of e and what it represents is directly related to fundamental questions from differential and integral calculus.

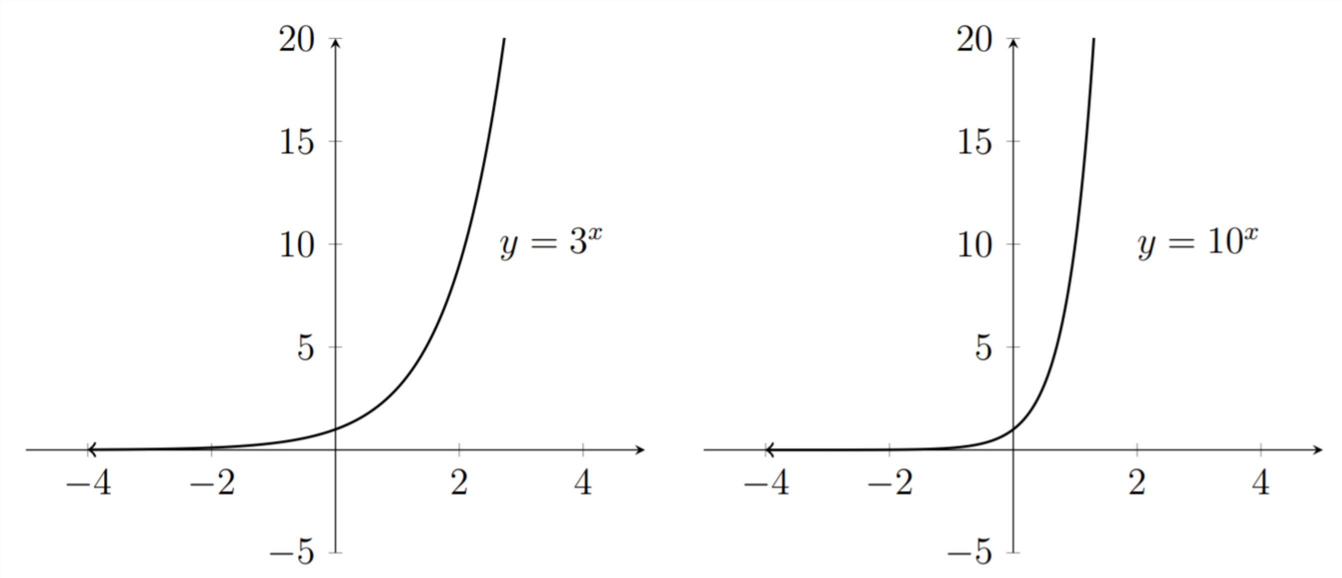
Differential Calculus is concerned primarily with the question of slopes. We discussed earlier that a linear relationship has a constant slope. Polynomial and exponential relationships have slopes that depend on the value of x and / or y. This is what makes them curves rather than lines. If we consider the slopes of some different exponential relationships, we can see one aspect of where the value for e comes from.

Consider the graphs of the following relationships:

$$y=1.5^x$$
$$y=2^x$$
$$y=3^x$$
$$y=10^x$$

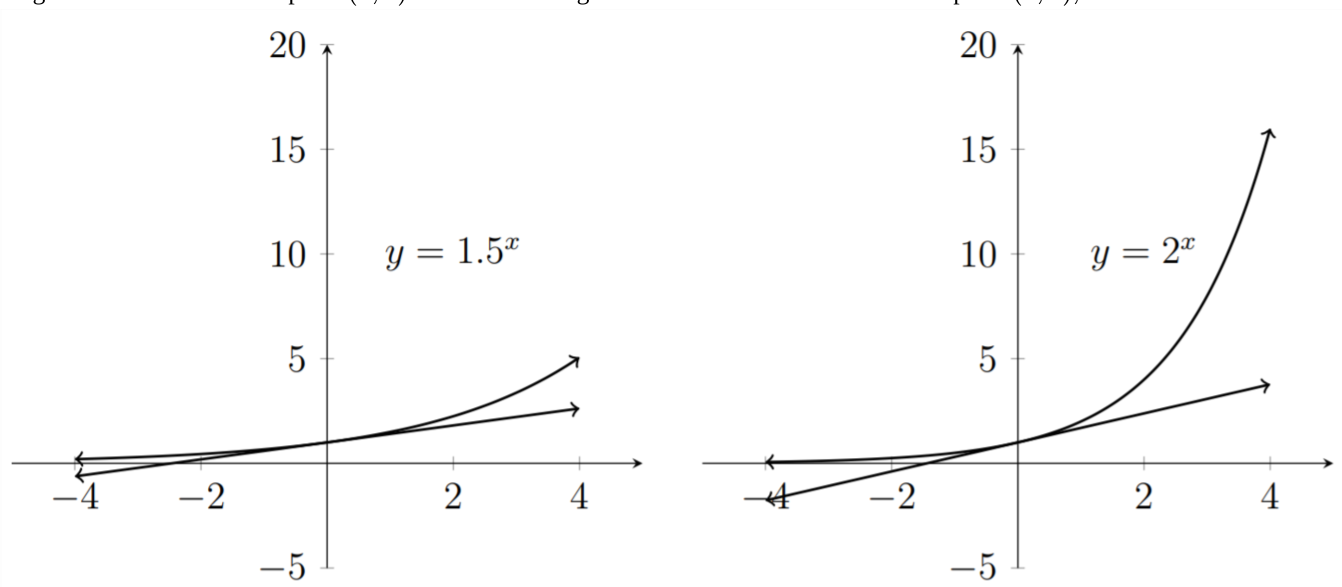
Let's look at the graphs for these functions:

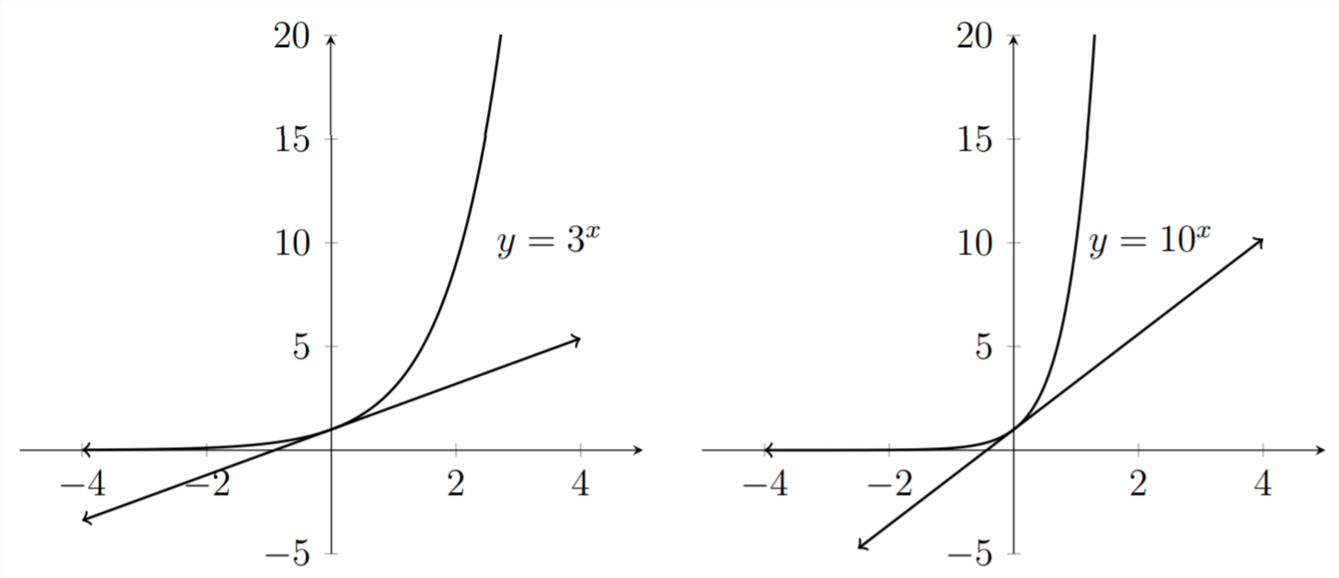




We can see that these graphs demonstrate slightly different behavior and different x and y values. One thing that they all have in common is that they all pass through the point (0,1) on the graph. This is because $1.5^0 = 1, 2^0 = 1, 3^0 = 1$ and $10^0 = 1$. Therefore the point where x is 0 and y is 1 is on all four of the graphs.

Although all four of the graphs pass through the point $(0, 1)$, they each do this in a different way. Let's look at the slope of a line tangent to each curve at the point $(0, 1)$. This is the straight line that touches the curve at the point $(0, 1)$, but nowhere else:

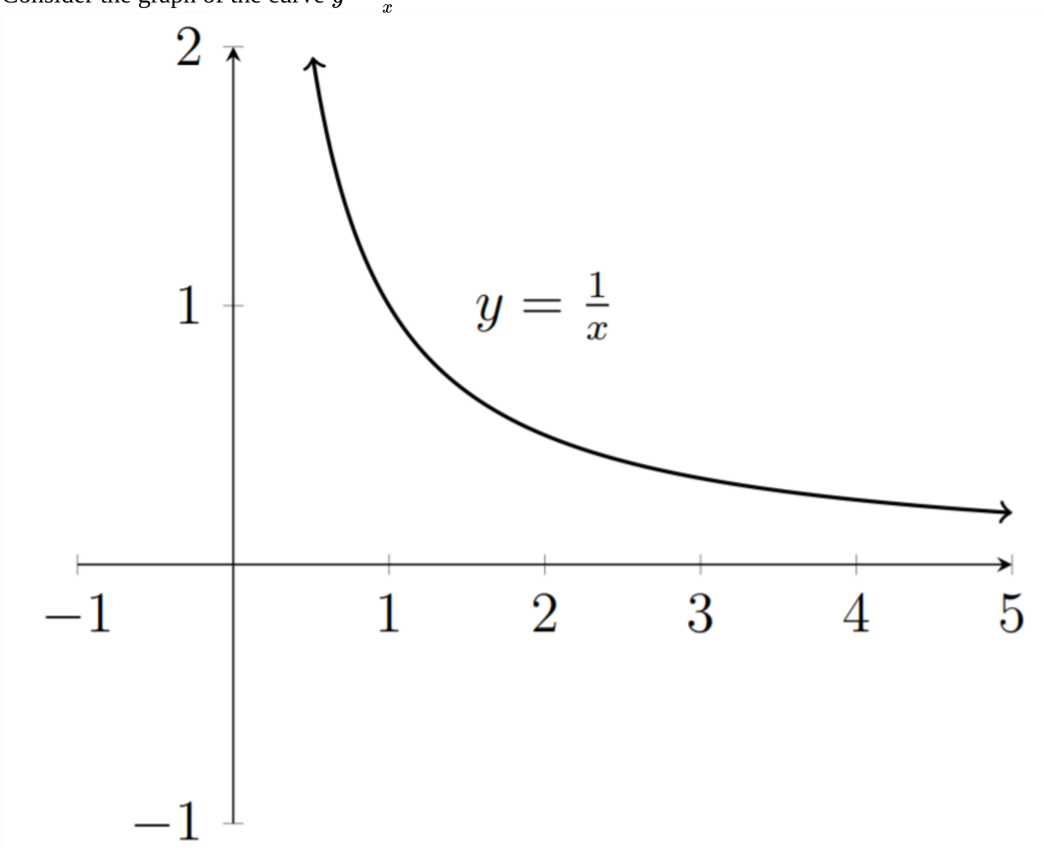




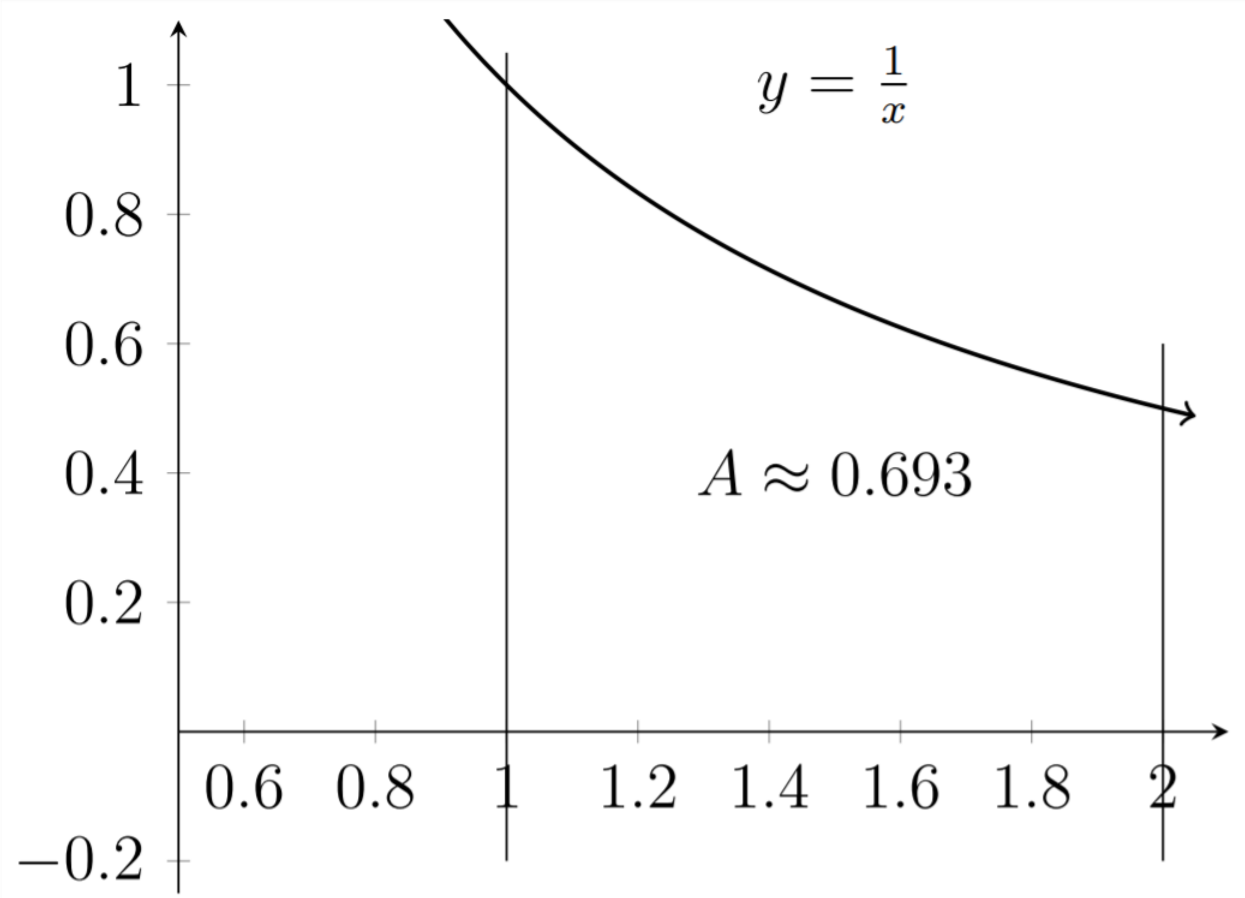
We can see that the slopes of these tangent lines are all different. In the case of $y=2^{x}$, the slope of the tangent line at (0,1) is about 0.7, while for the graph of $y=3^{x}$, the slope of the tangent line at (0,1) is about 1.1

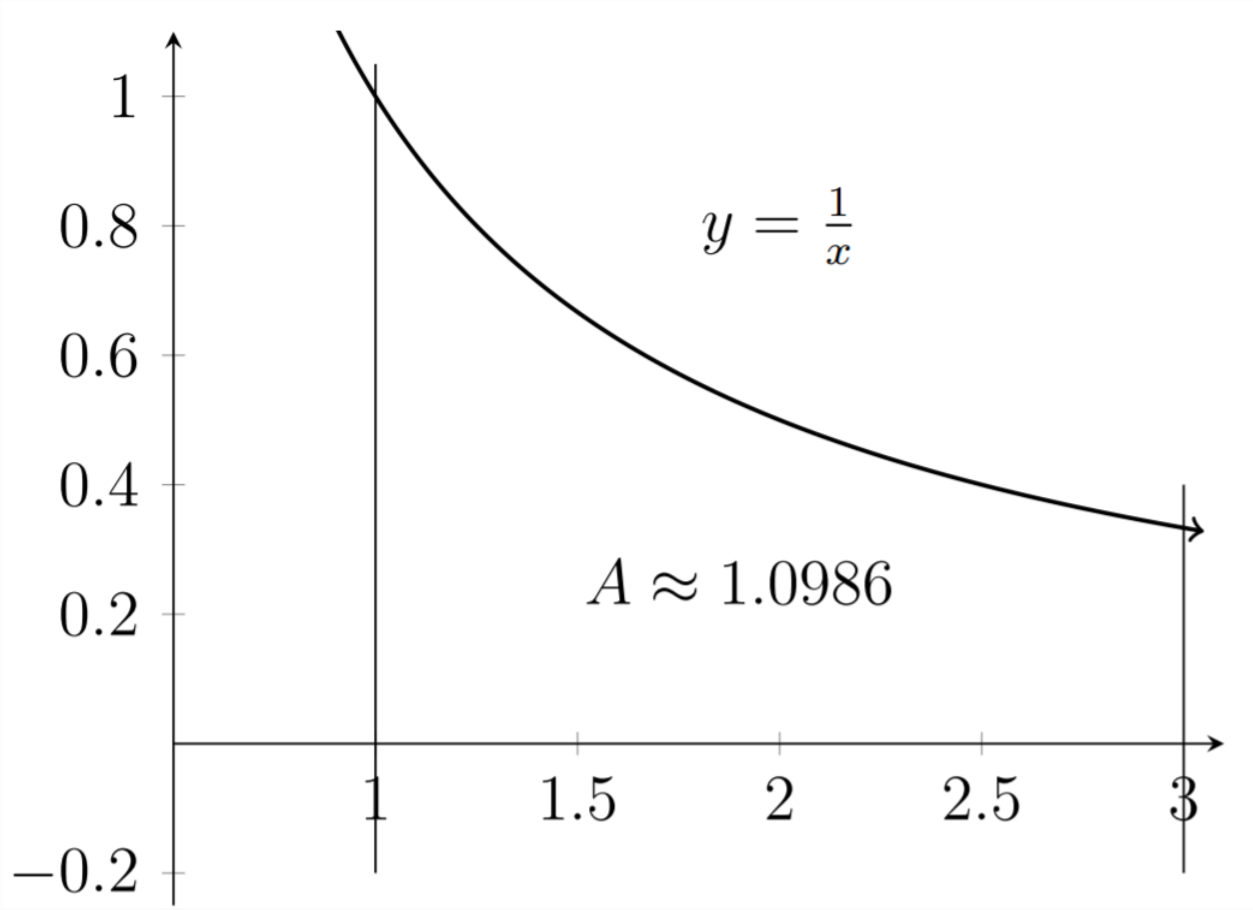
As mathematicians examined these graphs during the 17 th and 18 th centuries, they began to question what the value of the base " b'' should be in the equation $y=b^{x}$ so that the slope of the tangent line at the point (0,1) would be equal to exactly 1. The answer was $e \approx 2.71828$

Another way to derive the value of e uses Integral Calculus. Integral Calculus is often concerned with finding the area under a curve. This process can then be generalized and used to make many other types of calculations that are similar to finding area. Consider the graph of the curve $y=\frac{1}{x}$



We can delineate borders on the x values and determine the area of the resulting region using the techniques of calculus:


$y = \frac{1}{x}$
$A \approx 0.693$
1
0.8
0.6
0.4
0.2
−0.2
0.6
0.8
1
1.2
1.4
1.6
1.8
2

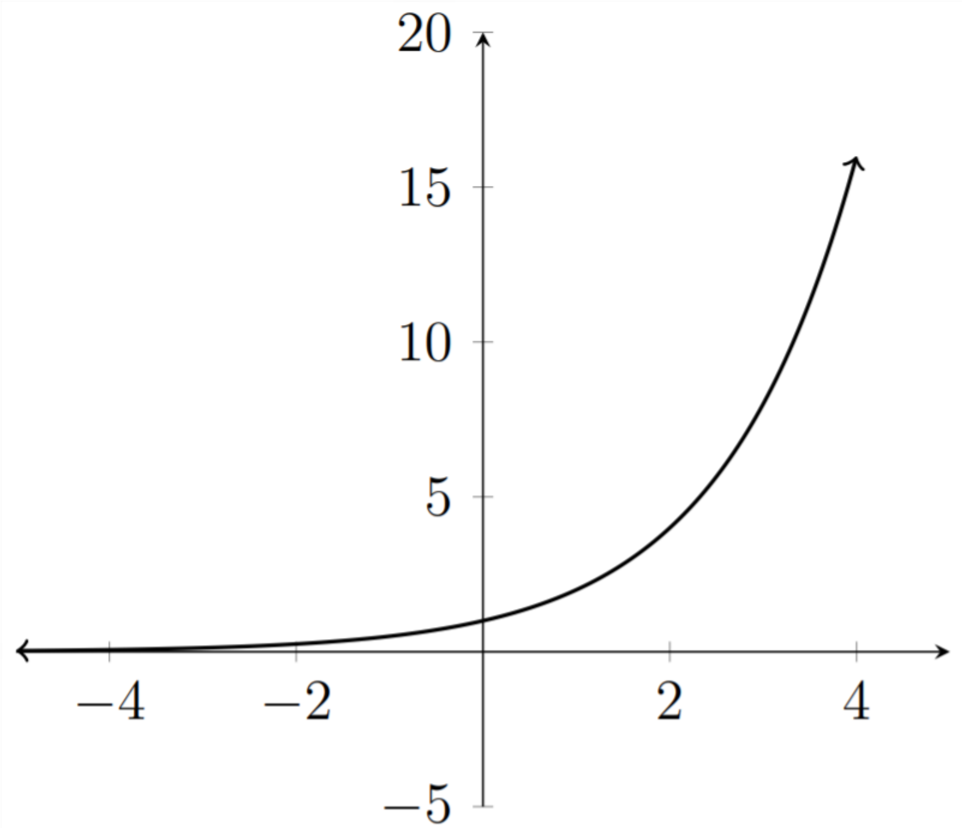


These values for the area under the curve are actually the same values as those for the slope of the tangent line in the previous graphs. If you ask the question, "Where should you draw the second vertical line so that the area under the curve is equal to exactly $1?''$ then just like the slope question, the answer is $e \approx 2.71828$

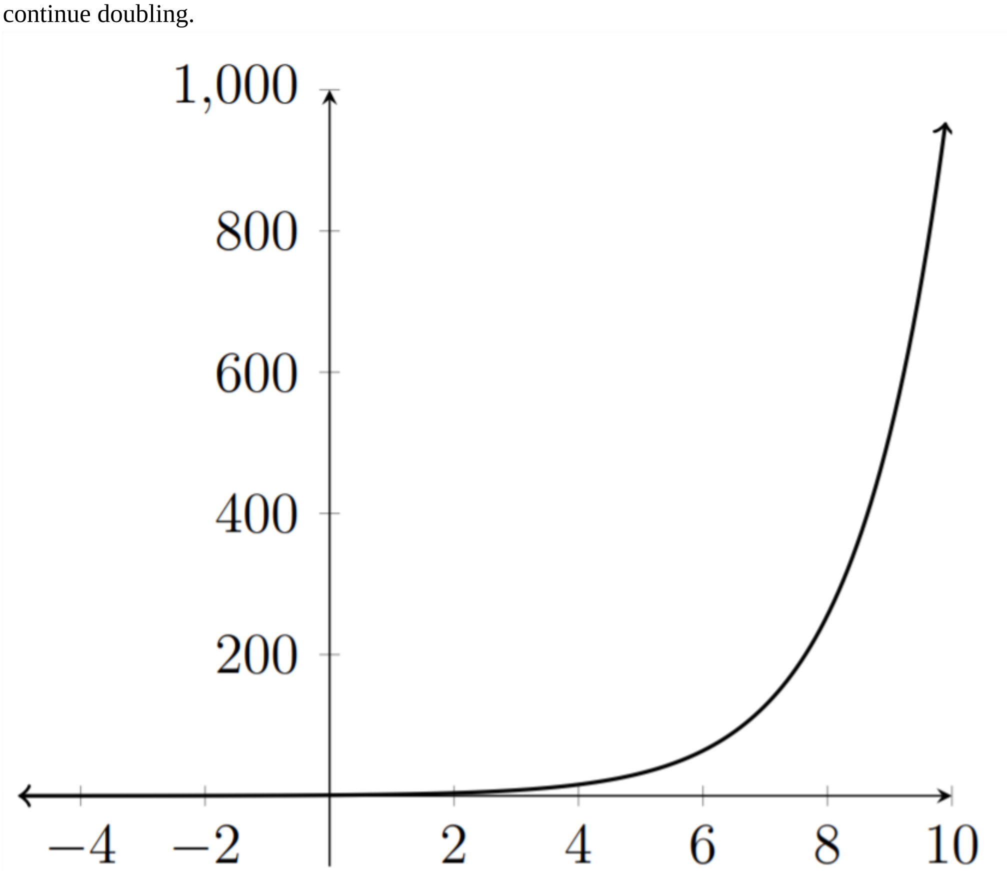
This is how the value of e was determined and why it is used to represent these exponential relationships.

Logistic Relationships

Let's consider the graph of $y = 2^x$:



If we extend the x -axis out further past $x = 4$, we would see that the y values for this relationship will grow very quickly, as they continue doubling.



Some phenomena in the natural world exhibit behavior similar to the growth of this function. However, in the natural world, few, if any, things can grow unconstrained. Most growth of any kind is limited by the resources that fuel the growth. Populations often grow exponentially for a period of time, however, populations are dependent on natural resources to continue growing. As a result, the simple exponential function is only useful for modeling real-world behavior if the x -values are limited.

It was this problem with the simple exponential function that led French mathematician Pierre Verhulst to slightly adjust the differential equation that gives rise to the exponential function to make it more realistic.
The original differential equation said:

$$y' = k * y \tag{3.1.3}$$

This says that the rate of growth of y is always directly proportional to the value of
y. In other words the larger a population gets, the faster it will grow - forever. Verhulst changed this to say:

$$y' = k * y \left(1 - \frac{y}{N}\right) \tag{3.1.4}$$

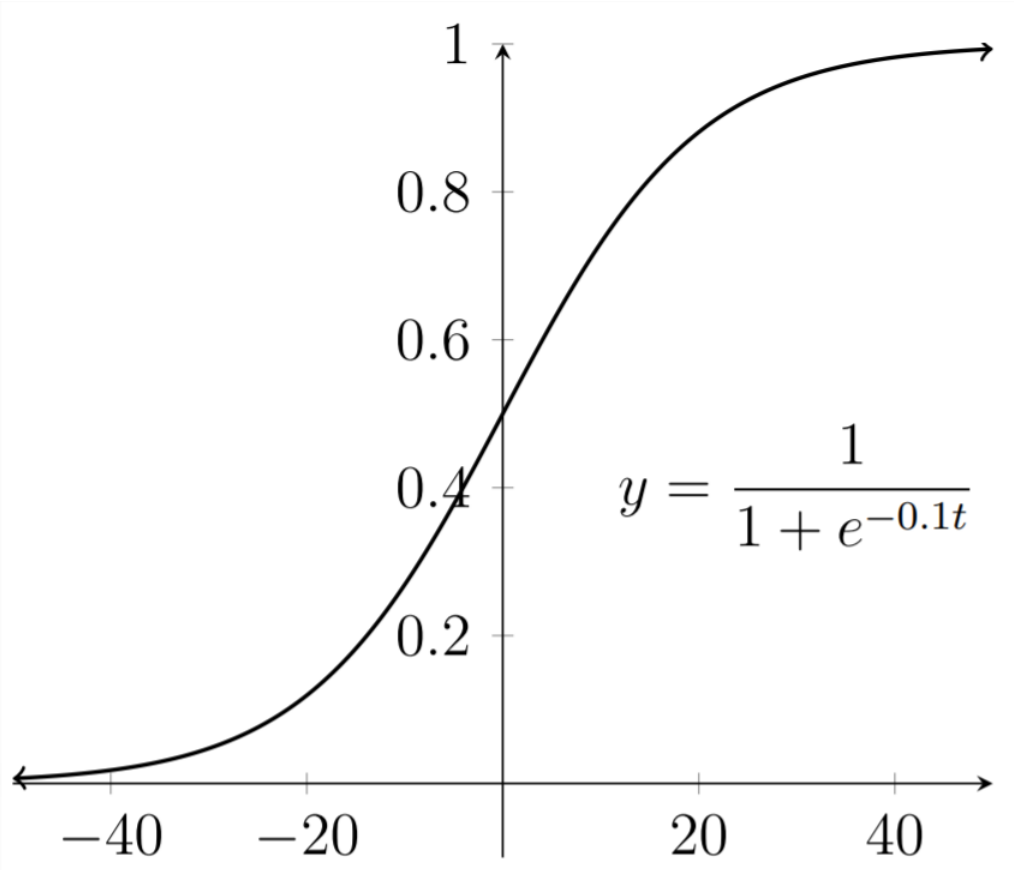
This is the defining relationship for the Logistic function. Notice that when values of y are small, this is essentially the same as the simple exponential. If y is small, then the $\left(1 - \frac{y}{N}\right)$ term will be very close to 1 and will produce behavior very much like the simple exponential.

The N in the formula is a theoretical "maximum population." As the value of y approaches this maximum value, $\frac{y}{N}$ will approach 1 and $\left(1 - \frac{y}{N}\right)$ will get smaller and smaller. As it gets smaller, the factor of $\left(1 - \frac{y}{N}\right)$ will slow down the growth of the function to model the pressure that is put on the resources that are driving the growth.

The solution of the Logistic equation is quite complicated and results in a standard form of:

$$y = \frac{N}{1 + be^{-kt}} \tag{3.1.5}$$

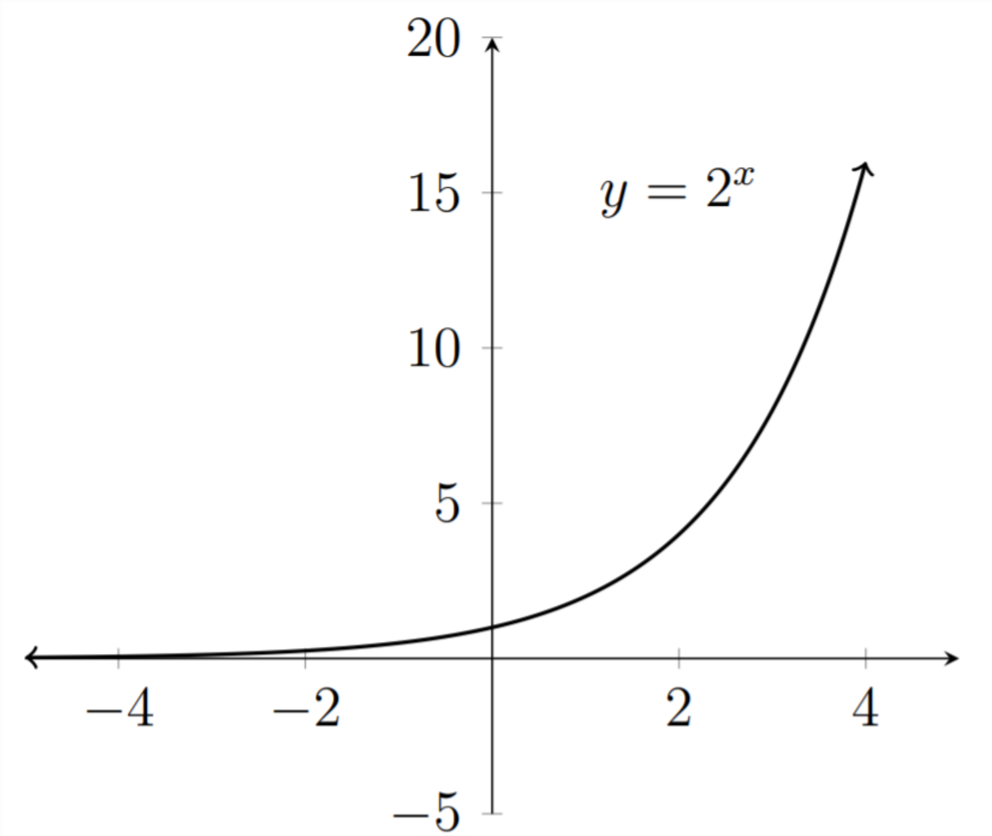
The graph for a sample logistic relationship is shown below:



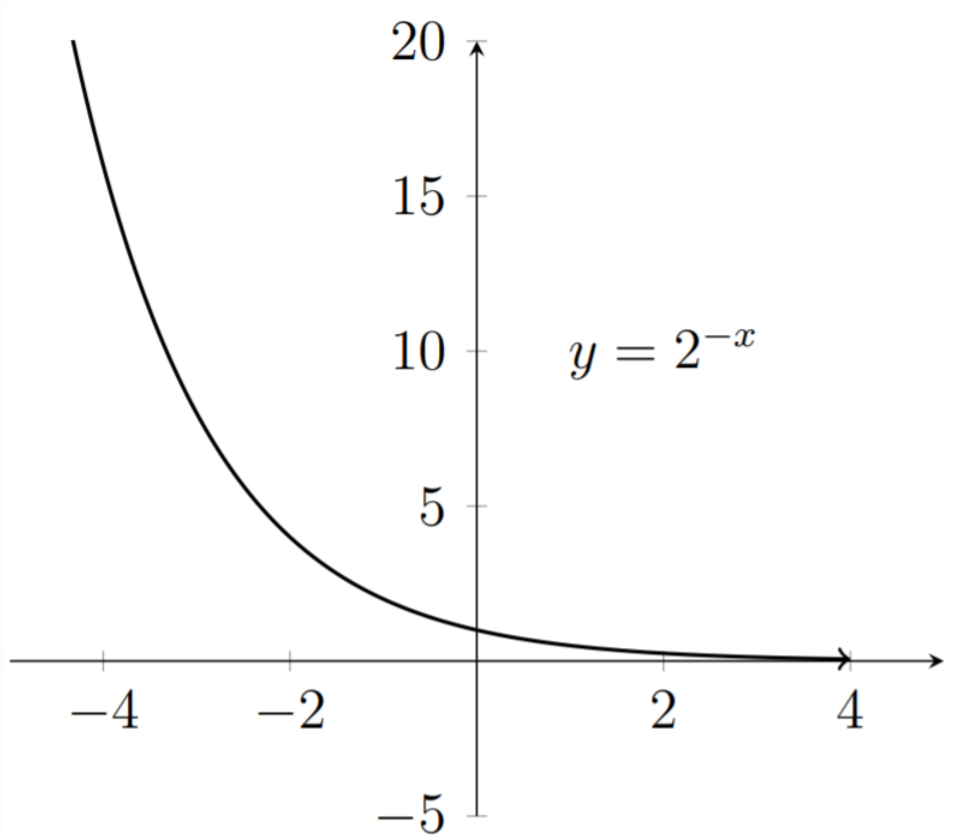
The "lazy-s" shape is characteristic of the logistic function. In the early stages, the relationship shows growth very similar to the simple exponential function but, as the function grows larger, the growth decreases and the function values stabilize. The maximum y value of N is always the horizontal asymptote for the logistic function.

Negative Exponential Relationships

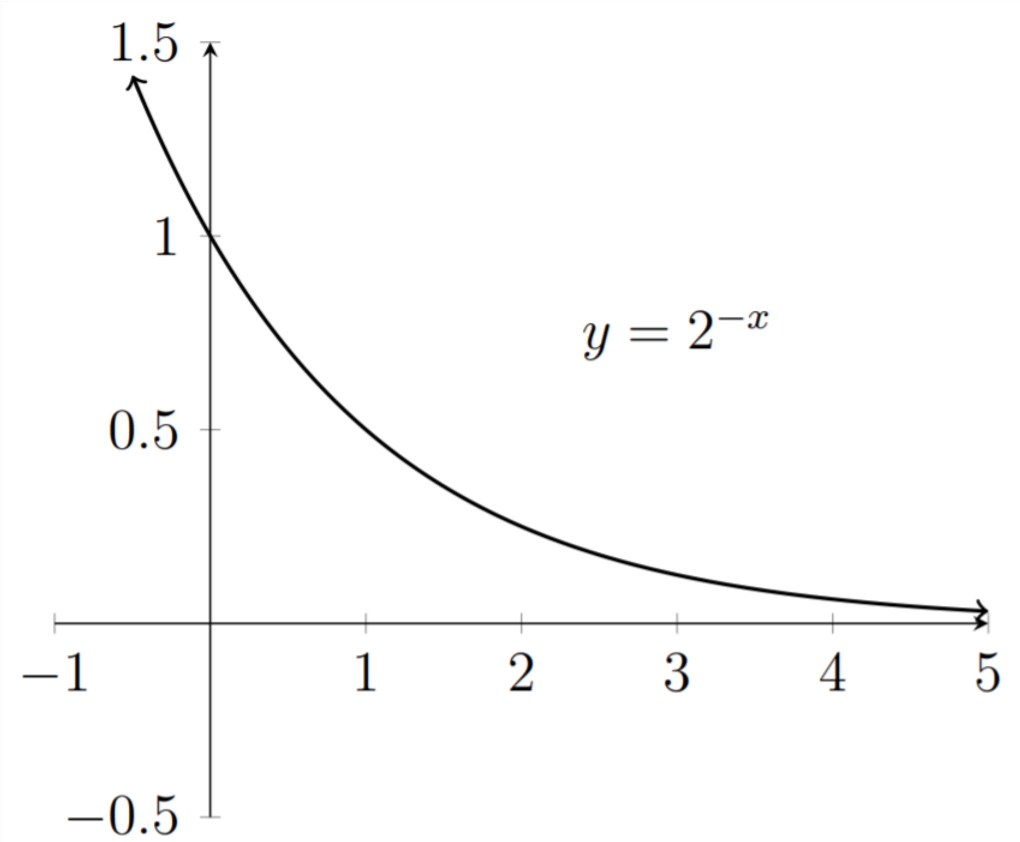
The Logistic function is very useful for modeling phenomena from the natural world. Although the simple exponential function is somewhat limited in modeling natural phenomena, the negative exponential is quite useful. Looking back to the graph of $y = 2^x$:



If we turn the graph around by changing x to $-x$ in the formula, then we will be working with the decaying tail of the graph:



Let's zoom in on the portion of the graph in the Quadrant I:



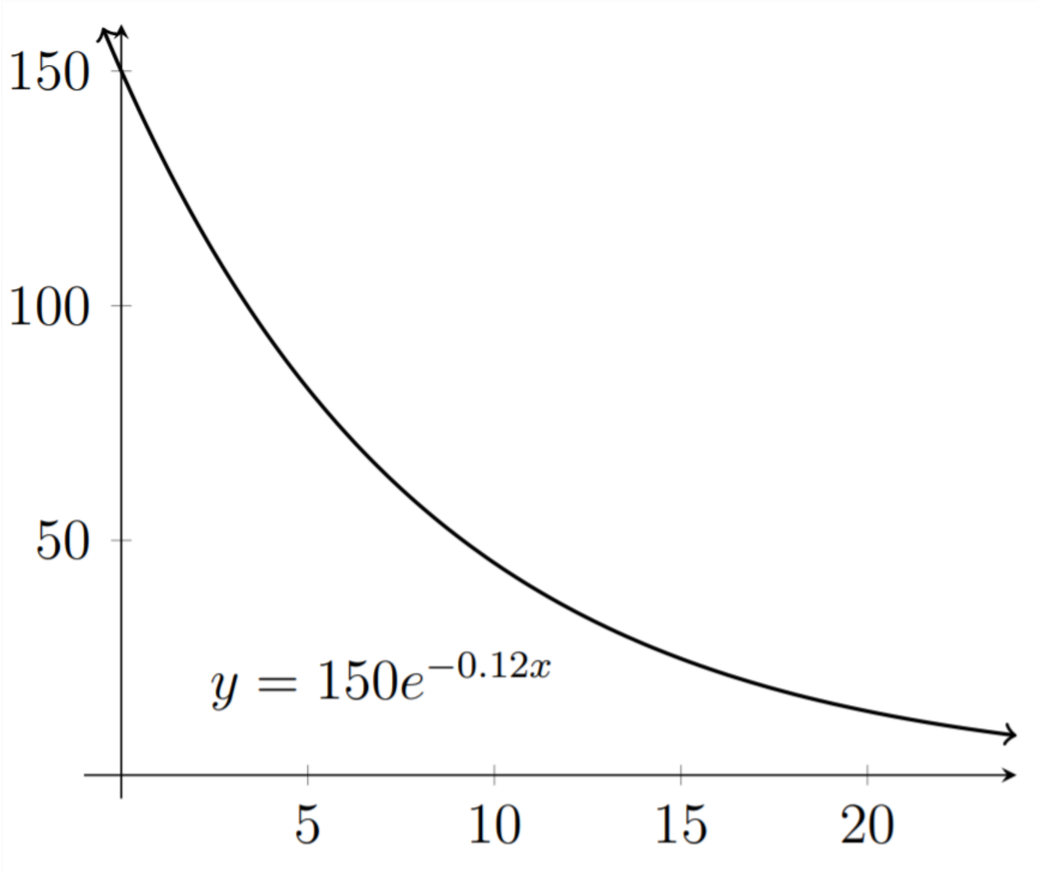
The behavior shown in the graph is quite useful for modeling radioactive decay, processes of heating and cooling, and assimilation of medication in the bloodstream. Let's look at an example.

Example
If a patient is injected with 150 micrograms of medication, the amount of medication still in the bloodstream after t hours is given by the function:

$$A(t) = 150e^{-0.12t} \tag{3.1.6}$$

a) Find the amount of medication in the bloodstream after 3 hours.
Round your answer to the nearest 100 th.
b) How long will it take for the amount of medication to reach 60 micrograms?
Round your answer to the nearest 10 th of an hour.

In this section we will focus on solving these problems using the graphing calculator. In later sections, we will cover processes that can be used to solve these problems algebraically. It is helpful to understand both methods of solution.
First, let's graph the function given in the problem:



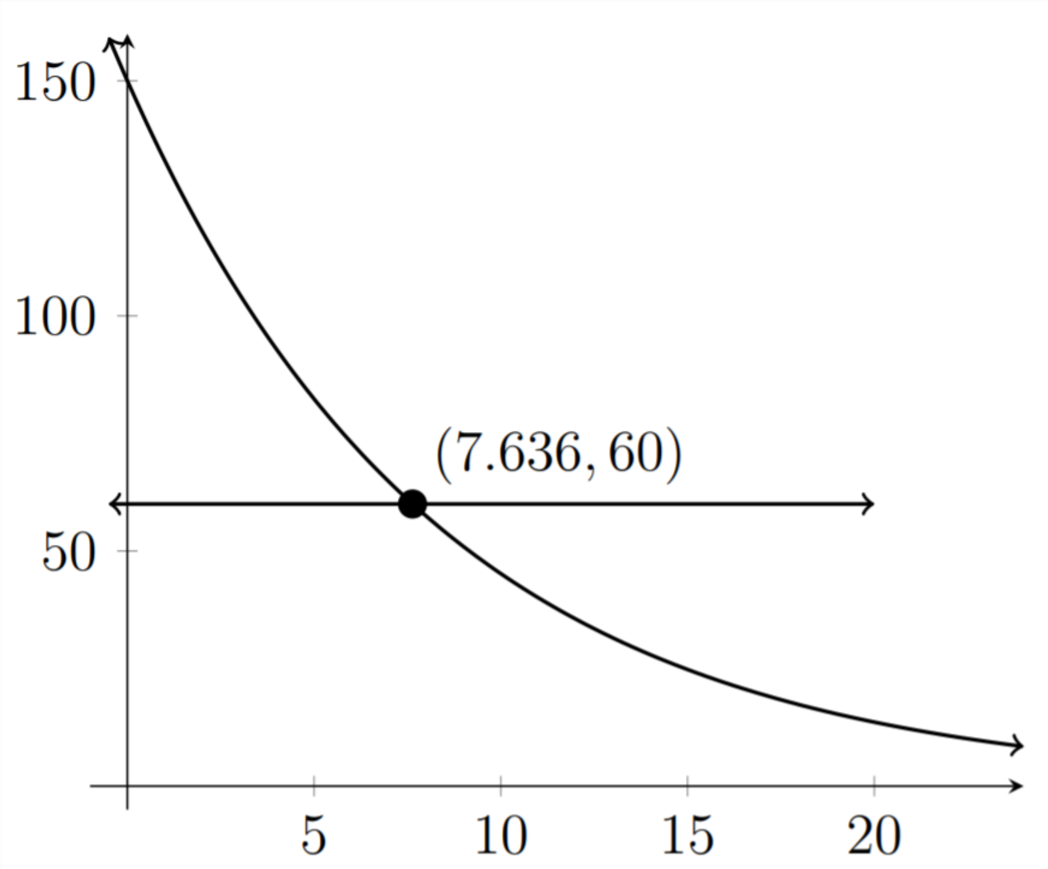
We can directly calculate the answer for part

(a) by plugging the value of 3 for t or we can use the table in the graphing calculator to find this value:

x	y
0	150
1	133.04
2	117.99
3	104.65

We can see that after 3 hours there are approximately 104.65 micrograms of medication in the patient's bloodstream.

To answer part (b) graphically, we'll graph the original function along with the horizontal line $y=60$. When we find the intersection of these two graphs, we'll know how long it takes for there to be 60 micrograms of medication in the patient's bloodstream:



Here, we can see that it would take about 7.6 hours (or about 7 hours 36 minutes), for there to be 60 micrograms of medication in the patient's bloodstream.

Example
The deer population on a nature preserve can be modeled using the equation:

$$y=\frac{8000}{1+9e^{-0.2t}} \tag{3.1.7}$$

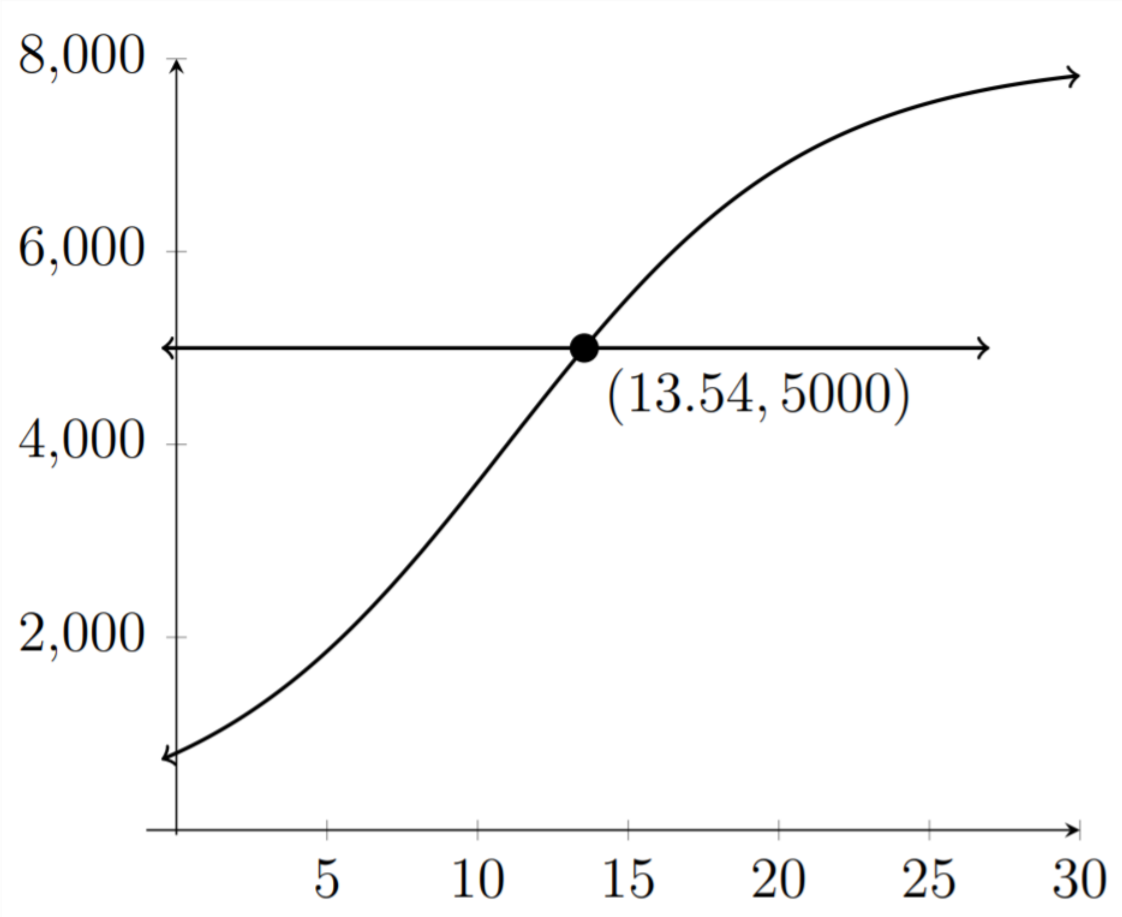
y indicates the number of deer living in the nature preserve and t represents the number of years that have passed since the initial population of deer were established there.
a) How many deer were in the initial population?
b) What is the deer population after 10 years?
Round your answer to the nearest whole number.
c) How long does it take for the population to reach $5,000$?
Round your answer to the nearest 10 th of a year.
Parts
(a) and (b) can both be answered from a table of values for the function. Before we look at the table, let's consider the question in Part (a). The problem is asking what the initial population of deer was. The means that we're looking to find out what y is when $t=0$
Let's see what happens when we plug zero in for t in the formula:

$$\begin{aligned} y &= \frac{8000}{1+9e^{-0.2*0}} = \frac{8000}{1+9e^{0}} \\ &= \frac{8000}{1+9*1} = \frac{8000}{10} \\ &= 800 \end{aligned}$$

Let's look at the table of values:

t	y
0	800
5	1855.8
10	3606.8

To the nearest whole number, the deer population after 10 years is 3,607
In Part (c), we'll need to graph a horizontal line at $y = 5,000$ and find the intersection.



To the nearest 10 th, it takes about 13.5 years for the deer population to reach 5,000

Exercises 3.1

1) Medication in the bloodstream

The function $A(t) = 200e^{-0.014t}$ gives the amount of medication, in milligrams, in a patient's bloodstream t minutes after the medication has been injected.

a) Find the amount of medication in the bloodstream after 45 minutes.

Round your answer to the nearest milligram.

b) How long will it take for the amount of medication to reach 50 milligrams?

Round your answer to the nearest minute.

2) Medication in the bloodstream

The function $D(t) = 50e^{-0.2t}$ gives the amount of medication, in milligrams, in a patient's bloodstream t hours after the medication has been injected.

a) Find the amount of medication in the bloodstream after 3 hours.

Round your answer to the nearest milligram.

b) How long will it take for the amount of medication to reach 10 milligrams?

Round your answer to the nearest minute.

3) Fish Population

The number of bass in a lake can be modeled using the given equation:

$$P(t) = \frac{3600}{1+7e^{-0.05t}} \tag{3.1.8}$$

where t is the number of months that have passed since the lake was stocked with bass. In each question below, round your answer

to the nearest whole number.
a) How many bass were in the lake immediately after it was stocked?
b) How many bass were in the lake 1 year after it was stocked?

4) Bird Population
The population of a certain species of bird is limited by the type of habitat required for nesting. The population can be modeled using the following equation:

$$P(t)=\frac{5600}{0.5+27.5e^{-0.044t}} \tag{3.1.9}$$

where t is the number of years. In each question below, round your answer to the nearest whole number.
a) Find the initial bird population.
b) What is the population 100 years later?

5) A Temperature Model
A cup of coffee is heated to $180^{\circ}\mathrm{F}$ and placed in a room that maintains a temperature of $65^{\circ}\mathrm{F}$. The temperature of the coffee after t minutes have passed is given by:

$$F(t)=65+115e^{-0.042t} \tag{3.1.10}$$

a) Find the temperature of the coffee 10 minutes after it is placed in the room.
Round your answer to the nearest degree.
b) When will the temperature of the coffee be $100^{\circ}\mathrm{F}$?
Round your answer to the nearest tenth of a minute.

6) A Temperature Model
Soup at a temperature of $170^{\circ}\mathrm{F}$ is poured into a bowl in a room that maintains a constant temperature. The temperature of the soup decreases according to the model given by:

$$F(t)=75+95e^{-0.12t} \tag{3.1.11}$$

where t is the number of minutes that have passed since the soup was poured.
a) What is the temperature of the soup after 2 minutes?
Round your answer to the nearest 10 th of a degree.
b) A certain customer prefers that the soup be cooled to $110^{\circ}\mathrm{F}$.
How long will this take?
Round your answer to the nearest 10 th of a minute.

7) Radioactive Decay
A radioactive substance decays in such a way that the amount of mass remaining after t days is given by the equation:

$$m(t)=13e^{-0.015t} \tag{3.1.12}$$

where the amount is measured in kilograms.
a) Find the mass at time $t=0$
b) How much of the mass remains after 45 days?
c) How long does it take for there to be 5 kg. left?

8) Radioactive Decay
Radioactive iodine is used by doctors as a tracer in diagnosing certain thyroid gland disorders. This type of iodine decays in such a way that the mass remaining after t days is given by:

$$m(t)=6e^{-0.087t} \tag{3.1.13}$$

where the amount is measured in grams.
a) Find the mass at time $t=0$
b) How much remains after 20 days?
c) How long does it take for there to be 2 grams left?
Round your answer to the nearest tenth of a day.

9) Fish population
The function:

$$P(t)=\frac{12}{1+e^{-t}} \tag{3.1.14}$$

gives the size of a fish population in thousands at time t, measured in years.
a) Find the initial population of fish at time $t=0$
Find the population of fish after 2 years, time $t=2$
b) How long will it take for the population to be $10,000$?
What appears to be the maximum population for this particular model?

10) Fish population
The function:

$$P(t)=\frac{70}{1+2e^{-t}} \tag{3.1.15}$$

gives the size of a fish population in thousands at time t, measured in years.
a) Find the initial population of fish at time $t=0$
Find the population of fish after 1.5 years, $t=1.5$
b) How long will it take for the population to be $50,000$?
What appears to be the maximum population for this particular model?

11) Fish population
The function:

$$P(t)=\frac{20}{1-0.4e^{-0.35t}} \tag{3.1.16}$$

gives the size of a fish population in thousands at time t, measured in years.
a) Find the initial population of fish at time $t=0$
Find the population of fish after 5 years, time $t=5$
b) How long will it take for the population to be $25,000$?
What happens to this population over time?

12) Fish population
The function:

$$P(t)=\frac{10}{1-0.73e^{-0.08t}} \tag{3.1.17}$$

gives the size of a fish population in thousands at time t, measured in years.
a) Find the initial population of fish at time $t=0$
Find the population of fish after 5 years, time $t=5$
b) How long will it take for the population to be $25,000$?
What happens to this population over time?

13) Continuous Mixing
A 100 gallon tank of pure water has salt water with 0.5 lb per gallon added to it at a rate of 2 gallons per minute. The brine solution is mixed thoroughly and drained at rate of 2 gallons per minute.

The equation for how many pounds of salt are in the tank at time t is given by:

$$y = 50 - 50e^{-0.02t} \tag{3.1.18}$$

where y is measured in pounds and t is measured in minutes.
a) How many pounds of salt are in the tank after 20 minutes?
b) How long does it take for there to be 30 lbs of salt in the tank?
Round your answer to the nearest tenth of a minute.

14) Continuous Mixing
A 20 gallon tank of pure water has salt water with 0.4 lb per gallon of salt added to it at a rate of 3 gallons per minute. The brine solution is mixed thoroughly and drained at rate of 3 gallons per minute.

The equation for how many pounds of salt are in the tank at time t is given by:

$$y = 8 - 8e^{-0.15t} \tag{3.1.19}$$

where y is measured in pounds and t is measured in minutes.
a) How many pounds of salt are in the tank after 15 minutes?
b) How long does it take for there to be 6 lbs of salt in the tank?
Round your answer to the nearest tenth of a minute.

15) Continuous Mixing
A tank contains 200 gallons of a 2% solution of HCl. A 5% solution of HCl is added at 5 gallons per minute. The well mixed solution is being drained at 5 gallons per minute.
The amount of HCl in the tank at any given time t in minutes is:

$$y = 10 - 6e^{-0.025t} \tag{3.1.20}$$

a) How many gallons of HCl are in the tank at $t = 25$ minutes?
b) When does the concentration of HCl in the solution reach 4%?

16) Continuous Mixing
A tank contains 100 gallons of salt water which contains a total of 25 lbs of salt. Salt water containing 0.4 lbs per gallon is added to the tank at a rate of 5 gallons per minute and the well-mixed solution is drained at the same rate.
The amount of salt in pounds in the tank at any given time t in minutes is:

$$y = 40 - 15e^{-0.05t} \tag{3.1.21}$$

a) How many pounds of salt are in the tank at $t = 20$ minutes?
b) How long does it take for there to be 30 lbs of salt in the tank?

3.2: Logarithmic Notation

A Logarithm is an exponent. In the early $1600'$ s, the Scottish mathematician John Napier devised a method of expressing numbers in terms of their powers of ten in order simplify calculation. since the advent of digital calculators, the methods of calculation using logarithms have become obsolete, however the concept of logarithms continues to be used in many area of mathematics.

The fundamental idea of logarithmic notation is that it is simply a restatement of an exponential relationship. The definition of a logarithm says:

$$\log_b N = x \rightarrow b^x = N \tag{3.2.1}$$

The notation above would be read as "log to the base b of N equals x means that b to the x power equals N." In this section we will focus mainly on becoming familiar with this notation. In later sections, we will learn to use this process to solve equations.

Example
Express the given statement using exponential notation:

$$\log_2 32 = 5 \tag{3.2.2}$$

If $\log_2 32 = 5$, then $2^5 = 32$

Example
Express the given statement using exponential notation:
$\log_7 4 \approx 0.7124$
If $\log_7 4 \approx 0.7124$, then $7^{0.7124} \approx 4$

If the logarithm notation appears without a base, it is usually assumed that the base should be 10

Example
Express the given statement using exponential notation:

$$\begin{gathered} \log 100 = 2 \\ \text{If } \log 100 = 2, \text{ then } 10^2 = 100 \end{gathered} \tag{3.2.3}$$

The notation $\ln N = x$ is typically used to indicate a logrithm to the base e. This means that:

$$\ln N = x \rightarrow e^x = N \tag{3.2.4}$$

Example
Express the given statement using exponential notation:
$\ln 15 \approx 2.708$
If $\ln 15 \approx 2.708$, then $e^{2.708} \approx 15$
In some cases, we would want to change an exponential statement into a logarithmic statement.

Example
Express the given statement using logarithmic notation:
$12^4 = 20,736$
If $12^4 = 20,736$ then $\log_{12} 20,736 = 4$

Example
Express the given statement using logarithmic notation:
$10^{2.5} \approx 316.23$
If $10^{2.5} \approx 316.23$ then $\log 316.23 \approx 2.5$

Example
Express the given statement using logarithmic notation:
$e^6 \approx 403.4$
If $e^6 \approx 403.4$, then $\ln 403.4 \approx 6$

Exercises 3.2

Rewrite each of the following using exponential notation.

1) $t = \log_5 9$
2) $h = \log_7 10$
3) $\log_5 25 = 2$
4) $\log_6 6 = 1$
5) $\log 0.1 = -1$
6) $\log 0.01 = -2$
7) $\log 7 \approx 0.845$
8) $\log 3 \approx 0.4771$
9) $\log_2 35 \approx 5.13$
10) $\log_{12} 50 \approx 1.5743$
11) $\ln 0.25 \approx -1.3863$
12) $\ln 0.989 \approx -0.0111$

Rewrite each of the following using logarithmic notation.

13) $10^2 = 100$
14) $10^4 = 10,000$
15) $4^{-5} = \frac{1}{1024}$
16) $5^{-3} = \frac{1}{125}$
17) $16^{\frac{3}{4}} = 8$
18) $8^{\frac{1}{3}} = 2$
19) $10^{1.3} \approx 20$
20) $10^{0.301} = 2$
21) $e^3 \approx 20.0855$
22) $e^2 \approx 7.3891$
23) $e^{-4} \approx 0.0183$
24) $e^{-2} \approx 0.1353$

3.3: Solving Exponential Equations

Because of the fact that logarithms are exponents, the rules for working with logarithms are similar to those that govern exponential expressions. One very helpful rule of equality for working with logarithms is related to the exponential rule for raising a power to a power. We recall one of the rules of exponents as:

$$(b^x)^y = b^{x*y} \tag{3.3.1}$$

in other words

$$\left(5^2\right)^4 = \left(5^2\right)\left(5^2\right)\left(5^2\right)\left(5^2\right) = 5^{2*4} = 5^8 \tag{3.3.2}$$

In logarithmic notation, this rule works out as:

$$\log_b M^p = p * \log_b M \tag{3.3.3}$$

The reason for this comes from the rule for exponents. Let's say that $\log_b M = x$
Then:

$$\log_b M = x \tag{3.3.4}$$

this means that

$$b^x = M \tag{3.3.5}$$

and

$$\begin{gathered}(b^x)^p = (M)^p \\ \text{so} \\ b^{p*x} = M^p\end{gathered} \tag{3.3.6}$$

Now, we come back to the question of $\log_b M^p = ?$. This expression $(\log_b M^p)$ is asking the question "What power do we raise b to in order to get an answer of
M^p? The result on the previous page shows that:

$$b^{px} = M^p \tag{3.3.7}$$

This means that we must raise b to the px power to get an answer of M^p. Remember that $x = \log_b M.$ This means that:

$$\begin{gathered}b^{px} = M^p \\ \text{so} \\ \log_b M^p = px = p * \log_b M\end{gathered} \tag{3.3.8}$$

This statement of equality is useful if we are trying to solve equations in which the variable is an exponent.
Example
Solve for x

$$4^x = 53 \tag{3.3.9}$$

We start by taking a logarithm on both of the equation. Just as we can add to both sides of an equation, or multiply on both sides of an equation, or raise both sides of an equation to a power, we can also take the logarithm of both sides. So long as two quantities are equal, then their logarithms will also be equal.

$$\begin{aligned} 4^x &= 53 \\ \log 4^x &= \log 53 \\ x \log 4 &= \log 53 \\ x &= \frac{\log 53}{\log 4} \approx 2.864 \end{aligned}$$

since the log base 10 and log base e are both programmed into most calculators, these are the most commonly used bases for logarithms.
Example
Solve for x

$$5^{2x+3} = 17 \tag{3.3.10}$$

We start this problem in the same fashion, but this time we will use a logarithm to the base e :

$$\begin{aligned} 5^{2x+3} &= 17 \\ \ln 5^{2x+3} &= \ln 17 \\ (2x+3) \ln 5 &= \ln 17 \end{aligned} \tag{3.3.11}$$

There are several possibilities for finishing the problem from this point. We will focus on two of them that are the most useful for solving more complex problems. First, we will distribute the $\ln 5$ into the parentheses and then get the x by itself.

$$\begin{aligned} (2x+3) \ln 5 &= \ln 17 \\ x * 2 \ln 5 + 3 \ln 5 &= \ln 17 \\ -3 \ln 5 &= -3 \ln 5 \\ x * 2 \ln 5 &= \ln 17 - 3 \ln 5 \\ x &= \frac{\ln 17 - 3 \ln 5}{2 \ln 5} \approx -0.620 \end{aligned}$$

And we can check the answer by plugging it back in:

$$5^{2*(-0.0620)+3} \approx 5^{1.760} \approx 16.9897 \approx 17 \tag{3.3.12}$$

We can also approximate the logarithms in the problem and solve for an approximate answer:

$$\begin{aligned} (2x+3) \ln 5 &= \ln 17 \\ x * 2 \ln 5 + 3 \ln 5 &= \ln 17 \\ 3.2189x + 4.8283 &\approx 2.8332 \\ -4.8283 &\approx -4.8283 \\ 3.2189x &\approx -1.9951 \\ x &\approx -0.620 \end{aligned}$$

If you use the method of approximating, it's important to make a good approximation. At least $4-5$ decimal places are necessary for an accurate answer.
Let's look at an example that has variables on both sides of the equation:

Example
Solve for x

$$4^{3x} = 9^{2x-1} \tag{3.3.13}$$

We'll use log base 10 in this problem.

$$\begin{aligned} 4^{3x} &= 9^{2x-1} \\ \log 4^{3x} &= \log 9^{2x-1} \\ 3x * \log 4 &= (2x-1)\log 9 \\ x * 3\log 4 &= x * 2\log 9 - \log 9 \end{aligned}$$

If we collect like terms, we'll end up with:

$$\begin{aligned} x * 3\log 4 &= x * 2\log 9 - \log 9 \\ \log 9 &= x * 2\log 9 - x * 3\log 4 \end{aligned}$$

At this point, if we want to get the x by itself, we need to factor out the x on the right-hand side:

$$\begin{aligned} \log 9 = x * 2\log 9 - x * 3\log 4 \\ \log 9 = x(2\log 9 - 3\log 4) \end{aligned} \tag{3.3.14}$$

Then divide on both sides by the coefficient in parentheses:

$$\begin{aligned} \frac{\log 9}{2\log 9 - 3\log 4} = \frac{x\ \cancel{(2\log 9 - 3\log 4)}}{\cancel{2\log 9 - 3\log 4}} \\ \frac{\log 9}{2\log 9 - 3\log 4} = x \\ 9.327 \approx x \end{aligned} \tag{3.3.15}$$

Again, we can check our answer by plugging it back into the equation:

$$4^{3*9.327} \approx 4^{27.981} \approx 7.0184 * 10^{16} \tag{3.3.16}$$

$$9^{2*9.327-1} \approx 9^{17.654} \approx 7.0177 * 10^{16} \tag{3.3.17}$$

We could also have solved this equation by approximating the logarithms in the beginning.

$$\begin{aligned} 4^{3x} &= 9^{2x-1} \\ \log 4^{3x} &= \log 9^{2x-1} \\ 3x * \log 4 &= (2x-1)\log 9 \\ 3x(0.60206) &\approx (2x-1)0.95424 \\ 1.80618x &\approx 1.9085x - 0.95424 \\ 0.95424 &\approx 0.10232x \\ 9.326 &\approx x \end{aligned}$$

This answer is less accurate than the other approximation $(9.326036 \text{ vs. } 9.327424)$The accuracy of an answer depends upon the original approximations for the logarithms.

Solve for the indicated variable.

1) $2^x = 5$
2) $2^x = 9$
3) $3^x = 7$
4) $3^x = 20$
5) $2^{x+1} = 6$
6) $7^{x+1} = 41$
7) $5^{x+1} = 36$
8) $8^{x-2} = 6$
9) $4^{2x+3} = 50$
10) $4^{x+2} = 5^x$

11) $5^{2x+1}=9$
12) $6^{x+4}=10^{x}$
13) $7^{y+1}=3^{y}$
14) $2^{x+1}=3^{x-2}$
15) $6^{y+2}=5^{y}$
16) $7^{x-3}=3^{x+1}$
17) $6^{2x+1}=5^{x+2}$
18) $9^{1-x}=12^{x+1}$
19) $5^{2x-1}=3^{x-3}$
20) $3^{x-2}=4^{2x+1}$
21) $8^{3x-2}=9^{x+2}$
22) $2^{2x-3}=5^{-x-1}$
23) $10^{3x+2}=5^{x+3}$
24) $5^{3x}=3^{x+4}$
25) $3^{x+4}=2^{1-3x}$
26) $4^{2x+3}=5^{x-2}$
27) $3^{2-3x}=4^{2x+1}$
28) $2^{2x-3}=5^{x-2}$

3.4: Solving Logarithmic Equations

In the previous section, we took exponential equations and used the properties of logarithms to restate them as logarithmic equations. In this section, we will take logarithmic equations and use properties of logarithms to restate them as exponential equations. In the previous section, we used the property of logarithms that said $\log_b M^p = p\log_b M.$ In this section, we will make use of two additional properties of logarithms:

$$\log_b(M*N) = \log_b M + \log_b N \tag{3.4.1}$$

and

$$\log_b \frac{M}{N} = \log_b M - \log_b N \tag{3.4.2}$$

Just as our previous property of logarithms was simply a restatement of the rules of expoenents, these two properties of logarithms depend on the rules of exponents as well. since we're interested in $\log_b M$ and $\log_b N$, let's restate these in terms of exponents:
If $\log_b M = x$ then $b^x = M$ and if $\log_b N = y$ then $b^y = N$
The properties of logarithms we're interested in justifying have to do with $M*N$ and $\frac{M}{N}$, so let's look at those expressions in terms of exponents:

$$\begin{gathered} M*N = b^x * b^y = b^{x+y} \\ \text{and} \\ \frac{M}{N} = \frac{b^x}{b^y} = b^{x-y} \end{gathered} \tag{3.4.3}$$

If we're interested in $\log_b(M*N)$, then we're asking the question "What power do we raise b to in order to get $M*N?$" We can see above that raising b to the $x+y$ power gives us $M*N.$ since $x = \log_b M$ and $y = \log_b N$ then $x+y = \log_b M + \log_b N$, so:

$$\log_b(M*N) = x+y = \log_b M + \log_b N \tag{3.4.4}$$

Likewise, if we're interested in $\log_b \frac{M}{N}$, we're asking the question "What power do we raise b to in order to get $\frac{M}{N}?$" since raising b to the $x-y$ power gives us $\frac{M}{N}$ and $x-y = \log_b M - \log_b N,$ then:

$$\log_b \frac{M}{N} = x-y = \log_b M - \log_b N \tag{3.4.5}$$

Let's look at an example to see how we'll use this to solve equations:

Example
Solve for x

$$\log_2 x + \log_2(x-4) = 2 \tag{3.4.6}$$

The first thing we can do here is to combine the two logarithmic statements into one. since $\log_b(M*N) = \log_b M + \log_b N,$ then $\log_2 x + \log_2(x-4) = \log_2[x(x-4)]$

$$\begin{aligned} \log_2 x + \log_2(x-4) &= 2 \\ \log_2[x(x-4)] &= 2 \end{aligned}$$

Then we'll restate the resulting logarithmic relationship as an exponential relationship:

$$\begin{aligned} 2^2 &= x(x-4) \\ 4 &= x^2 - 4x \\ 0 &= x^2 - 4x - 4 \\ 4.828, -0.828 &\approx x \end{aligned}$$

Most textbooks reject answers that result in taking the logarithm of a negative number, such as would be the case for $x \approx -0.828$. However, the logarithms of negative numbers result in complex valued answers, rather than an undefined quantity. For that reason, in this text, we will include all answers.

If a problem involves a difference of logarithms, we can use the other property of logarithms introduced in this section.
Example
Solve for x

$$\log(5x-1)-\log(x-2)=2 \tag{3.4.7}$$

Again, our first step is to restate the difference of logarithms using the property $\log_b \frac{M}{N}=\log_b M-\log_b N$:

$$\begin{gathered}\log(5x-1)\ -\log(x-2)=2\\ \log\left[\frac{5x-1}{x-2}\right]=2\end{gathered}$$

We're working with a logarithm in base 10 in this problem, so in our next step we'll say:

$$\begin{gathered}\log\left[\frac{5x-1}{x-2}\right]=2\\ \frac{5x-1}{x-2}=10^2\end{gathered} \tag{3.4.8}$$

Then multiply on both sides by $x-2$

$$\begin{gathered}10^2=\frac{5x-1}{x-2}\\ (x-2)*100=\frac{5x-1}{\cancel{(x-2)}}*\ \cancel{(x-2)}\end{gathered} \tag{3.4.9}$$

And, solve for x

$$\begin{gathered}100x-200=5x-1\\ 95x=199\\ x=\frac{199}{95}\end{gathered} \tag{3.4.10}$$

In some equations, all of the terms are stated using logartihms. These equations often come out in a form that says $\log_b x=\log_b y$. If this is the case, we can then conclude that $x=y$

It seems reasonable that if the exponent we raise b to in order to get x is the same exponent that we raise b to in order to get y, then x and y are the same thing.
Assume:

$$\log_b x=\log_b y \tag{3.4.11}$$

then

$$b^a=x \text{ and } b^a=y \tag{3.4.12}$$

if both x and y are equal to b^a, then $x=y$

Example
Solve for x
$\log_5(4-x)=\log_5(x+8)+\log_5(2x+13)$

First, let's use the properties of logarithms to restate the equation so that there is only one logarithm on each side.

$$\begin{gathered}\log_5(4-x)=\log_5(x+8)+\log_5(2x+13)\\ \log_5(4-x)=\log_5[(x+8)(2x+13)]\end{gathered} \tag{3.4.13}$$

Then, we'll use the property of logarithms we just discussed:

$$\text{If } \log_b x = \log_b y \tag{3.4.14}$$

then

$$\begin{gathered} x = y \\ \log_5(4-x) = \log_5[(x+8)(2x+13)] \\ 4-x = (x+8)(2x+13) \\ 0 = 2x^2+29x+104 \\ 0 = 2(x+5)(x+10) \\ -5, -10 = x \end{gathered} \tag{3.4.15}$$

Exercises 3.4

Solve for the indicated variable in each equation.

1) $\log_3 5 + \log_3 x = 2$
2) $\log_4 x + \log_4 5 = 1$
3) $\log_2 x = 2 + \log_2 3$
4) $\log_5 x = 2 + \log_5 3$
5) $\log_3 x + \log_3(x-8) = 2$
6) $\log_6 x + \log_6(x-5) = 1$
7) $\log(3x+2) = \log(x-4) + 1$
8) $\log(x-1) - \log x = -0.5$
9) $\log_2 a + \log_2(a+2) = 3$
10) $\log_3 x + \log_3(x-2) = 1$
11) $\log_2 y - \log_2(y-2) = 3$
12) $\log_2 x - \log_2(x+3) = 2$
13) $\log_3 x + \log_3(x+4) = 2$
14) $\log_4 u + \log_4(u+1) = 1$
15) $\log 5 + \log x = \log 6$
16) $\ln x + \ln 4 = \ln 2$
17) $\log_7 x - \log_7 12 = \log_7 2$
18) $\log 2 - \log x = \log 8$
19) $\log_3 x - \log_3(x-2) = \log_3 4$
20) $\log_6 2 - \log_6(x-2) = \log_6 9$
21) $\log_4 x - \log_4(x-4) = \log_4(x-6)$
22) $\log_9(2x+7) - \log_9(x-1) = \log_9(x-7)$
23) $2\log_2 x = \log_2(2x-1)$
24) $2\log_4 y = \log_4(y+2)$
25) $2\log(x-3) - 3\log 2 = 1$
26) $2\log_5 7 - \log_5(x+1) = \log_5(2x-5)$

3.5: Applications of the Negative Exponential Function

At the beginning of Chapter 3, we worked with application problems and solved them using the graphing calculator. In this section, we will revisit some of these application problems and use the solution methods discussed in the previous sections to solve these problems algebraically.

Radioactive Decay

The decay of a radioactive element into its non-radioactive form occurs following a time line dictated by the "half-life" of the element. The half-life is the amount of time that it takes for half of the existing radioactive material to decay to its non-radioactive form.

Consider the equation:

$$A(t)=A_0e^{-kt} \tag{3.5.1}$$

where $A(t)$ is the amount of material left at time t, A_0 is the amount present at $t=0$, and k is a constant that can be determined based on the half-life of the material.

If we know that after one half-life, there will 50% of the radioactive material remaining, then we can say that:

$$0.5A_0=A_0e^{-kt_h} \tag{3.5.2}$$

where t_h is the half-life. To solve this equation for k, we would first divide on both sides by A_0 :

$$\begin{aligned}\frac{0.5A_0}{A_0}&=\frac{A_0e^{-kt_h}}{A_0}\\0.5&=e^{-kt_h}\end{aligned} \tag{3.5.3}$$

Then take the natural logarithm of both sides and bring the exponent down in front of the expression as a coefficient:

$$\begin{aligned}0.5&=e^{-kt_h}\\\ln(0.5)&=\ln\left(e^{-kt_h}\right)\\\ln(0.5)&=-kt_h*\ln(e)\\\ln(0.5)&=-kt_h*1\\-\frac{\ln(0.5)}{t_h}&=k\end{aligned}$$

The value of k can then be used in the equation $A(t)=A_0e^{-kt}$ to determine the amount of material left after any time t

Example

The isotope Gold-198 $\left({}^{198}\mathrm{Au}\right)$ is a type of gold sometimes used in medical applications and has a half-life of 2.7 days. How much of a 65 gram sample of ${}^{198}\mathrm{Au}$ will be left after 6 days? How long would it take for there to be 10 grams left?

If we know the half-life, we can calculate the value of the constant k

$$\begin{aligned}k&=-\frac{\ln(0.5)}{t_h}\\k&=-\frac{\ln(0.5)}{2.7}\\k&\approx0.2567\end{aligned} \tag{3.5.4}$$

Now that we know the value of k, we can directly calculate the amount of ${}^{198}\mathrm{Au}$ left after 6 days:

$$\begin{aligned}A(t)&=A_0e^{-kt}\\&=65e^{-0.2567*6}\\&\approx13.932\end{aligned}$$

So, approximately 13.932 grams of ^{198} Au would be left after 6 days.

In order to calculate how long it takes for 10 grams of ^{198} Au to be left, we'll need to solve for t in the equation $A(t)=A_0e^{-kt}$ with $A(t)=10$:

$$\begin{aligned}A(t)&=A_0e^{-kt}\\10&=65e^{-0.2567t}\end{aligned}$$

First, we'll divide on both sides by 65

$$\begin{aligned}\frac{10}{65}&=\frac{65e^{-0.2567t}}{65}\\\frac{10}{65}&=e^{-0.2567t}\end{aligned}\tag{3.5.5}$$

Then, take the natural logarithm on both sides:

$$\ln\left(\frac{10}{65}\right)=\ln\left(e^{-0.2567t}\right)\tag{3.5.6}$$

We'll calculate an approximate value for $\ln\left(\frac{10}{65}\right)$ and restate the right hand side of the equation using the properties of logarithms:

$$\begin{aligned}-1.872&\approx-0.2567t*\ln e\\-1.872&\approx-0.2567t*1\\-1.872&\approx-0.2567t\\7.3&\approx t\end{aligned}$$

So, it would take about 7.3 days for the 65 grams of ^{198} Au to decay to 10 grams.

Newton's Law of Cooling

Newton's Law of Cooling states that the temperature of an object can be determined using the equation:

$$T=T_a+Ce^{-kt}\tag{3.5.7}$$

where T_a is the ambient temperature of the surrounding environment. The values of the constants C and k can often be calculated from given information.

Example

A bottle of soda at room temperature $(72^{\circ}\mathrm{F})$ is placed in a refrigerator where the temperature is $44^{\circ}\mathrm{F}$

After half an hour, the soda has cooled to $61^{\circ}\mathrm{F}$. What is the temperature of the soda after another half hour?

First, since the soda is placed in the refrigerator where the ambient temperature is $44^{\circ}\mathrm{F}$, then $T_a=44$. We also know that at $t=0, T=72$, which is the temperature of the soda can when it is first put in the refrigerator. This will allow us to calculate the constant C.

$$\begin{aligned}T&=T_a+Ce^{-kt}\\72&=44+Ce^{-k*0}\\72&=44+Ce^{0}=44+C*1\\72&=44+C\\28&=C\end{aligned}$$

Now we know that $T_a=44$ and $C=28$. We can use the other piece of information from the problem to calculate the value of k. The problem states that after 30 minutes the soda has cooled off to $61^{\circ}\mathrm{F}$. That means that when $t=30$ (or $t=0.5$ depending on which units you choose) the temperature T will be $61^{\circ}\mathrm{F}$. We can then set up the equation to reflect this and calculate the vlaue of k :

$$\begin{aligned}T&=44+28e^{-kt}\\61&=44+28e^{-k*30}\end{aligned}\tag{3.5.8}$$

First, we'll subtract 44 on both sides and divide by 28 :

$$\begin{aligned}61 &= 44+28e^{-k*30}\\ 17 &= 28e^{-30k}\\ \tfrac{17}{28} &= e^{-30k}\end{aligned} \tag{3.5.9}$$

Now, we'll take the natural logarithm on both sides to bring the $-30k$ down from the exponent:

$$\begin{aligned}\ln\left(\tfrac{17}{28}\right) &= \ln\left(e^{-30k}\right)\\ \ln\left(\tfrac{17}{28}\right) &= -30k*\ln e\\ -0.499 &\approx -30k\\ 0.01663 &\approx k\end{aligned} \tag{3.5.10}$$

Now, we have the full formula for calculating temperature in this scenario:

$$T = 44+28e^{-0.01663t} \tag{3.5.11}$$

To find out what happens after 60 minutes, we can simply plug in 60 for t

$$\begin{aligned}T &= 44+28e^{-0.01663t}\\ T &= 44+28e^{-0.01663\cdot 60}\\ T &= 44+28e^{-0.9978}\\ T &\approx 44+28*0.3687\\ T &\approx 44+10.3236 \approx 54.3^{\circ}\mathrm{F}\end{aligned} \tag{3.5.12}$$

Exercises 3.5

1) If the half-life of radioactive cesium- 137 is 30 years, find the value of k in the equation $A(t)=A_0e^{-kt}$

a) Given a 10 gram sample of cesium-137, how much will remain after 80 years?

b) How long will it take for only 2 grams of cesium- 137 to remain?

2) The half-life for radioactive thorium- 234 is about 25 days. Use this to find the value of k in the equation $A(t)=A_0e^{-kt}$

a) How much of a 40 gram sample will remain after 60 days?

b) After how long will only 10 grams of thorium- 234 remain?

3) Given a sample of strontium- 90 , it is known that after 18 years there are $32\mathrm{mg}$ remaining and after 65 years there are $10\mathrm{mg}$ remaining. Use this information to find out how much strontium- 90 was in the sample to begin with, and also determine the half-life of strontium- 90 .

4) A 12 mg sample of radioactive polonium decays to $7.26\mathrm{mg}$ in 100 days.

a) What is the half-life of polonium?

b) How much of the 12 mg sample remains after 180 days?

5) A hot bowl of soup is served at a dinner party. It starts to cool according to Newton's Law of Cooling so that its temperature at time t is given by:

$$y = 65+145e^{-0.05t} \tag{3.5.13}$$

where t is measured in minutes and y is measured in degrees Fahrenheit.

a) What is the initial temperature of the soup?

b) What is the temperature after 10 minutes?

c) After how long will the temperature be 100°?

6) Newton's Law of Cooling is used in homicide investigations to determine the time of death. The normal body temperature is $98.6^{\circ}\mathrm{F}$. Immediately following death, the body begins to cool. This process uses Newton's Law of Cooling:

$$y = T_a+Ce^{-kt} \tag{3.5.14}$$

If the ambient temperature is 60°, and the body has cooled to $72^{\circ}\mathrm{F}$ after 6 hours, use this information to determine the value of k in the equation.

7) The police discover the body of a murder victim. Critical to solving the crime is determining when the murder was committed. The coroner arrives at the murder scene at 12 Noon. She immediately takes the temperature of the body and finds it to be $94.6^{\circ}\mathrm{F}$. She then takes the temperature 1.5 hours later and finds it to be $93.4^{\circ}\mathrm{F}$. If the temperature of the room is $70^{\circ}\mathrm{F}$, when was the murder committed?

8) A roasted turkey is taken from the oven when its temperature has reached
$185^{\circ}\mathrm{F}$ and is placed on a table in a room where the temperature is $75^{\circ}\mathrm{F}$
a) If the temperature of the turkey is $150^{\circ}\mathrm{F}$ after 30 minutes, what is its temperature after 45 minutes?
b) When will the turkey cool off to $100^{\circ}\mathrm{F}$?

9) A kettle full of water is brought to a boil in a room with an ambient temperature of $20^{\circ}\mathrm{C}$. After 15 minutes, the temperature of the water has decreased from $100^{\circ}\mathrm{C}$ to $75^{\circ}\mathrm{C}$. Find an equation using Newton's Law of Cooling to represent to temperature at time t. Find the temperature of the water after 25 minutes.

10) A cup of coffee with a temperature of $105^{\circ}\mathrm{F}$ is placed in a freezer with a temperature of $0^{\circ}\mathrm{F}$. After 5 minutes, the temperature of the coffee is $70^{\circ}\mathrm{F}$. Find an equation using Newton's Law of Cooling to represent to temperature at time t What will the temperature be in 10 minutes?

CHAPTER OVERVIEW

4: Functions

A function is generally defined as a relation in which each x value corresponds to one and only one y value. This assigning of only one y value to each x is known as "univalence." The univalence requirement makes some manipulations in calculus easier to work with and can be important in certain applications, but many important relations (such as the hyperbola, circle and ellipse) are not univalent and thus not functions.

A function typically describes a relationship between two sets. In our consideration of functions these two sets will typically be real or complex numbers. Functions are usually defined in one of several ways. They can be defined by a graph, (2) an algebraic relationship,
(3) a rule or (4) a table of values. More than one of these methods can be used to describe the same function.

4.1: Function Notation

The notation for a function is generally the $f(x)$ notation. In learning about graphing in algebra we typically use the x and y notation, that is: $y=6x-1$ In function notation, the dependent variable y is replaced by the notation $f(x)$:

$$f(x)=6x-1 \tag{4.1.1}$$

Function values for particular values of the independent variable x can be found
by substituting the appropriate x value into the formula.

$$\begin{gathered} f(9)=6(9)-1=53 \\ f(9)=53 \end{gathered} \tag{4.1.2}$$

Find each of the following values for the given functions:

$f(0) \quad f(-1) \quad f(3) \quad f\left(\frac{1}{2}\right) \quad f(x+2) \quad f(x+h)$

1) $f(x)=2x^2-3x+1$
2) $f(x)=5x^2+x-7$
3) $f(x)=\frac{x}{x^2-1}$
4) $f(x)=\frac{x+3}{x^2+1}$

4.2: Domain and Range of a Function

The analysis of the behavior of functions addresses questions of when the function is increasing or decreasing, whether and where it has maximum or minimum values, where it crosses the x or y axis, and which values of x and y are to be included in the function.

The set of values available for the x, or independent variable is called the Domain of the function. The set of corresponding y values is called the Range of the function.

The linear function mentioned above $f(x)=6x-1$ has a domain of all real numbers and a range of all real numbers, $x \in \mathbb{R}$ and $y \in \mathbb{R}$. On the other hand, the function $f(x)=x^2$ has a domain of all real numbers, $x \in \mathbb{R}$, but its range is limited to the positive real numbers, $y \geq 0$

Considerations of the domain of a function typically refer to restrictions on which x values will generate real number values for y. The most common restrictions occur with the use of square root functions or rational functions.

The domain of the function $f(x)=\sqrt{x-7}$ would be the set of $x \geq 7$, so that no negative values are permitted under the square root. This secures the necessary real values for y. The range for this function is $y \geq 0$

The domain of the function $f(x)=\frac{x}{2x-3}$ would be $x \in \mathbb{R}$ (all real numbers), but $x \neq \frac{3}{2}$, thus avoiding a zero denominator which is an undefined value. The range for this function would be $y \in \mathbb{R}$ (all real numbers), but $y \neq \frac{1}{2}$, due to the horizontal asymptote at $y=\frac{1}{2}$

The questions of domain and range become more interesting when considered in relation to functions defined by graphs, or in applications. In an application involving perimeter in which the perimeter of a rectangle is given as 50 feet, we know that

$$2\ell+2w=50 \tag{4.2.1}$$

Rewriting this as a function of w, we can say that

$$\ell=f(w)=25-w \tag{4.2.2}$$

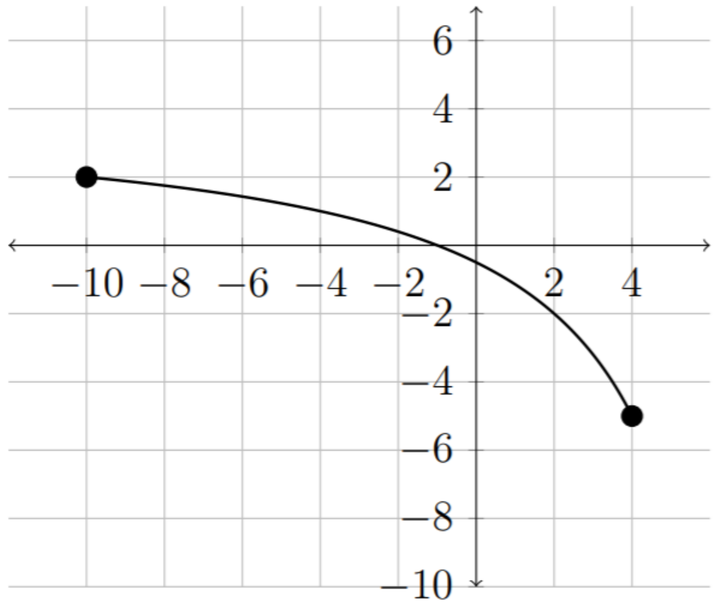
In this function relating the length and width based on a given perimeter, we can say the domain of the function is $0<w<25$. The width must be greater than 0 but less than 25, otherwise there would not be a rectangle. The same is true for the range or possible set of values for the length $0<\ell<25$ Exercises 4.2

Find the domain and range for each of the following functions:

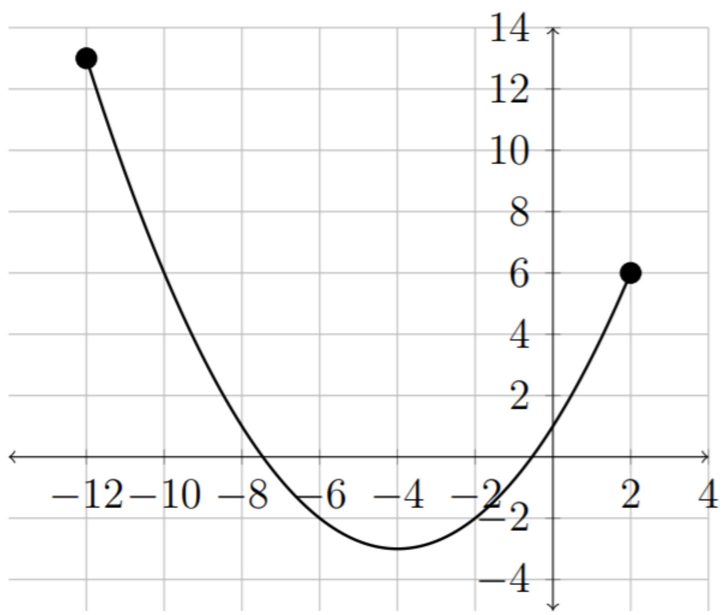
1) $f(x)=\sqrt{2x+1}$
2) $f(x)=\sqrt{3x-5}$
3) $\frac{x}{x+9}$
4) $f(x)=\frac{x+2}{2x-3}$

For the following graphs, assume that the endpoints for the domain and range are whole numbers.

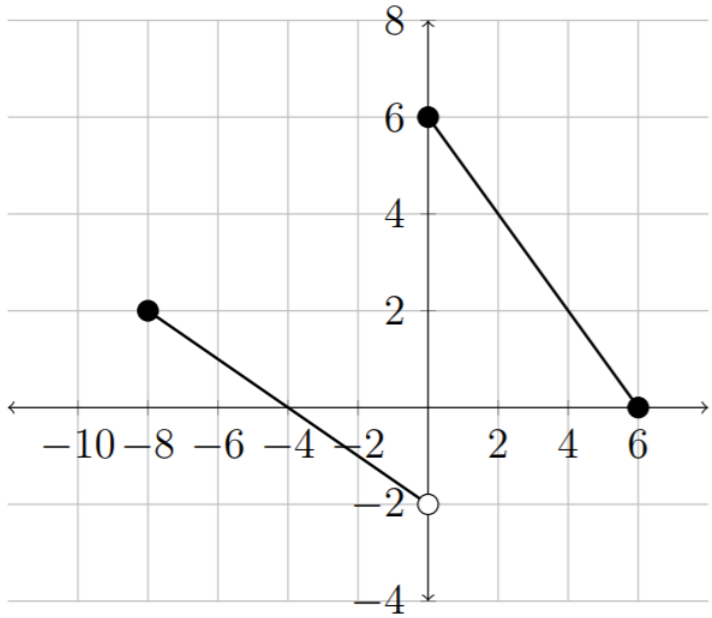
5)



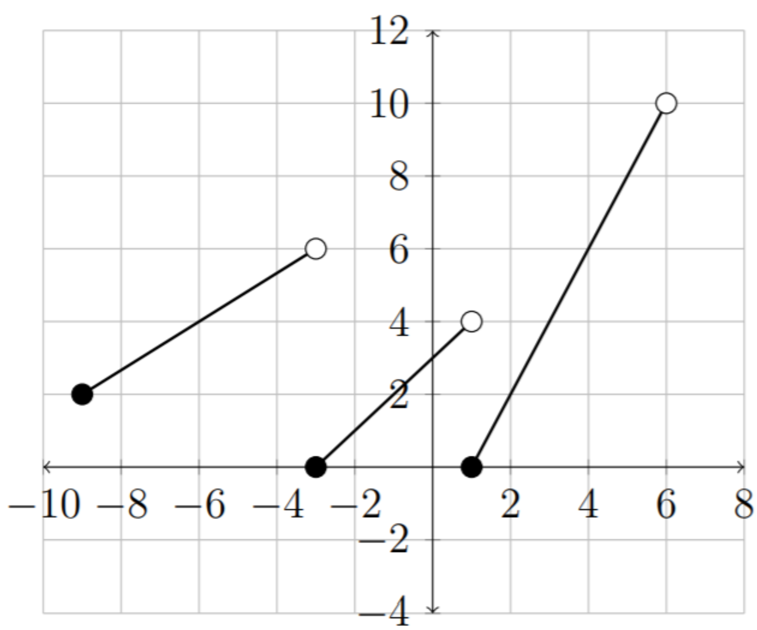
6)



7)



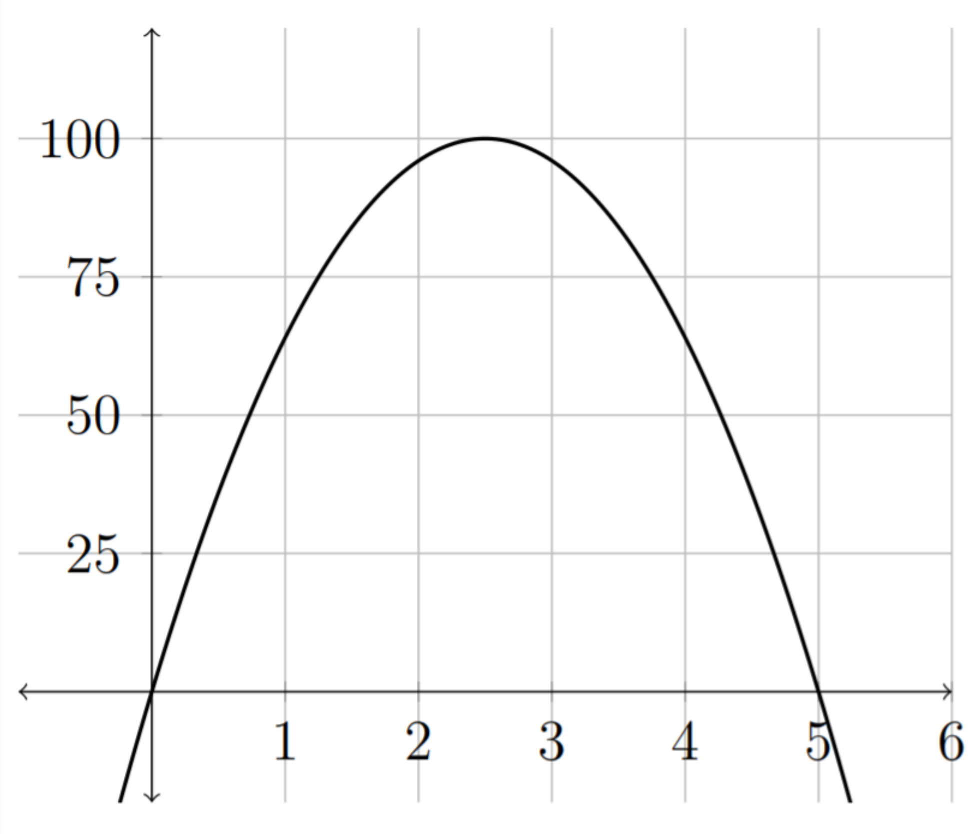
8)



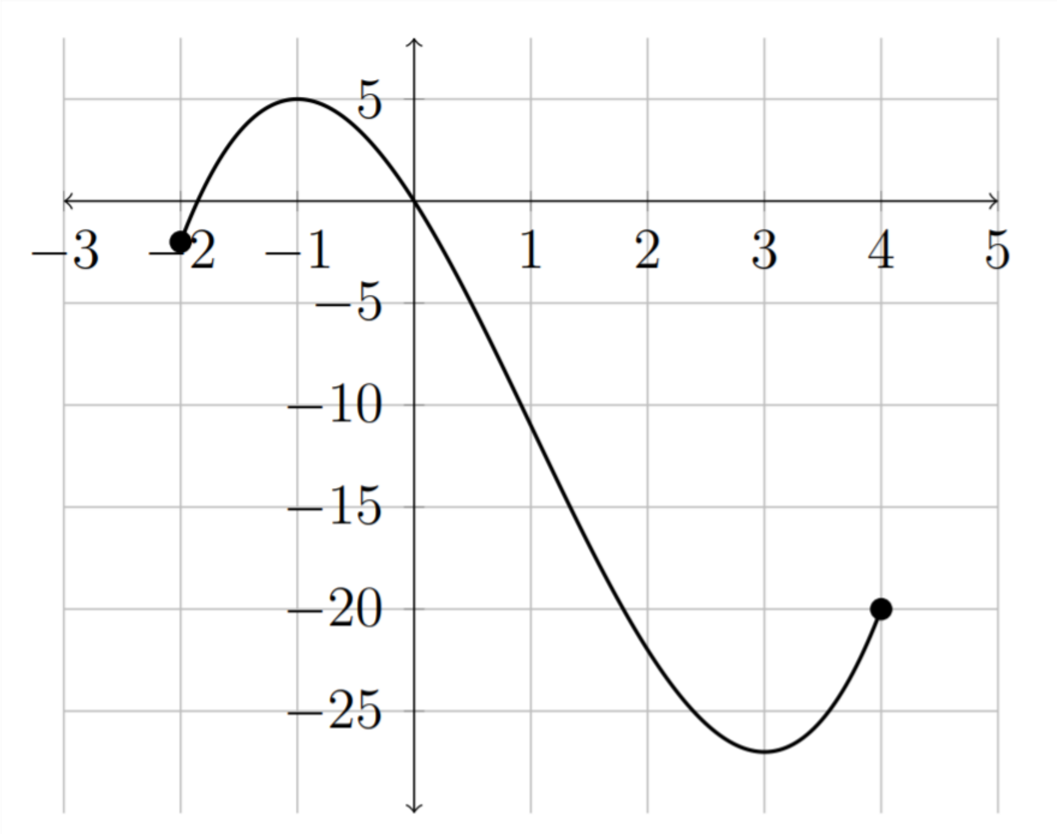
4.3: Maximum and Minimum Values

Finding the maximum and minimum values of a function can be very useful in applications. This is usually referred to as "optimization." Later on, we will examine the use of optimization in application problems. For now, we will use graphs to find the maximum and minimum values of the function.

In the example below, the maximum function value in the region shown is 100 . This occurs where $x = 2.5$

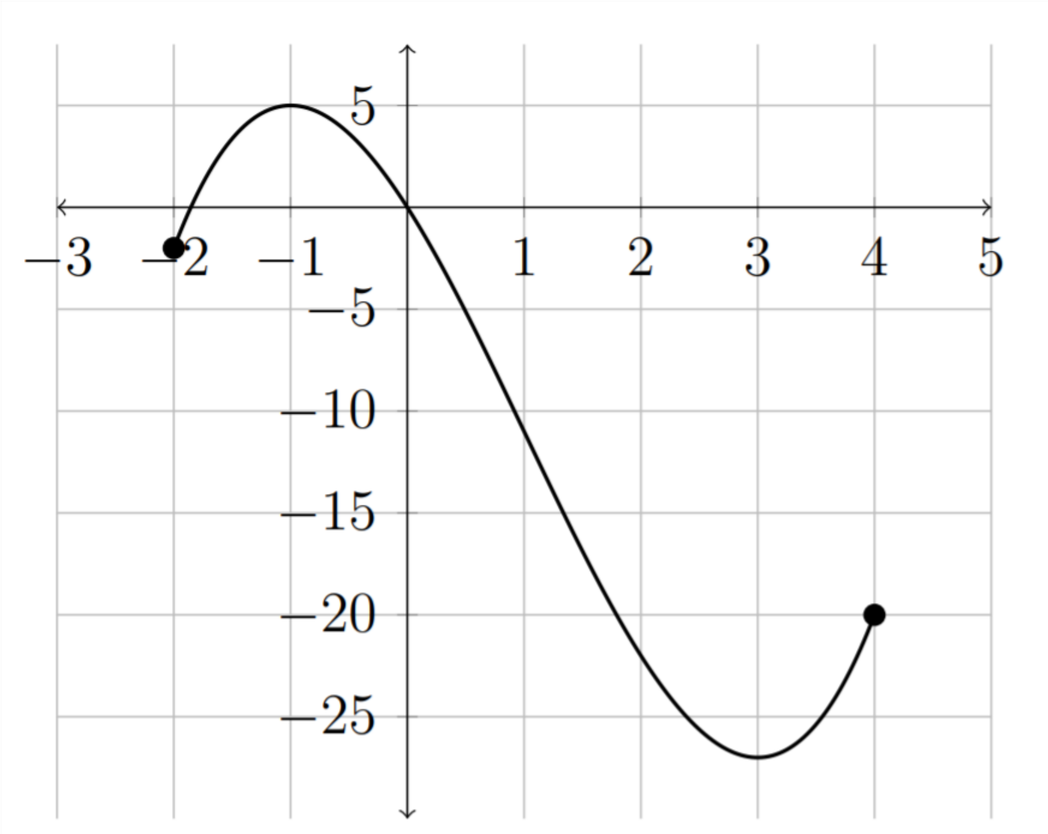


In the graph below, the function shows a maximum value of 5 at $x=-1$ and a minimum value of -27 at $x=3$



We can use the graphing calculator to find maximum and minimum values. In calculus the maximum and minimum values can be found algebraically.

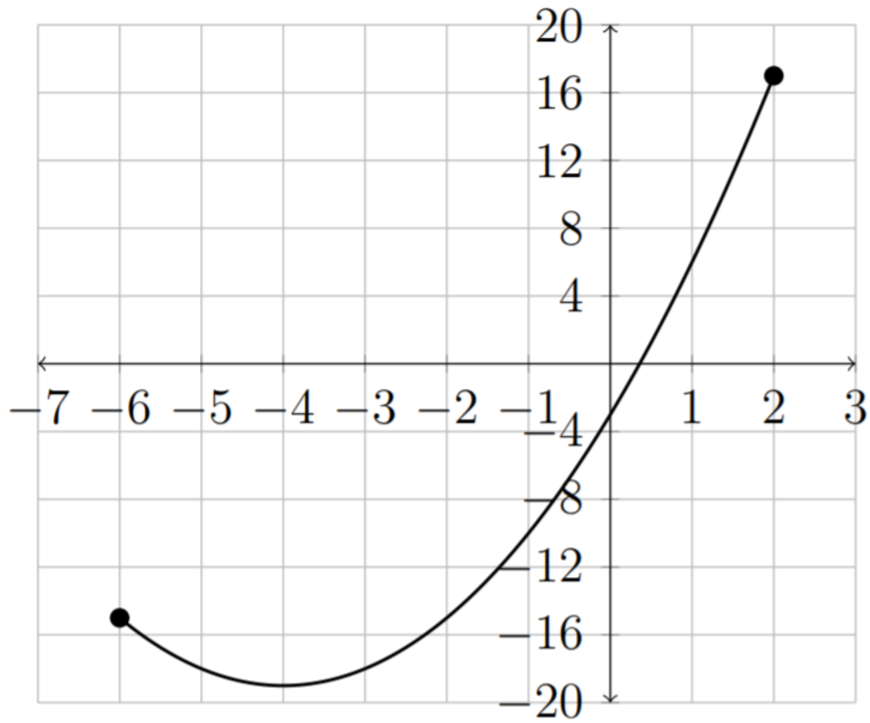
The maximum and minimum values are also helpful in determining where the function is increasing or decreasing. In the previous example:



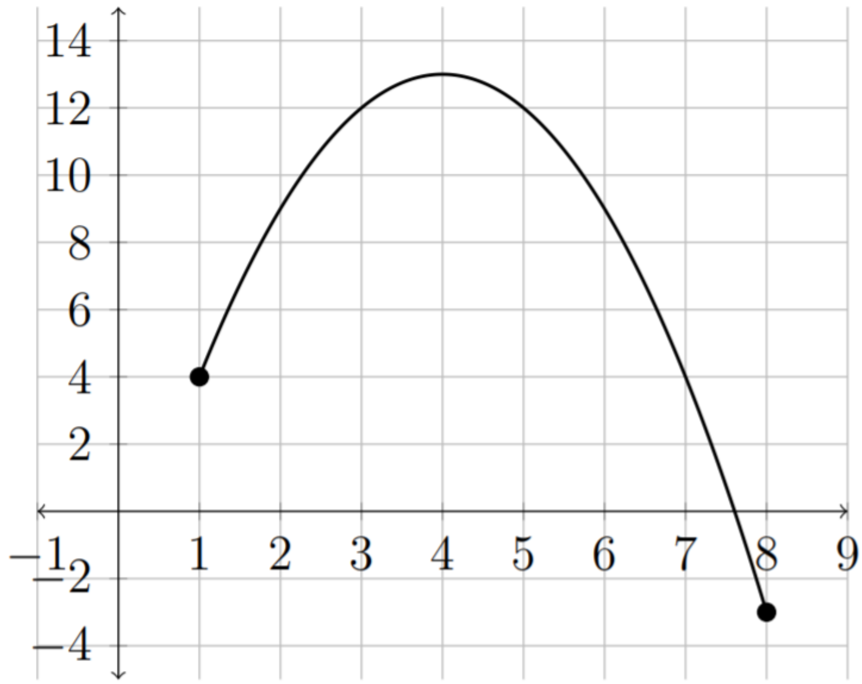
The function is increasing between $x=-2$ and $x=-1$, then again from $x=3$ to $x=4$. So we would say that the intervals in which the function is increasing are $-2 \leq x < -1$ and $3 < x \leq 4$ or, using interval notation, [-2,-1)∪$(3,4]$
The interval in which the function is decreasing is $-1 < x < 3$ or (-1,3).

Determine the domain and range for each function given. Then determine the maximum and minimum values for the function and the intervals in which the function is increasing or decreasing.

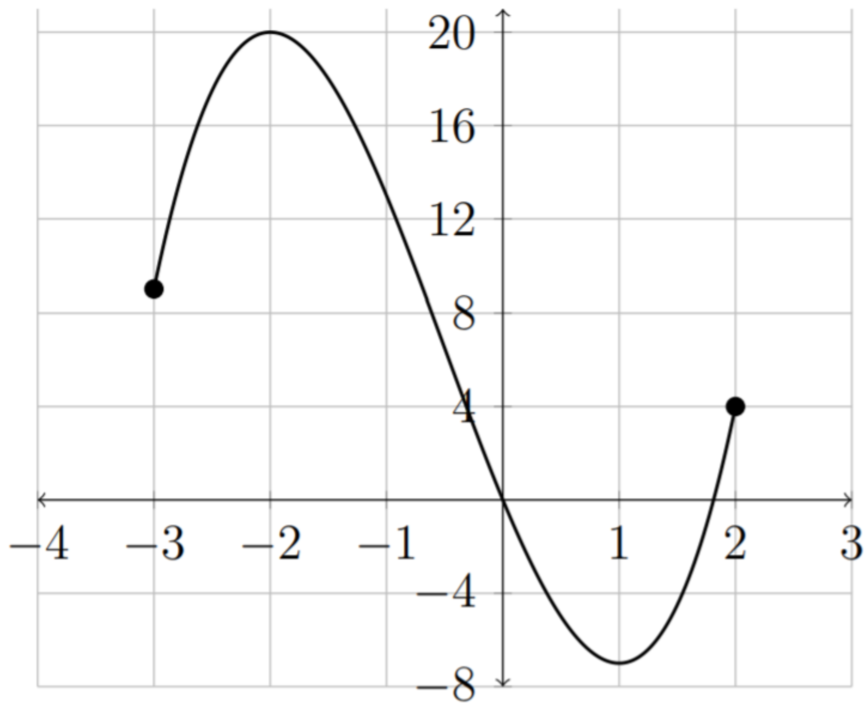
1)



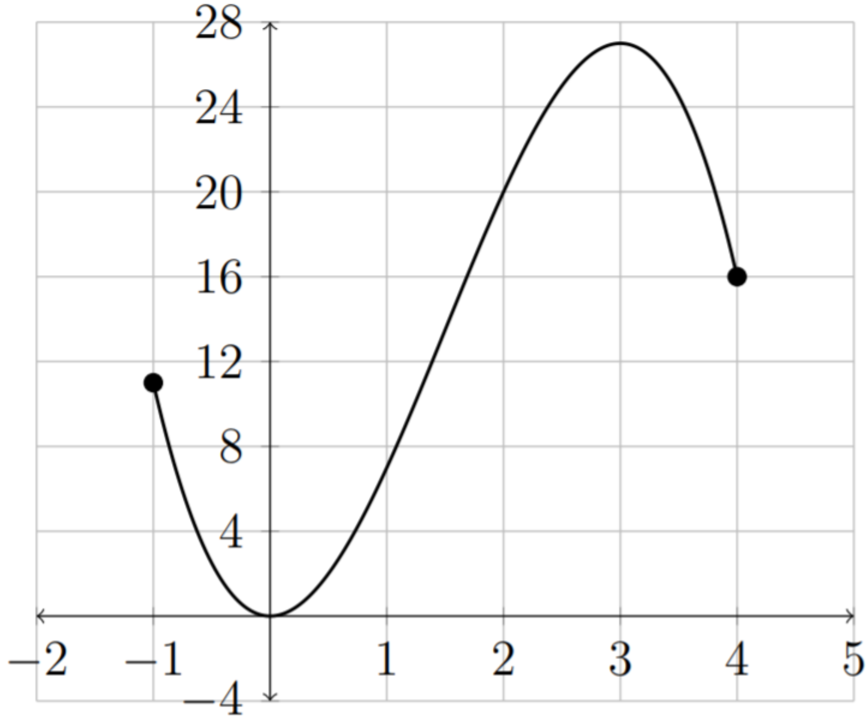
2)



3)



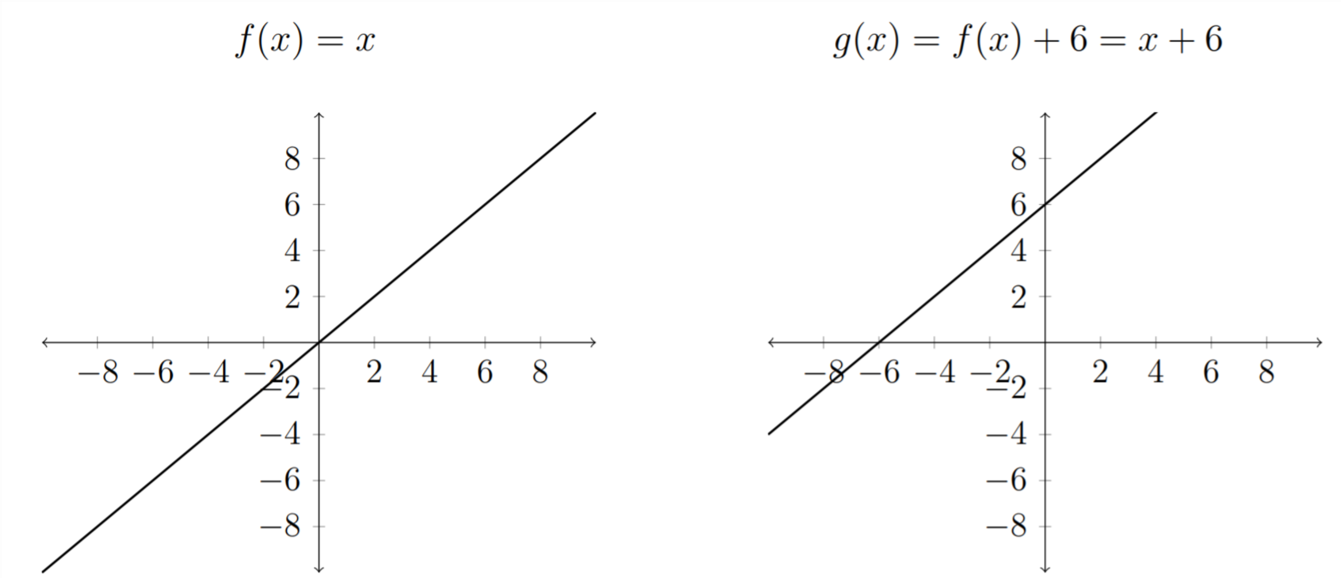
4)



4.4: Transformations

There are three major types of transformation that we will consider:
1) Horizontal and vertical shifts
2) Reflections over the x or y -axis
3) Horizontal and vertical stretches

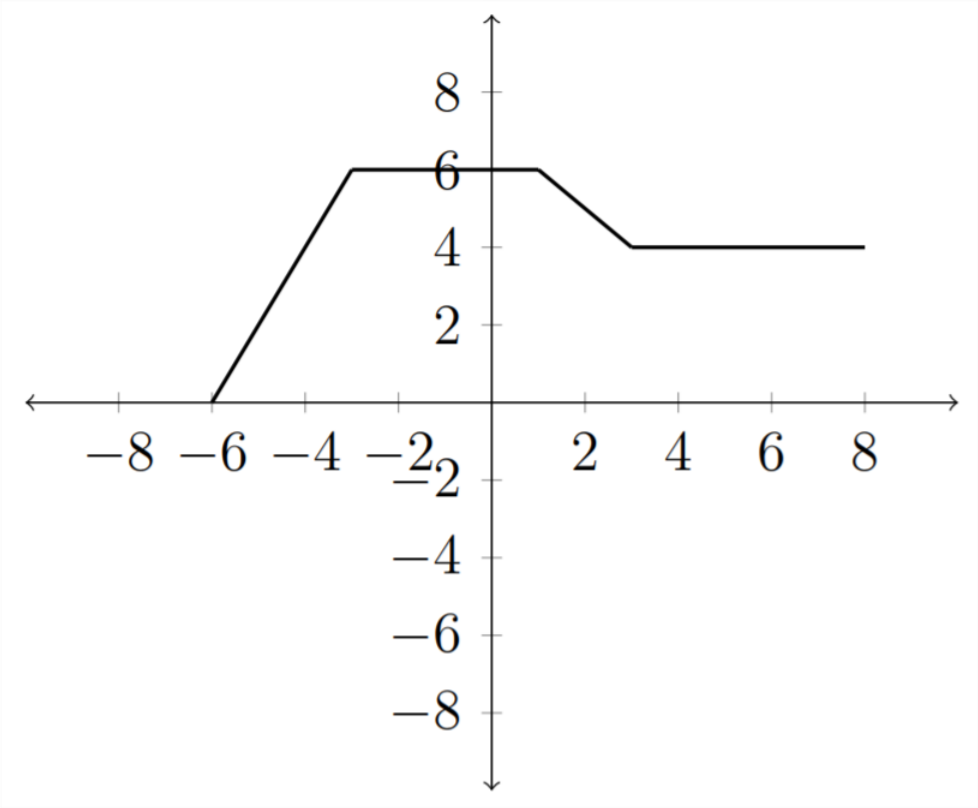
If we take a given function, let's say $f(x)=x$, then this has the graph we see below - a straight line with a slope of 1 and a y -intercept of 0. If we add to the function $f(x)+6=x+6$, then this will add 6 to all of the y -values which shifts the graph 6 places up.



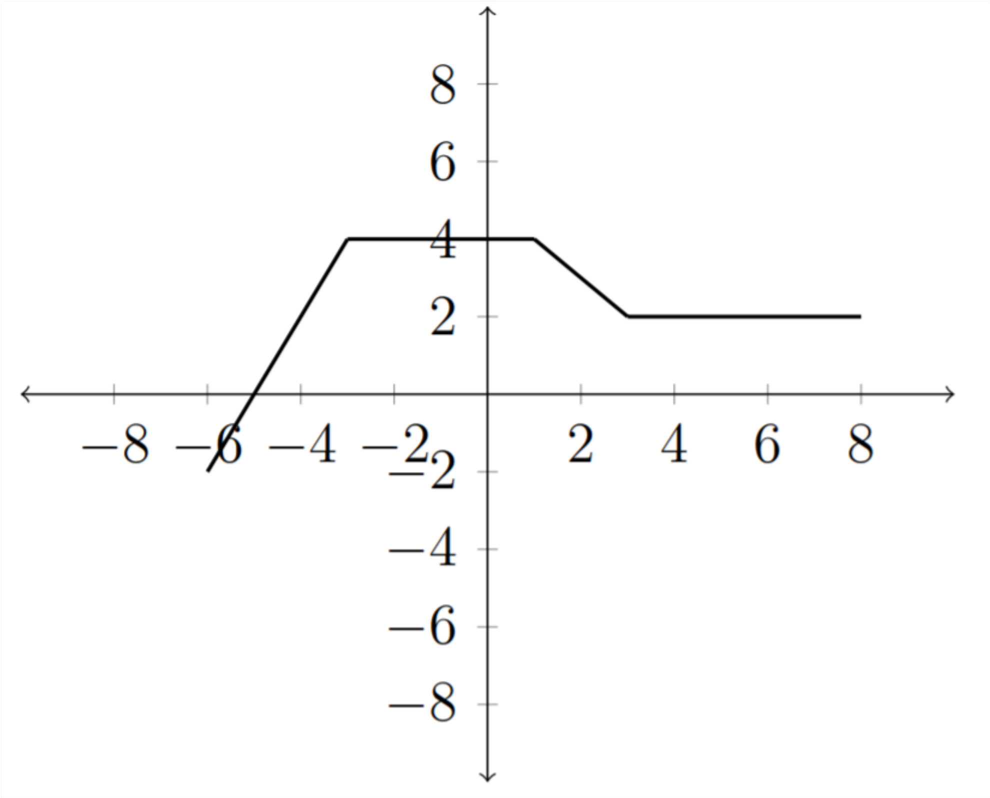
Vertical Shifts

So, if we have a general function $f(x)$ that is described by a graph, we can determine a graph for $f(x)+k$, where k is some number that will either shift $f(x)$ up (if $k>0$) or down (if $k<0$)

For example, consider the following graph for $f(x)$



Then, if we want a graph for $f(x)-2$, just shift all the y -values down 2 places:

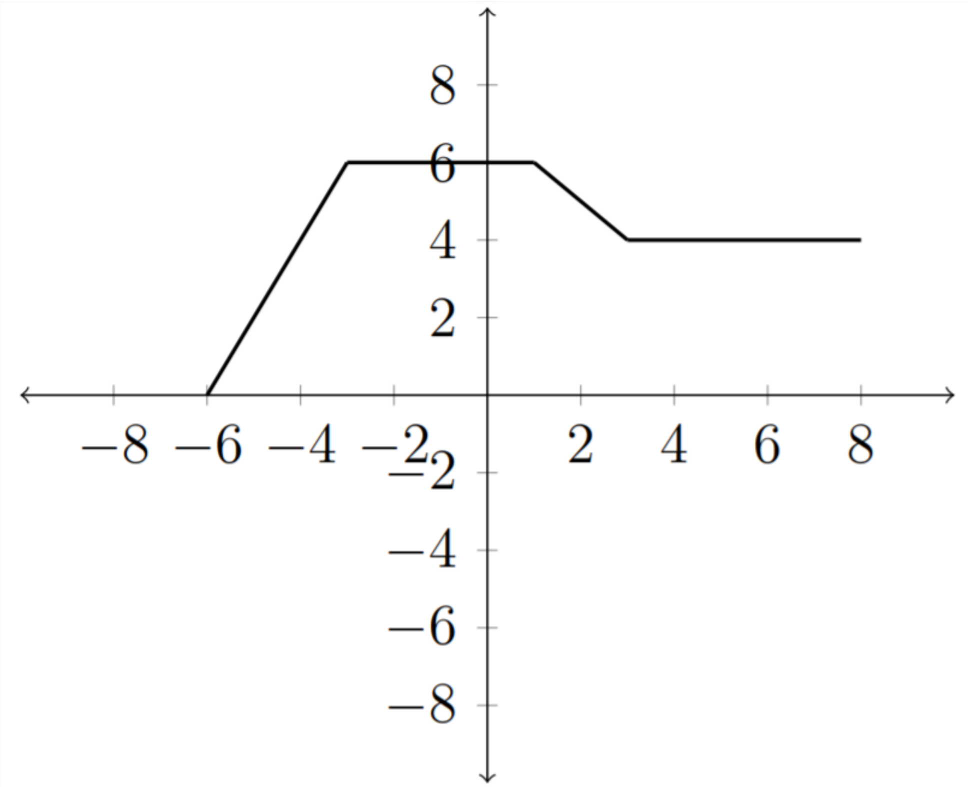


This is a standard vertical shift transformation of a function.

Horizontal Shifts

A function may also be shifted horizontally by adding or subtracting a number inside the parentheses.

If we start with our original function $f(x)$



Then the transformation $f(x-2)$ will shift the graph horizontally, except that it will move in the opposite direction of the sign. The $f(x-2)$ shift will move the graph 2 places to the right whereas a shift of $f(x+2)$ will move the graph 2 places to the left.

The reason why this happens will be more clear if we look at a table of values for the function:

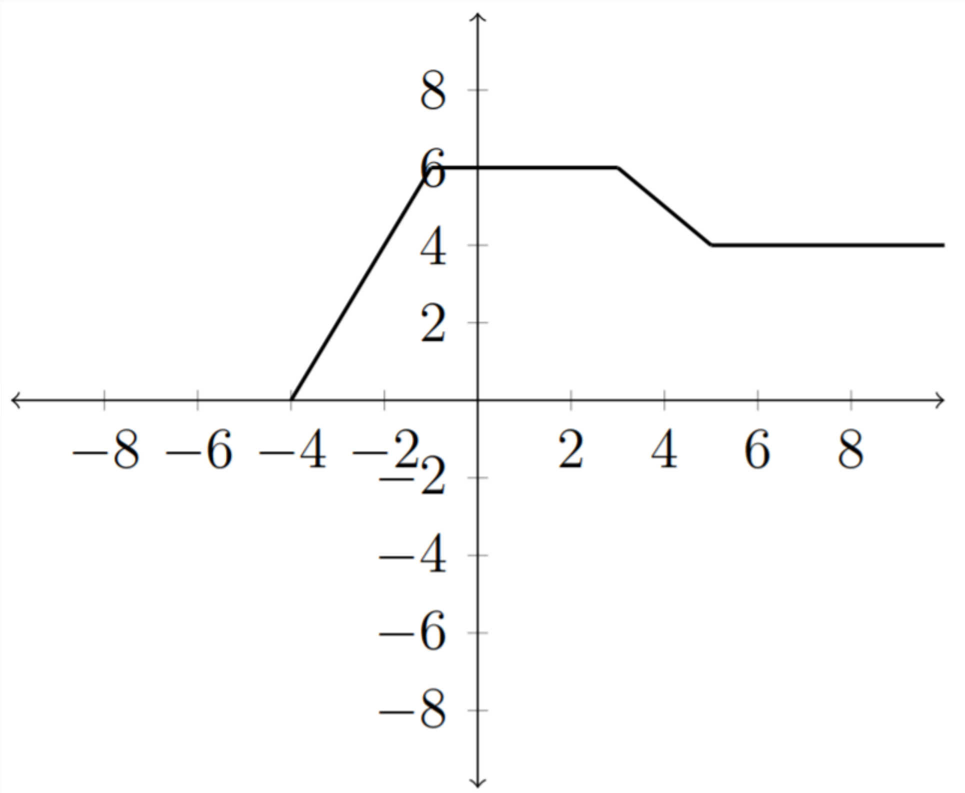
x	$f(x)$
-6	0
-3	6
1	6
3	4
8	4

Now, if, instead of $f(x)$, we want $f(x-2)$, then we add another column to the table:

x	$x-2$	$f(x-2)$
-4	-6	0
-1	-3	6
3	1	6
1	3	4
6	8	4

Notice that to have the same y -value as the original graph, we must go 2 places to the right so that after we subtract 2 from the x -

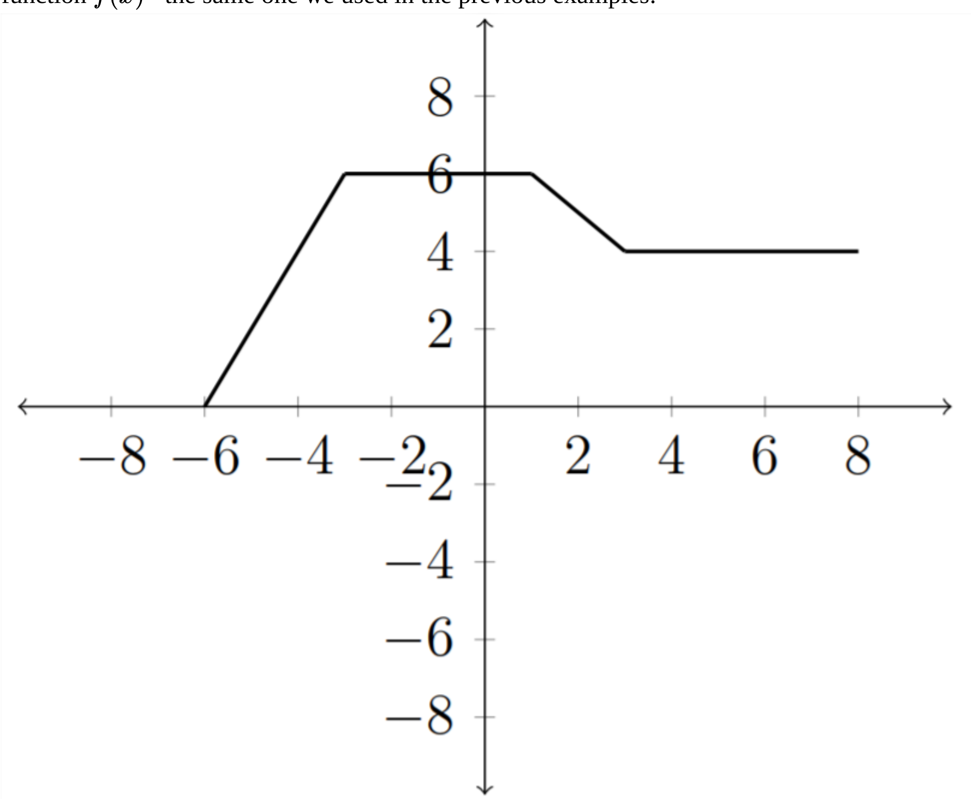
value, we arrive back at the original x -value. So the graph of $f(x-2)$ will look like this:



Notice how the values in the graph match up with the values in the table.

Reflections

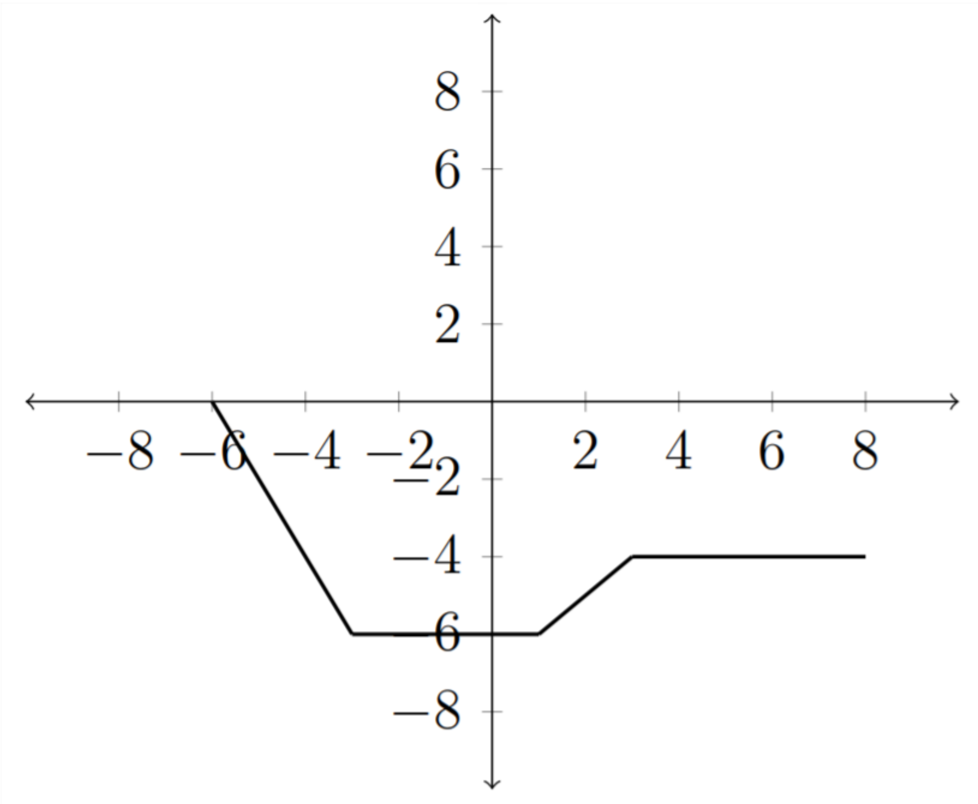
Negating the x or y values of a function will have the effect of reflecting the function over the y or x axis. If we consider the function $f(x)$ - the same one we used in the previous examples:



Then the graph of $y = -f(x)$ will be reflected over the x -axis. The negative sign in front of the function negates all of the y values, reflecting them over the x -axis. Lets look at this in a table of values:

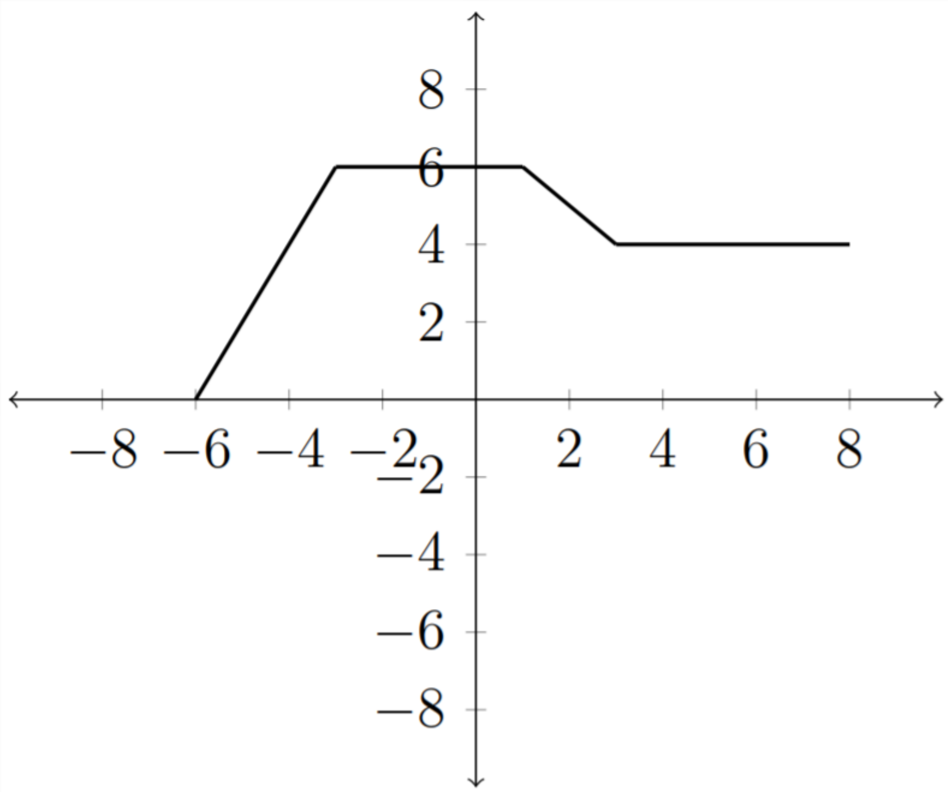
x	$f(x)$	$-f(x)$
-6	0	0
-3	6	-6
1	6	-6
3	4	-4
8	4	-4

So the graph would look like this:



On the other hand, if we negate the x -variable $(f(-x))$, then the function values previously associated with the positive values of

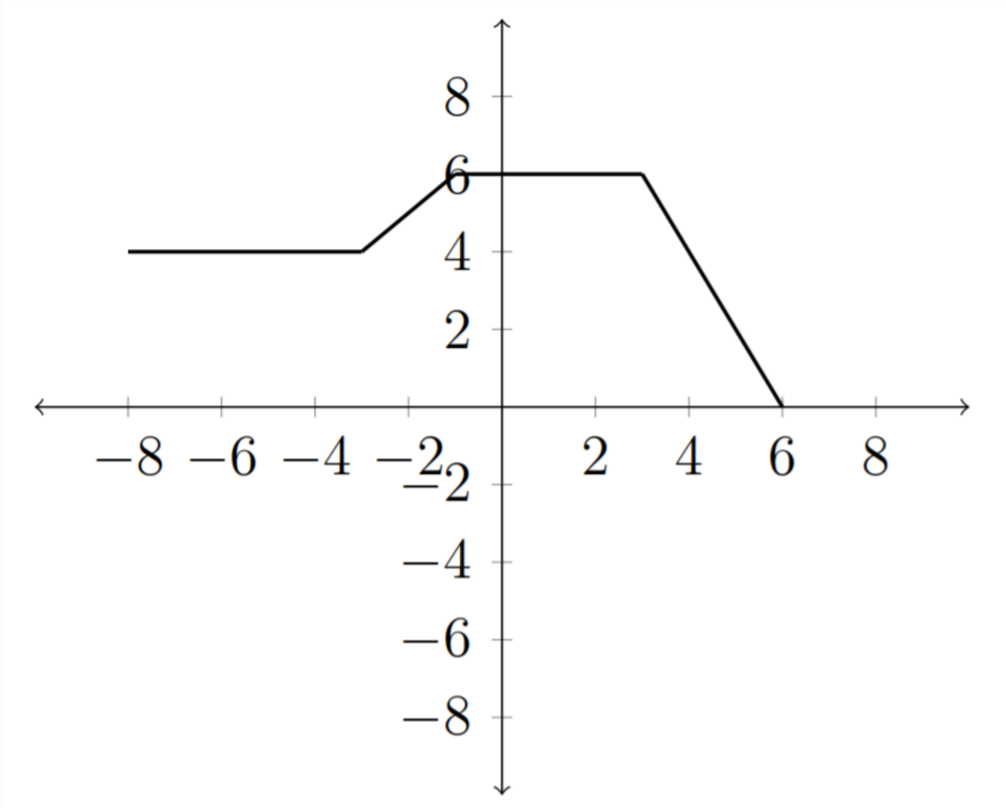
x would be associated with the negative values of x and vice versa. This would reflect the function over the y -axis.
If we again consider our original function and a table of values:



In the table:

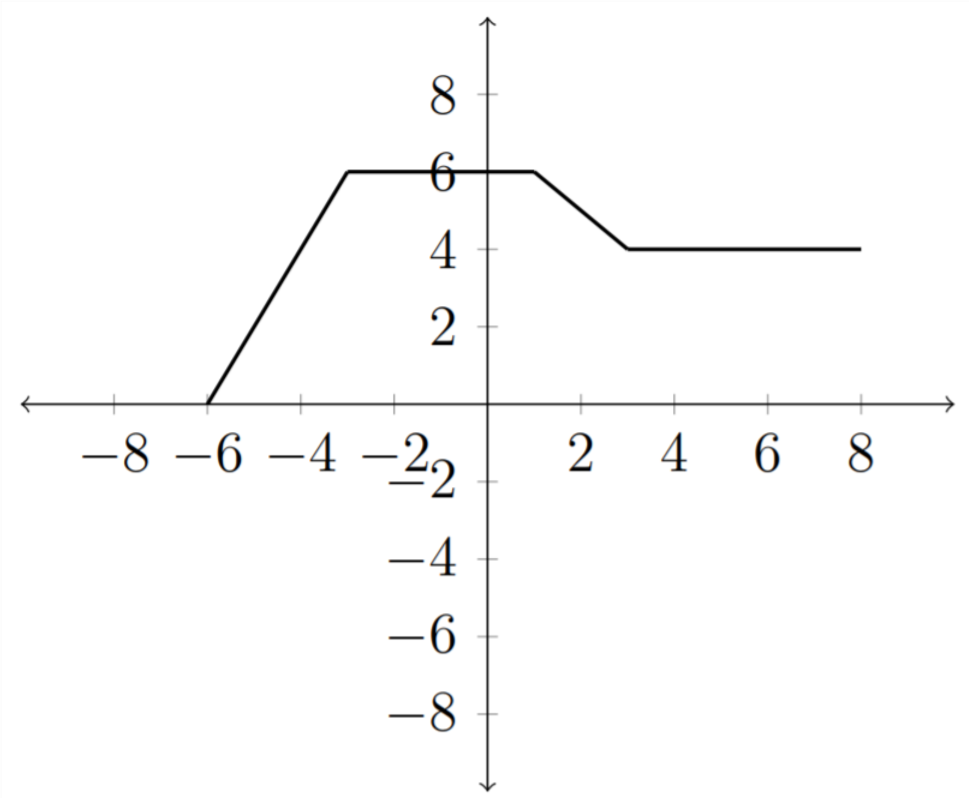
x	$-x$	$f(-x)$
-8	8	4
-3	3	4
-1	1	6
3	-3	6
6	-6	0

And the graph would be reflected over the y -axis:



Stretching and compressing graphs

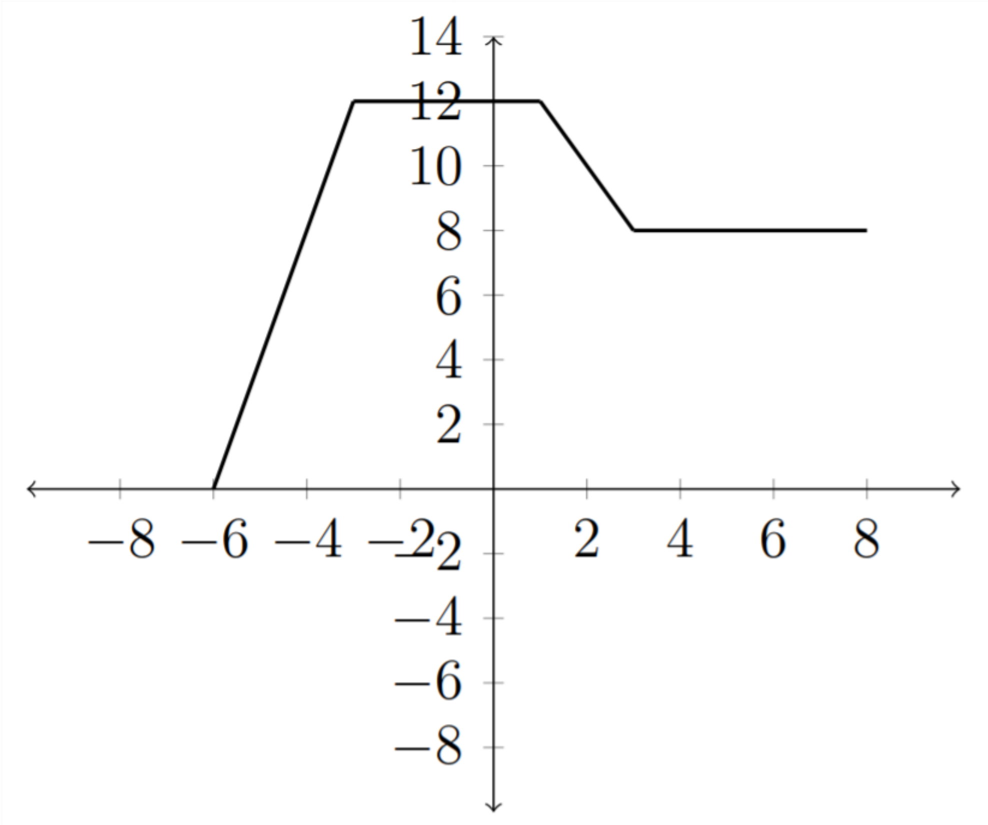
The last type of transformation we will examine is that of stretching or compressing a graph by multiplying inside or outside the parentheses. Starting with our familiar example function $y = f(x)$



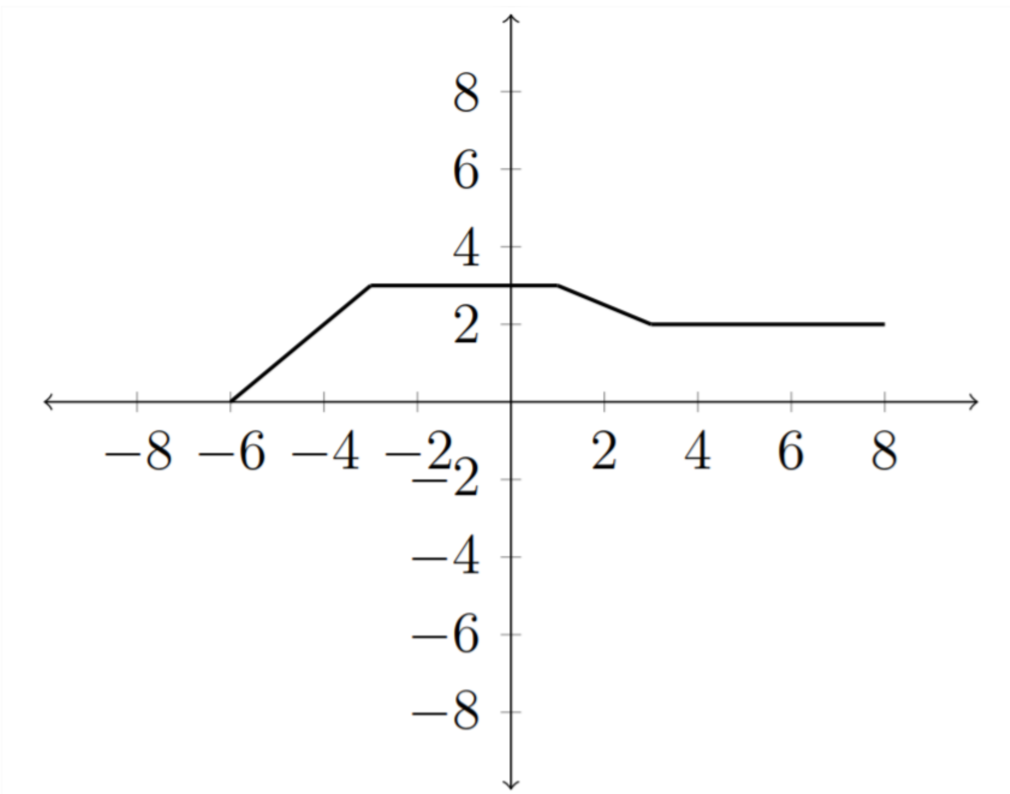
If we multiply the function by a constant outside of the parentheses: $y=2f(x)$ then this will have the effect of multiplying all of the y values by 2. In the table:

x	$f(x)$	$2f(x)$
-6	0	0
-3	6	12
1	6	12
3	4	8
8	4	8

The graph of $y=2f(x)$ would look like this:



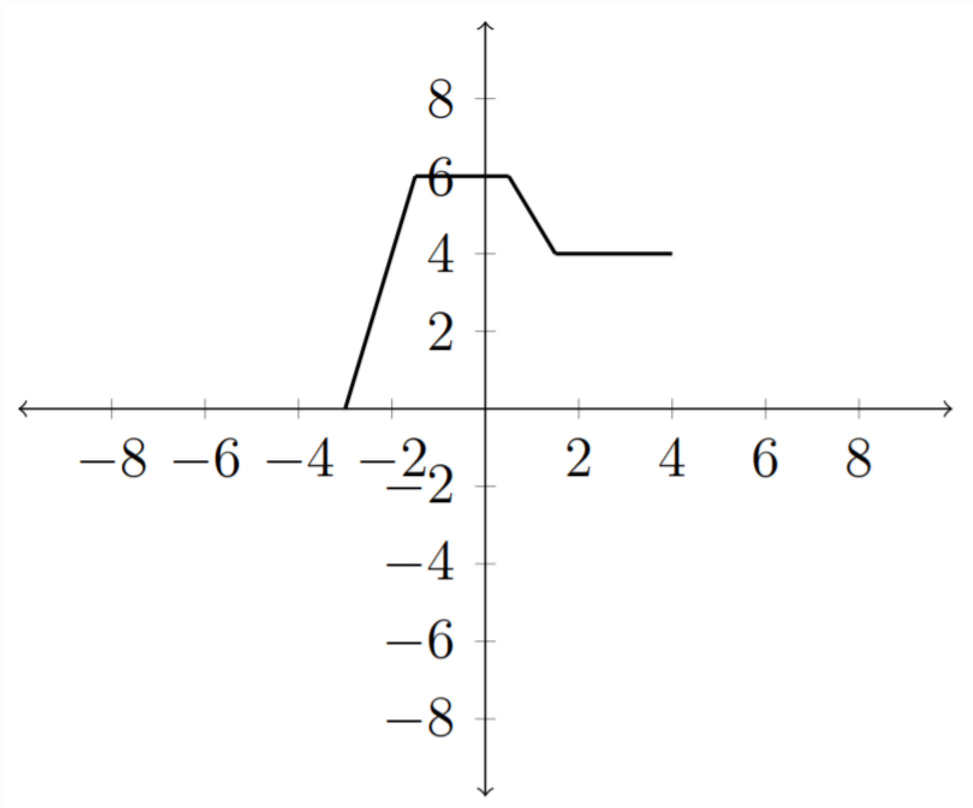
In a similar way, multiplying by a number less than 1 would compress the graph. The graph for $y = \frac{1}{2}f(x)$ is below:



Multiplying inside the parentheses affects the x variables.
If we consider the function $y = f(2x)$, then this will have the effect of compressing the graph along the x -axis:

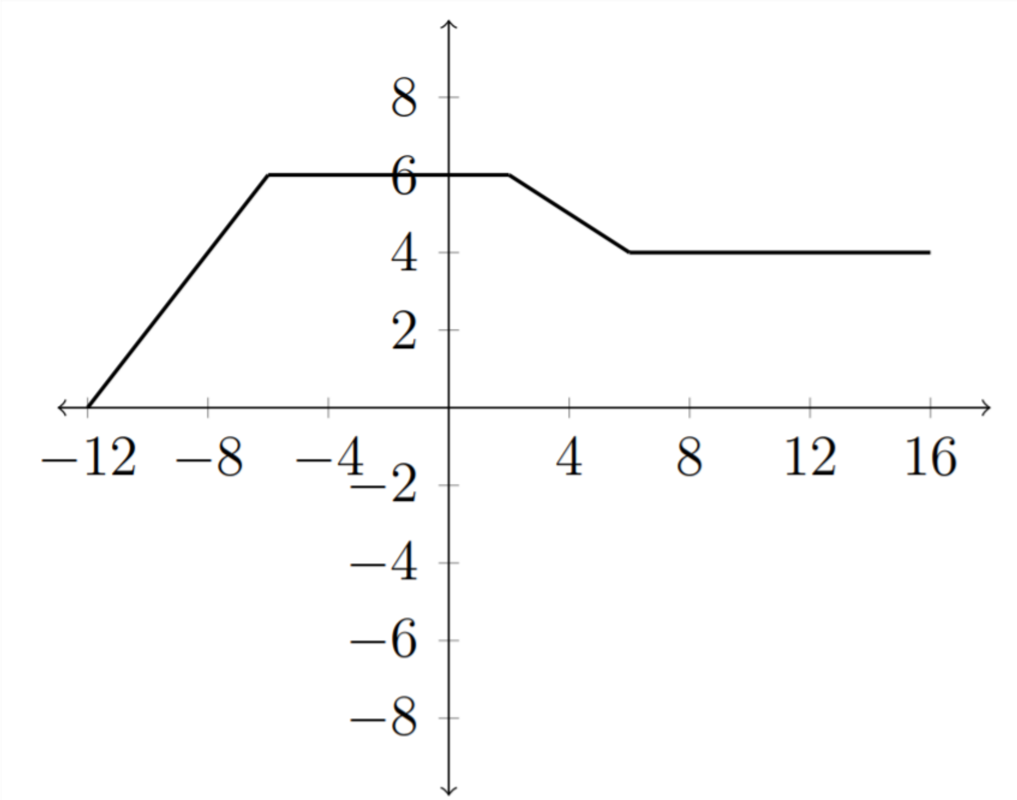
x	$2x$	$f(2x)$
-3	-6	0
-1.5	-3	6
0.5	1	6
1.5	3	4
4	8	4

Notice how each x value had to be cut in half so that when we multiplied it by 2 we ended up with the original x value. The graph would look like this:



Multiplying inside the parentheses by a number less than one would stretch the graph out.

x	$\frac{1}{2}x$	$f\left(\frac{1}{2}x\right)$
-12	-6	0
-6	-3	6
2	1	6
6	3	4
16	8	4



In these examples, we have considered only one transformation at a time. In the exercises you will need to consider the effect of several transformations at once.

Exercises 4.4

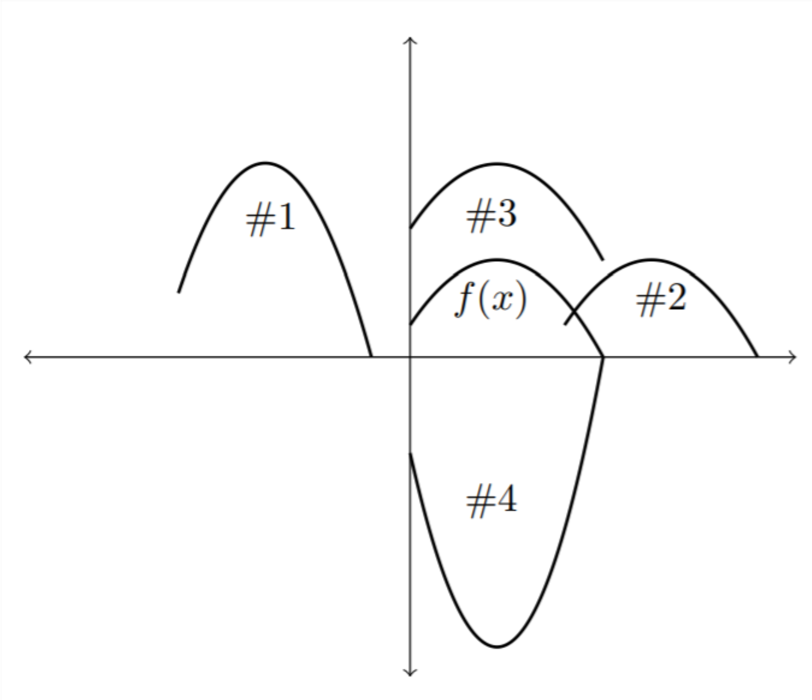
1) Match each of the functions on the graph to the appropriate transformation described below.

a) $f(x-4)$

b) $f(x)+3$

c) $-3f(x)$

d) $2f(x+6)$



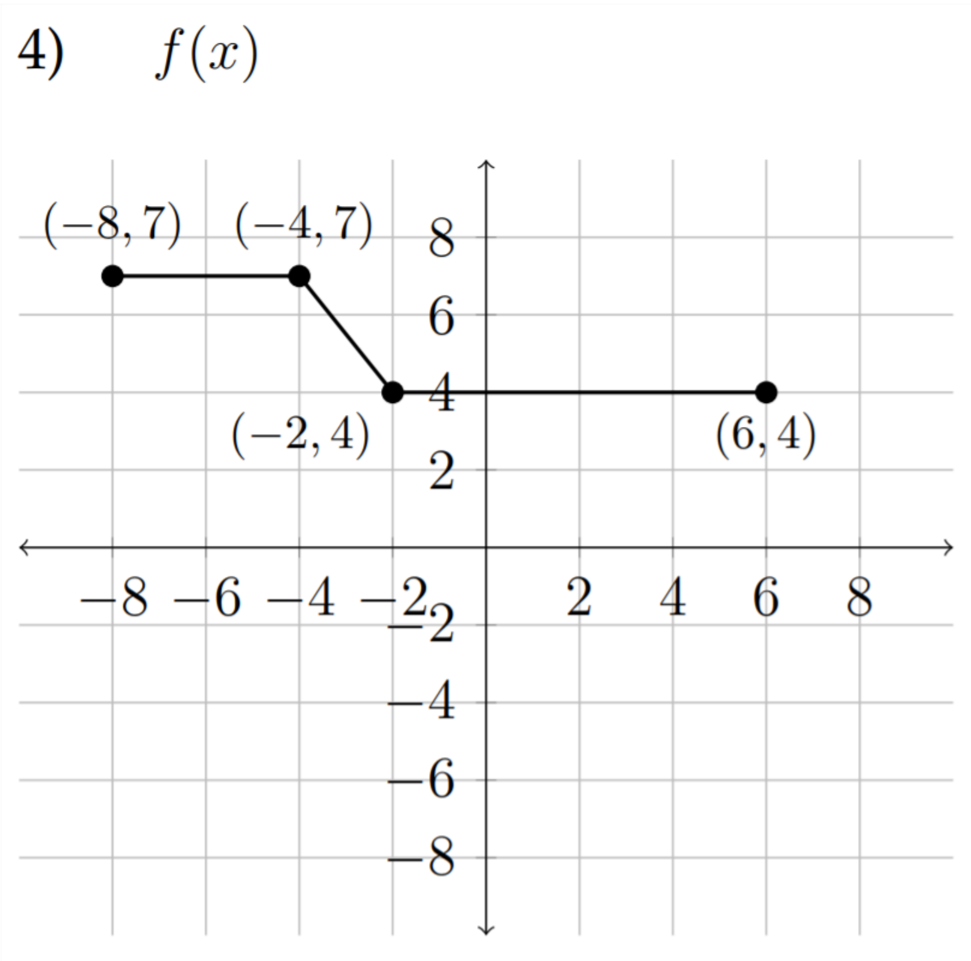
2) Match each of the descriptions with the appropriate function transformation.

$$\begin{array}{ll} a) \quad y=f(x-1) & \text{1)Shift left one unit} \\ b) \quad y=f(x)-1 & \text{2)Reflect over } x \text{ -axis, then shift left one unit} \\ c) \quad y=f(x)+1 & \text{3)Shift right one unit} \\ d) \quad y=f(x+1) & \text{4)Reflect over } x \text{ -axis, then shift up one unit} \\ e) \quad y=f(-x)+1 & \text{5)Reflect over } x \text{ -axis, then shift down one unit} \\ f) \quad y=f(-x)-1 & \text{6)Shift down one unit} \\ g) \quad y=-f(x)+1 & \text{7)Reflect over } x \text{ -axis, reflect over } y \text{ -axis, then shift up one unit} \\ h) \quad y=-f(x+1) & \text{8Shift left one unit, reflect over } y \text{ -axis, then shift up one unit} \\ i) \quad y=-f(x)-1 & \text{9)Shift up one unit} \\ j) \quad y=f(-x+1) & \text{10)Reflect over } y \text{ -axis, then shift up one unit} \\ k) \quad y=-f(-x) & \text{11)Reflect over } y \text{ -axis, then shift down one unit} \end{array} \tag{4.4.1}$$

3) Match each of the descriptions with the appropriate function transformation.

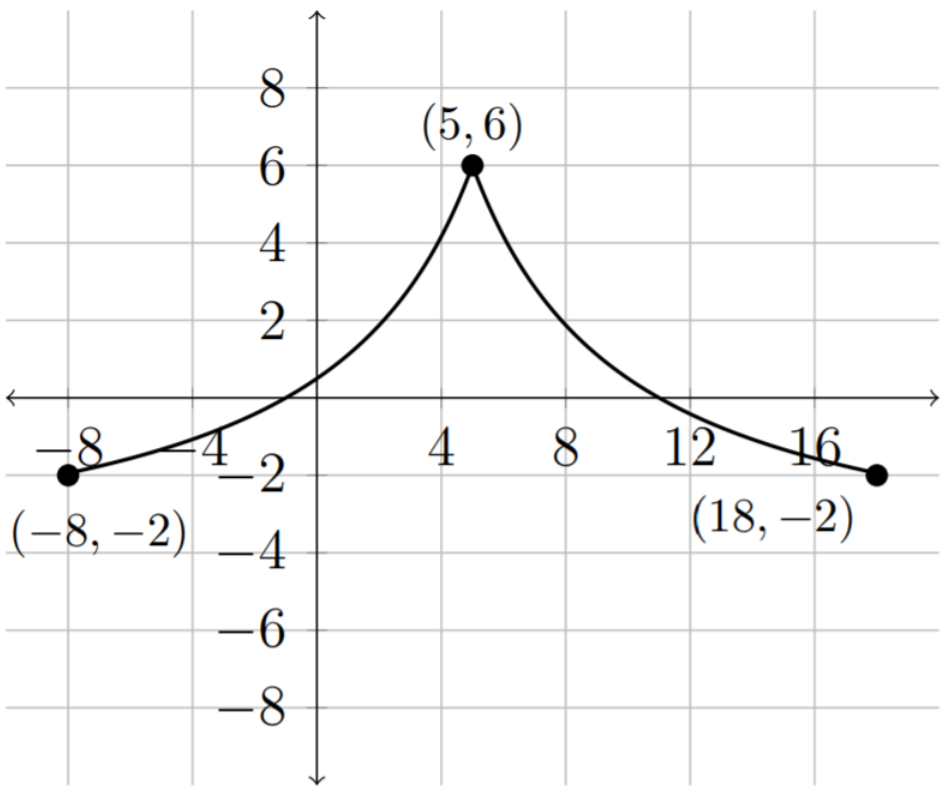
$$\begin{array}{ll} a) \quad y=f(x+2)+3 & 1)\text{Shift left 2 units, then shift down 3 units} \\ b) \quad y=f(x+3)+2 & 2)\text{Shift left 3 units, then shift up 2 units} \\ c) \quad y=f(x-2)+3 & 3)\text{Shift right 3 units, then shift up 2 units} \\ d) \quad y=f(x-2)-3 & 4)\text{Shift left 3 units, then shift down 2 units} \\ e) \quad y=f(x+2)-3 & 5)\text{Shift right 3 units, then shift down 2 units} \\ f) \quad y=f(x-3)+2 & 6)\text{Reflect over the } y\text{-axis, then shift up 2 units} \\ g) \quad y=f(x-3)-2 & 7)\text{Reflect over } x\text{-axis, then shift right 2 units} \\ h) \quad y=f(x+3)-2 & 8)\text{Reflect over } x\text{-axis, then shift left 2 units} \\ i) \quad y=-f(x+2) & 9)\text{Shift left 2 units, then reflect over the } y\text{-axis} \\ j) \quad y=-f(x-2) & 10)\text{Shift right 2 units, then shift up 3 units} \\ k) \quad y=f(2-x) & 11)\text{Shift left 2 units, then shift up 3 units} \\ \ell) \quad y=f(-x)+2 & 12)\text{Shift right 2 units, then shift down 3 units} \end{array} \tag{4.4.2}$$

Apply the transformations indicated for each function.

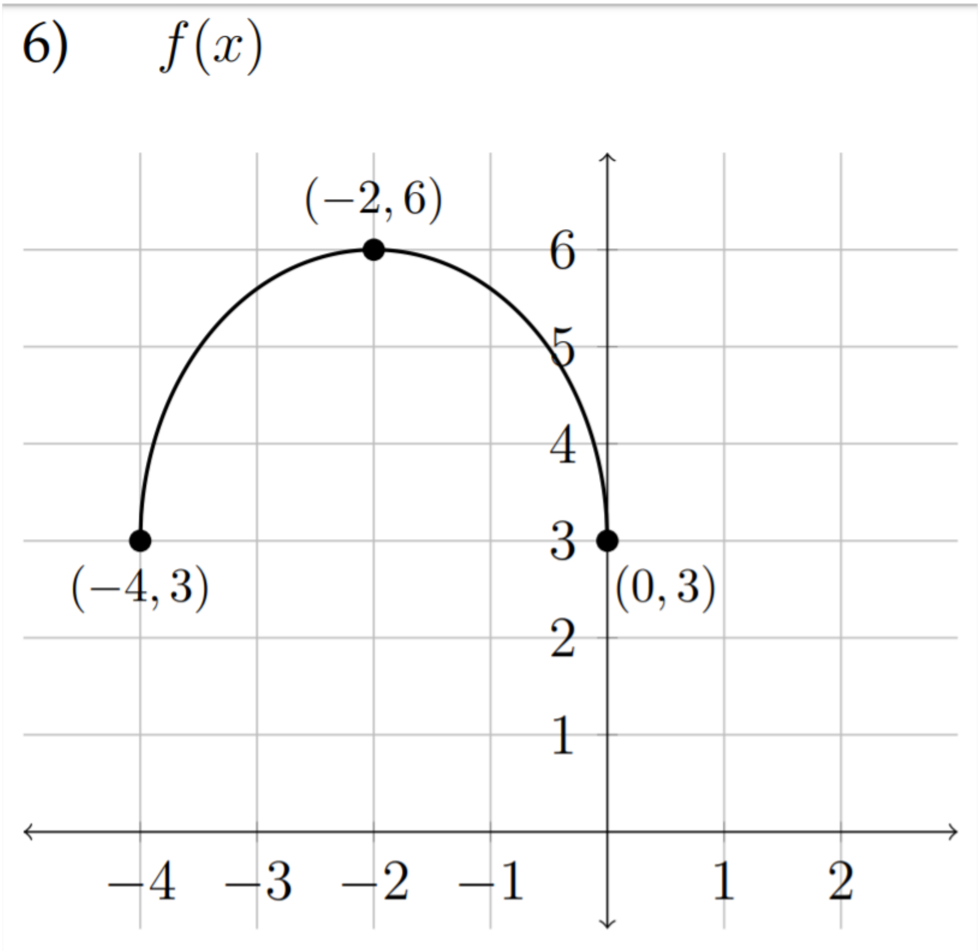


a) $f(x-3)$
b) $-f(x)+2$
c) $\frac{1}{2}f(x-1)$
d) $f(-x)+1$

5) $g(x)$



a) $g(x)-3$
b) $-g(x)-1$
c) $2g(x+2)$
d) $g(2x)-1$



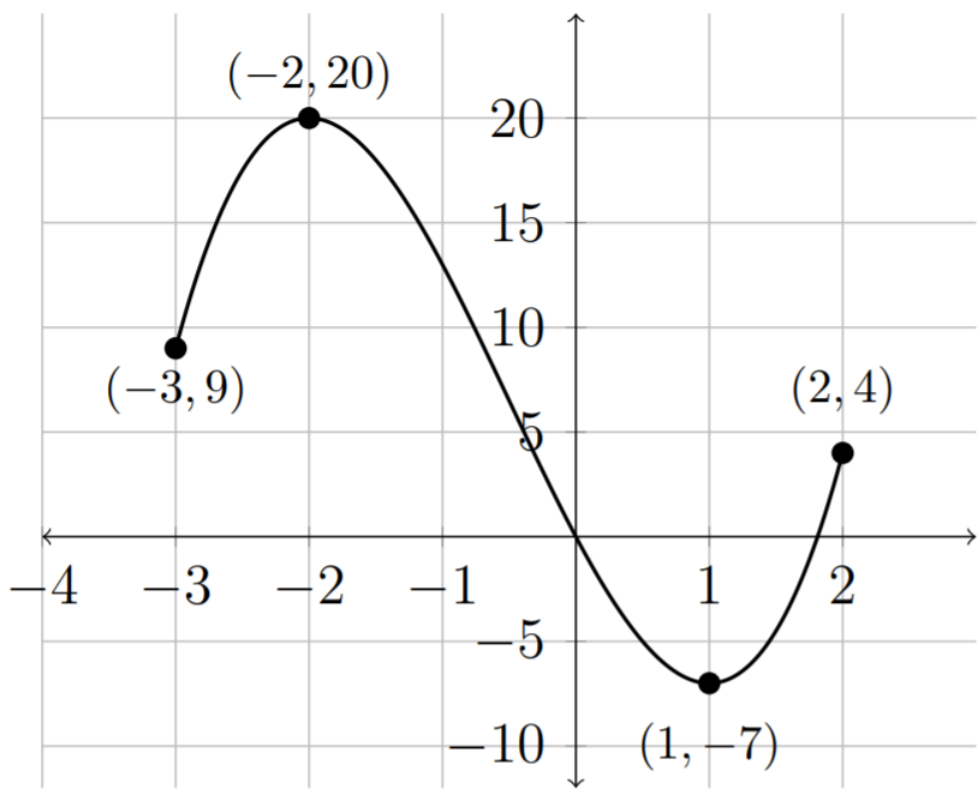
a) $f(x)+2$

b) $-f(x-1)$

c) $f(x-1)-3$

d) $\frac{1}{3}f(x)+4$

7) $g(x)$

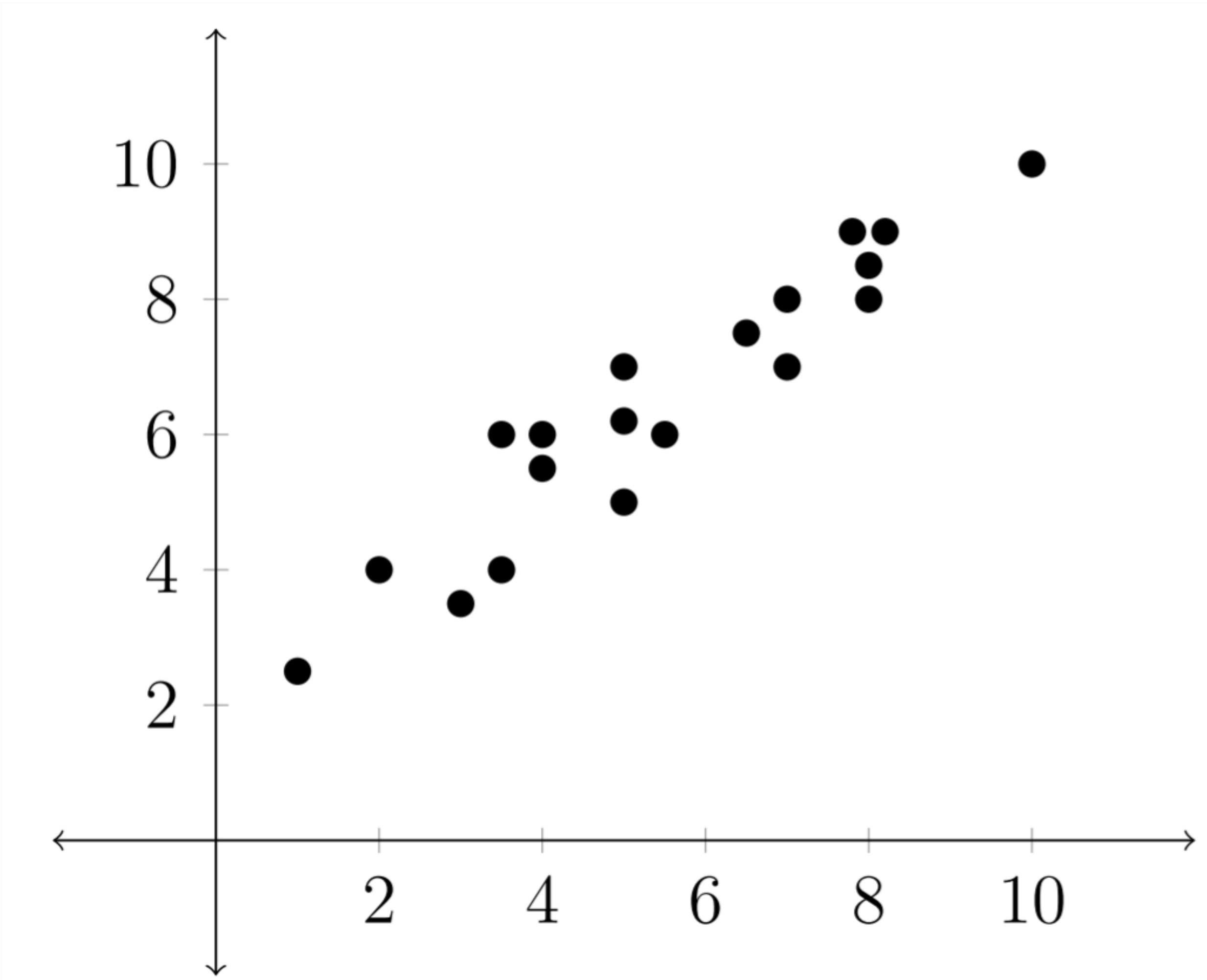


a) $g(x-2)$
b) $-g(x)+1$
c) $2g(x-1)$
d) $\frac{1}{2}g(x)-3$

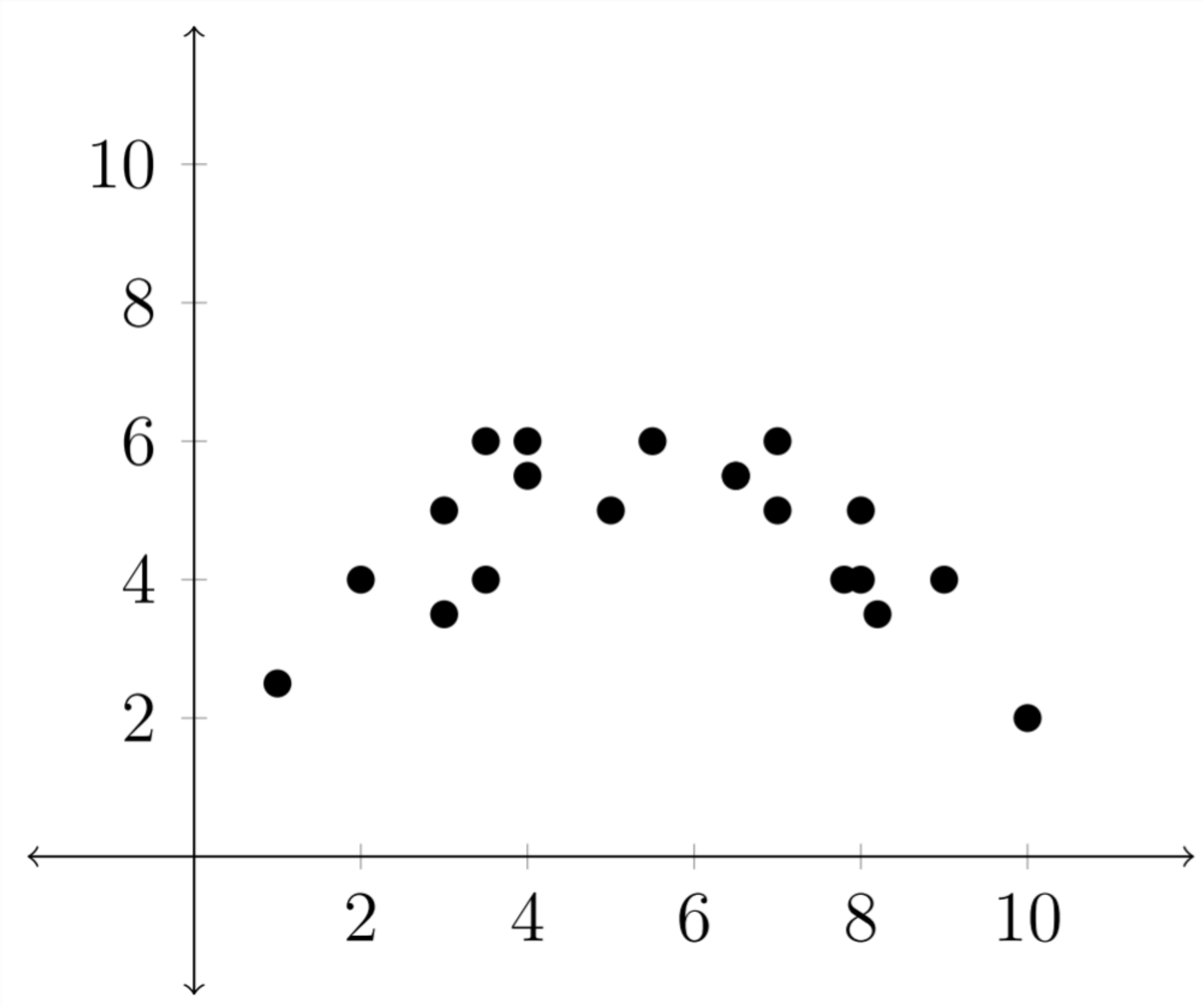
4.5: Toolbox Functions

Learning the general shapes of some common function families can be helpful in analyzing various problems. This can also be helpful in applying the ideas of statistical regression. Statistical regression typically gathers a collection of data points and tries to fit a mathematical function to the data points. Choosing the type of function that will best fit the data is an important step in determining a suitable regression function.

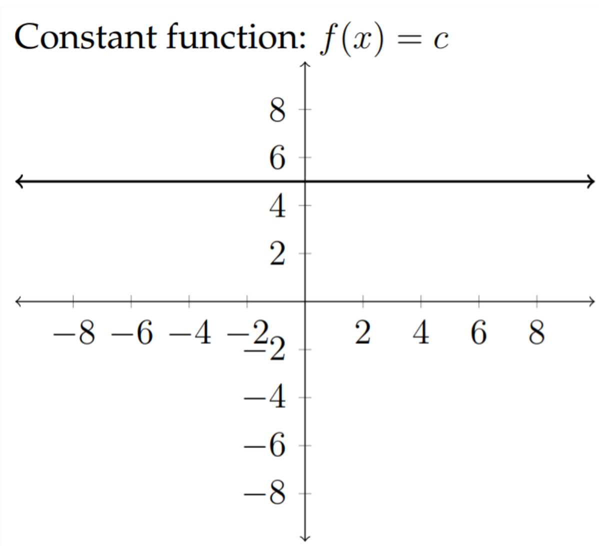
Below is an example of a Linear Regression:

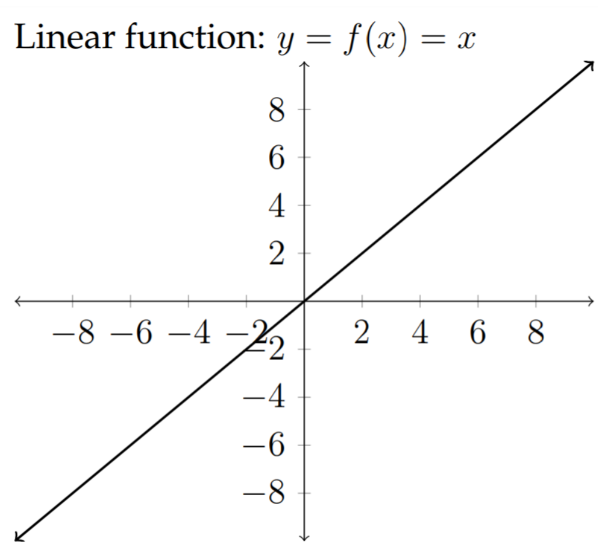


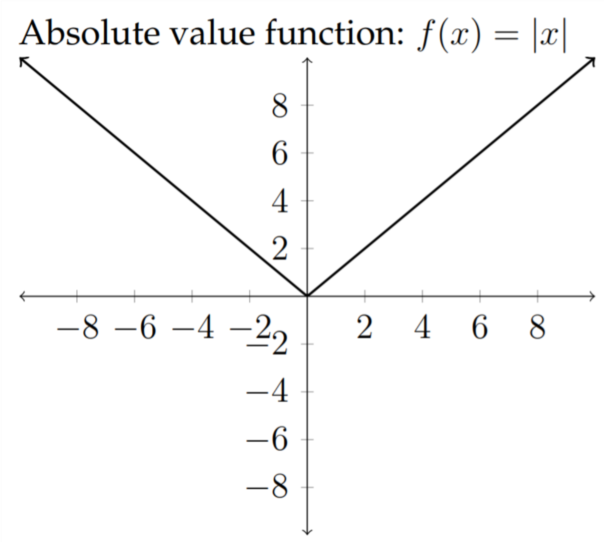
The graph below is an example of a Quadratic Regression:

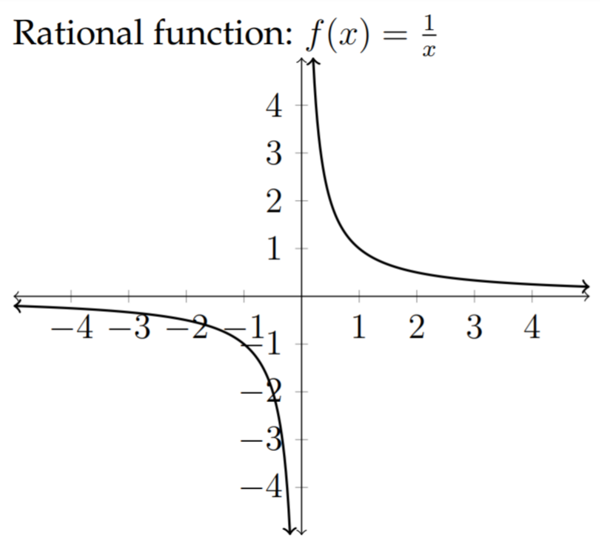


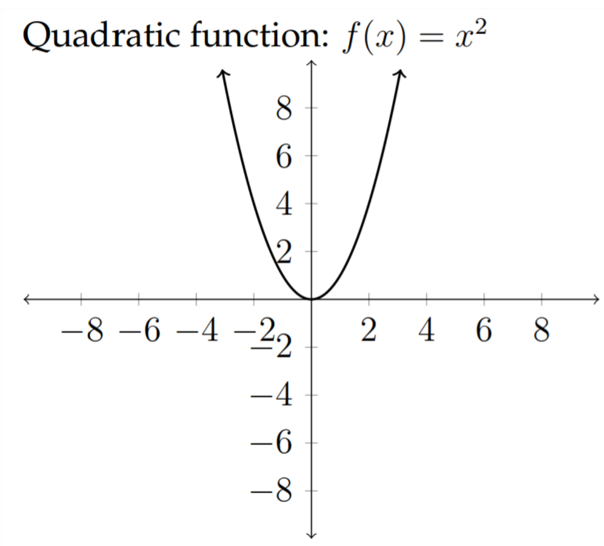
Being familiar with the typical shape of the various function families can help in analyzing experimental data.
The standard function families are:

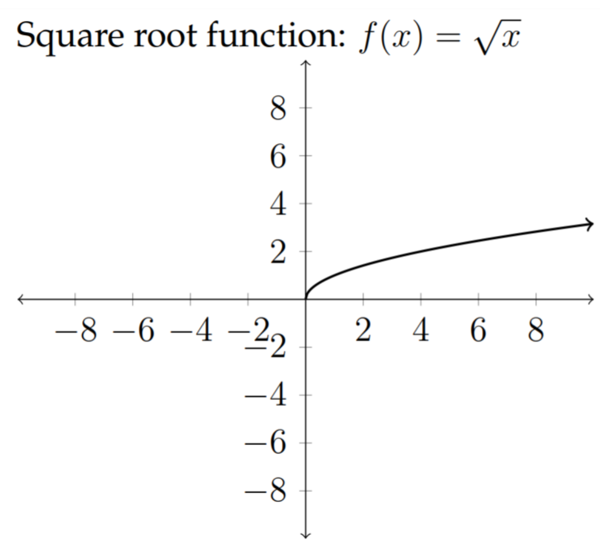


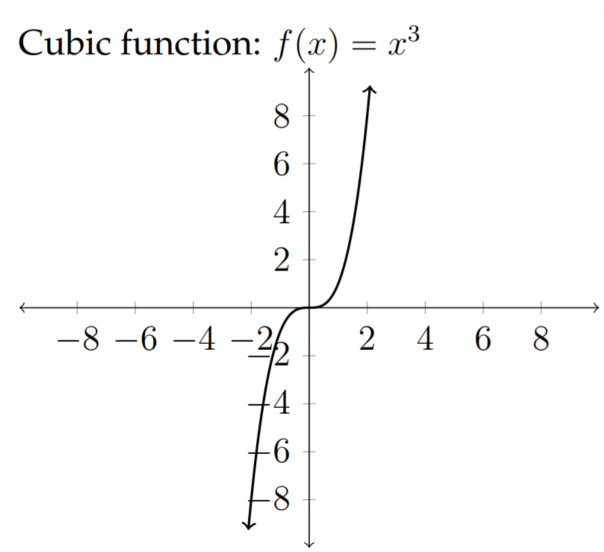


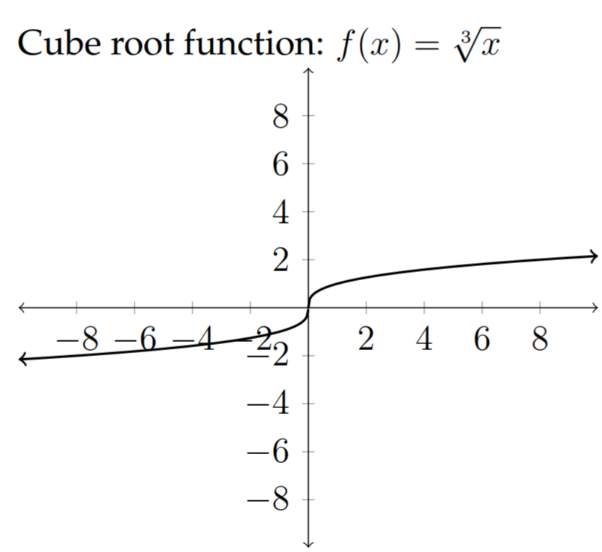












Some other function families that we won't discuss are:

Exponential function: $f(x) = a^x$

Logarithmic function: $f(x) = \log_b x$
Trigonometric function: $f(x) = \sin x$

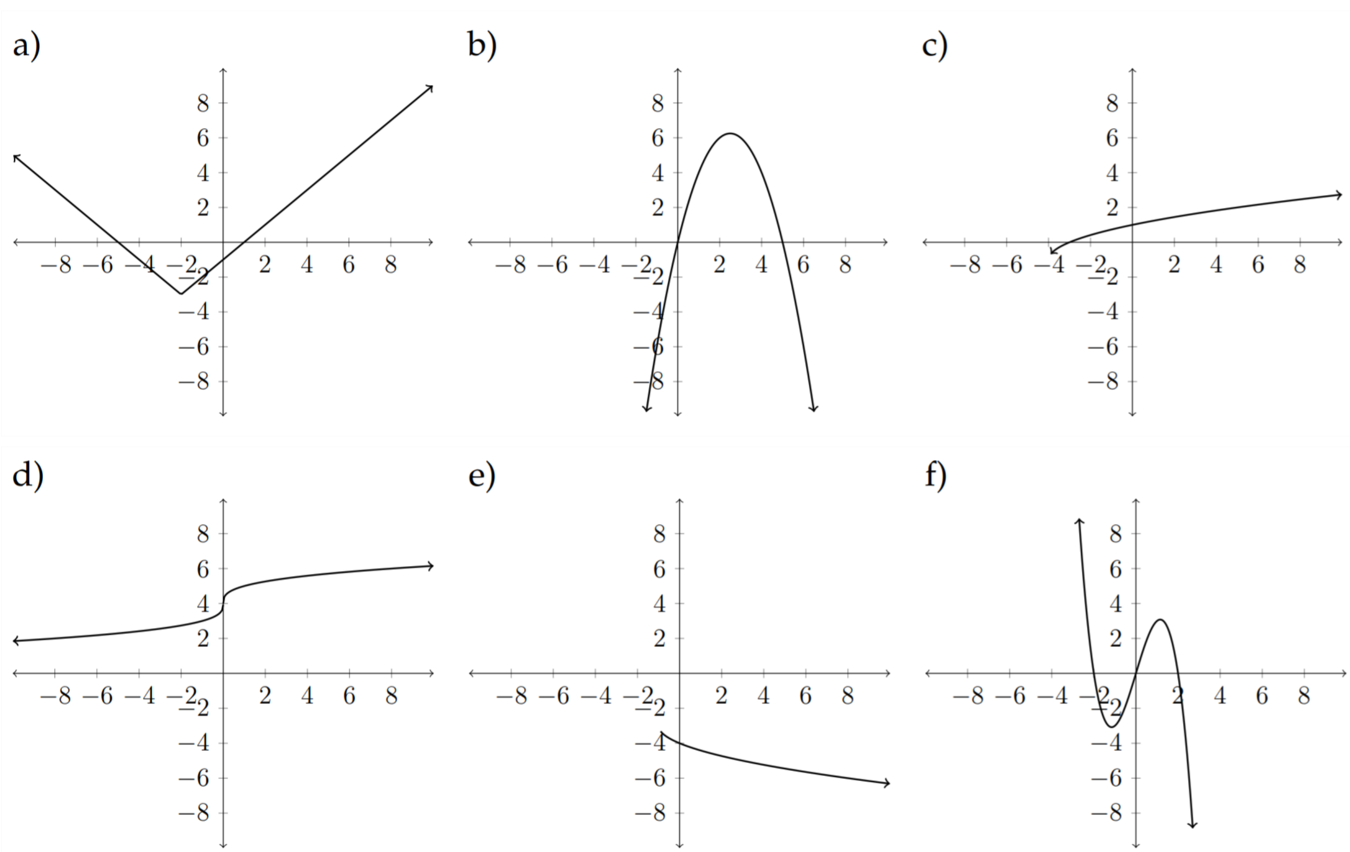
Exercises 4.5

Sketch the graph for each of the following transformations.

1) $f(x) = x^2 + 3$
2) $f(x) = x^2 - 4$
3) $f(x) = (x-5)^2 + 3$
4) $f(x) = (x+1)^2 - 4$
5) $f(x) = |x| - 2$
6) $f(x) = |x| + 5$
7) $f(x) = |x+3| - 2$
8) $f(x) = |x-1| + 5$

Match each of the following equations to the appropriate graph.

9) $f(x) = \sqrt{x+4} - 1$
10) $h(x) = x^2 + x - 6$
11) $g(x) = |x+2| - 3$
12) $f(x) = 4x - x^2$
13) $h(x) = -|x-2| + 1$
14) $g(x) = -\frac{3}{5}x + 2$
15) $h(x) = \sqrt[3]{x} + 4$
16) $f(x) = x^3 - 3x^2 + 3x + 2$
17) $g(x) = -\sqrt[3]{x+2}$
18) $h(x) = -\sqrt{x+1} - 3$
19) $f(x) = \frac{5}{4}x - 2$
20) $f(x) = 4x - x^3$



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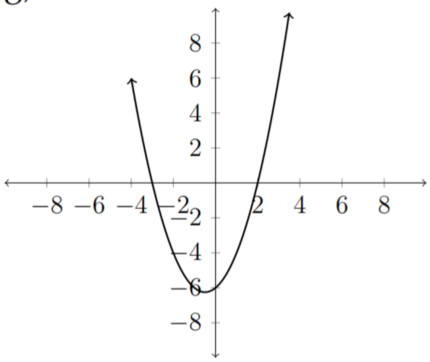
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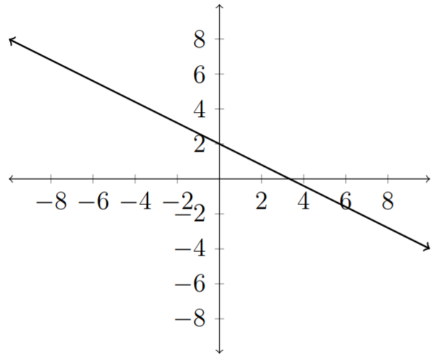
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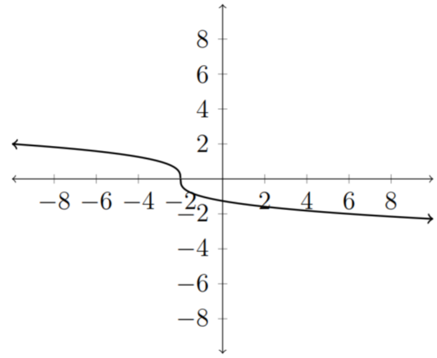
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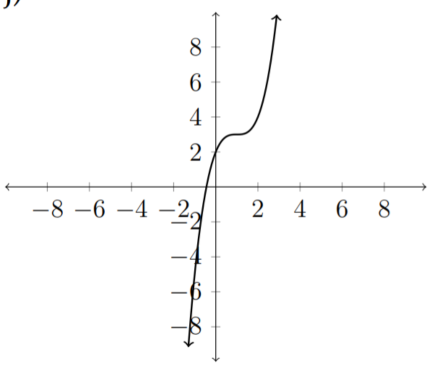
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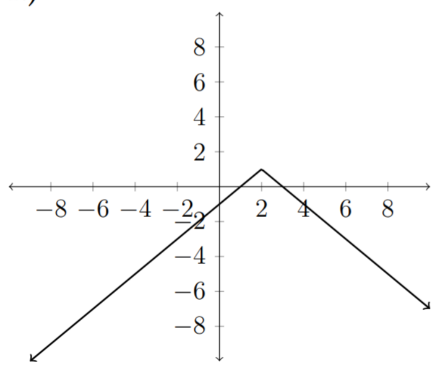
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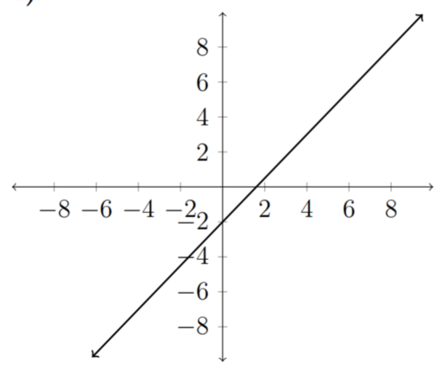
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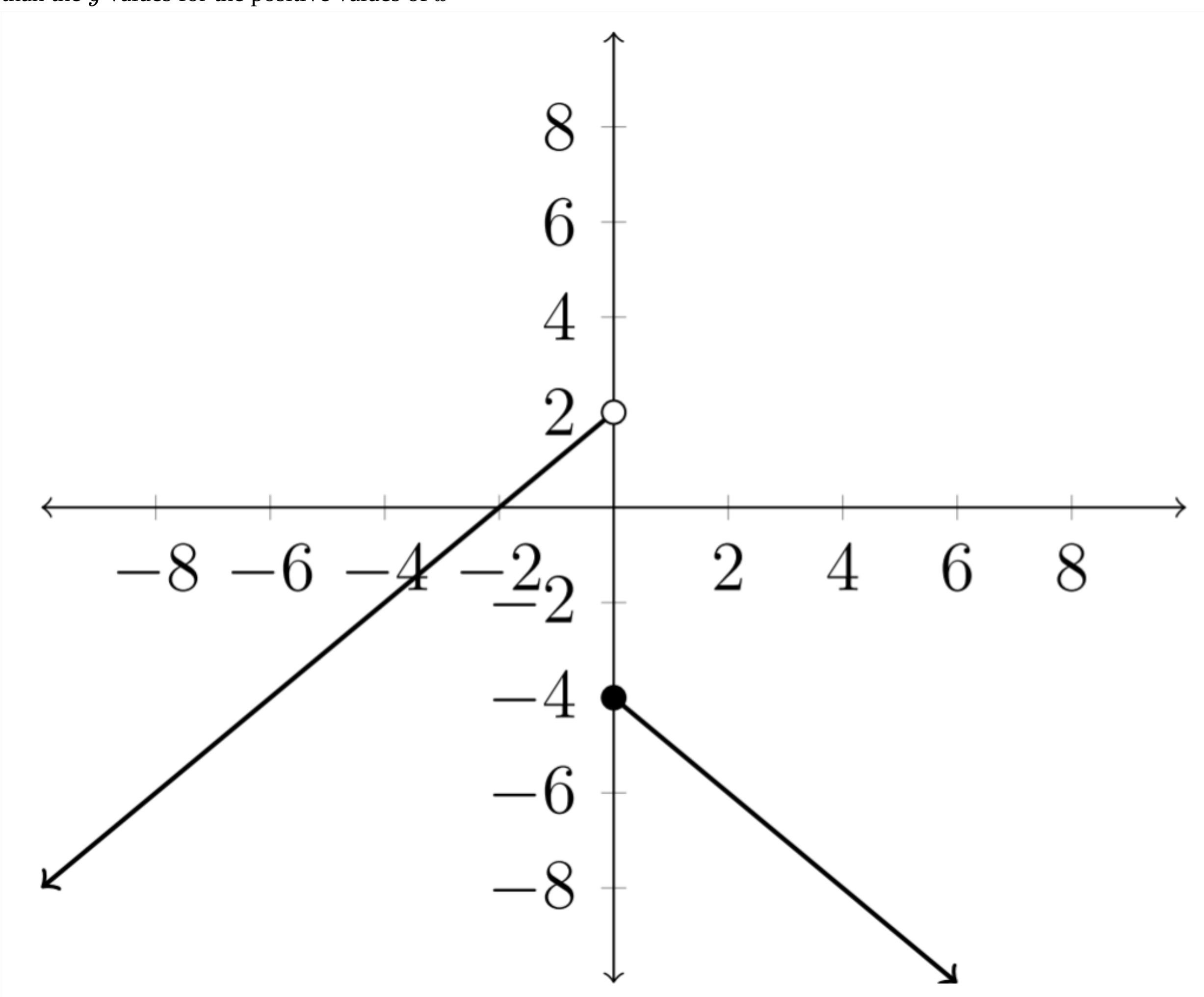
4.6: Piecewise-defined Functions

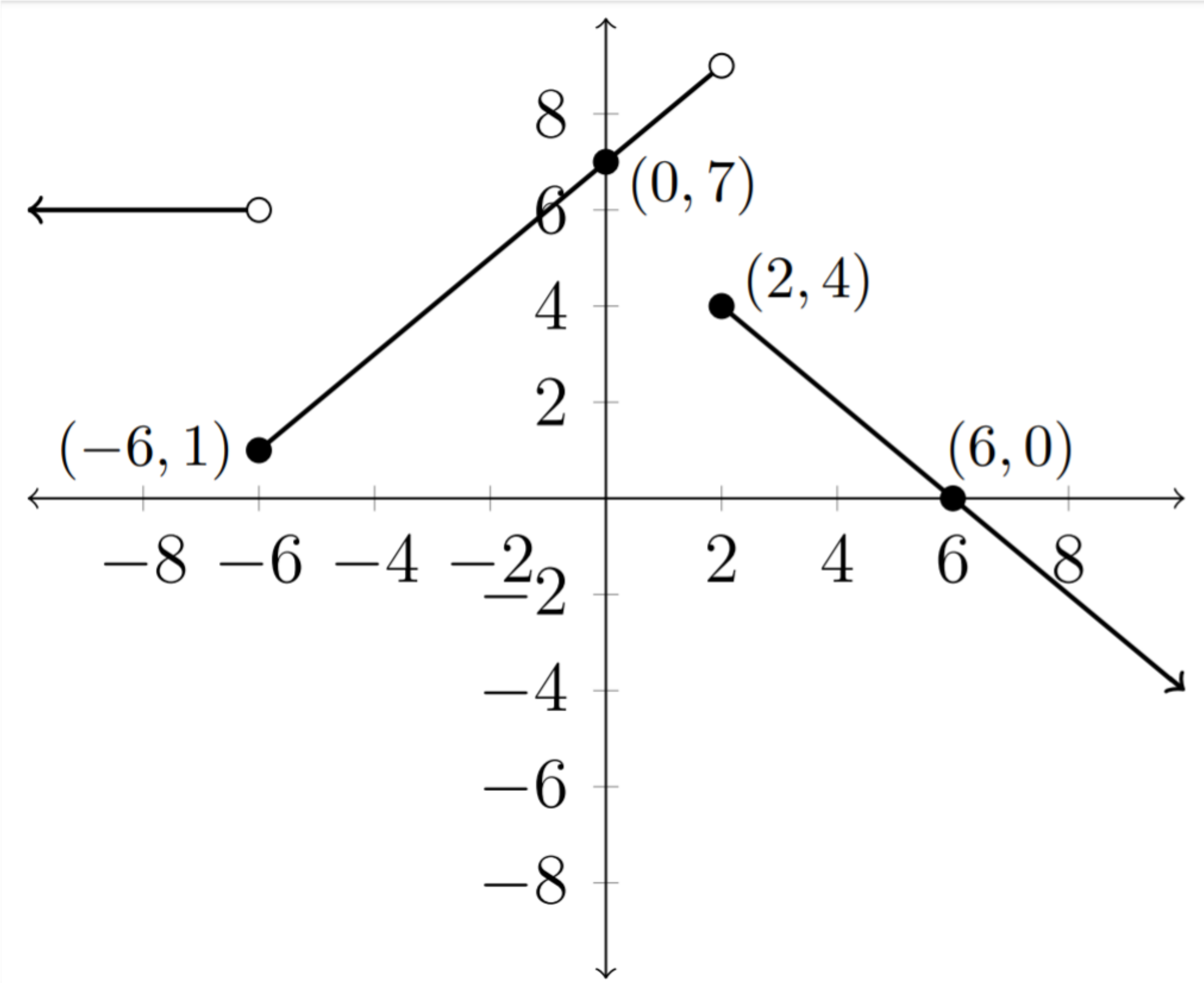
In certain situations a numerical relationship may follow one pattern of behavior for a while and then exhibit a different kind of behavior. In a situation such as this, it is helpful to use what is known as a piecewise defined function - a function that is defined in pieces.

$$f(x) = \begin{cases} x+2 & x<0 \\ -x-4 & x \geq 0 \end{cases}$$

In the above example of a piecewise defined function, we see that the y values for the negative values of x are defined differently than the y values for the positive values of x



Sometimes we are given a graph and need to write a piecewise description of the function it describes.



The piecewise function pictured above could be described as follows:

$$f(x) = \begin{cases} 6 & x < -6 \\ x + 7 & -6 \leq x < 2 \\ -x + 6 & x \geq 2 \end{cases}$$

Exercises 4.6

Sketch a graph for each of the piecewise functions described below.

1) $$f(x)=\begin{cases}1 & x<2\\ 2 & x\geq 2\end{cases}$$

2) $$f(x)=\begin{cases}0 & x\leq 1\\ x-1 & x>1\end{cases}$$

3) $$f(x)=\begin{cases}4 & x<2\\ x-2 & x\geq 2\end{cases}$$

4) $$f(x)=\begin{cases}2-x & x<-2\\ 4 & x\geq -2\end{cases}$$

5) $$f(x)=\begin{cases} x & x\leq 1 \\ x+1 & x>1 \end{cases}$$

6) $$f(x)=\begin{cases} 2x+3 & x<-2 \\ 3-x & x\geq -2 \end{cases}$$

7) $$f(x)=\begin{cases} -1 & x<-1 \\ 1 & -1\leq x\leq 1 \\ -1 & x>1 \end{cases}$$

8) $$f(x)=\begin{cases} -1 & x<-1 \\ x & -1\leq x\leq 1 \\ 1 & x>1 \end{cases}$$

9) $$f(x) = \begin{cases} x^2 & x < 0 \\ x+1 & x \geq 0 \end{cases}$$

10) $$f(x) = \begin{cases} x^2 + 2x & x \leq -1 \\ x & x > -1 \end{cases}$$

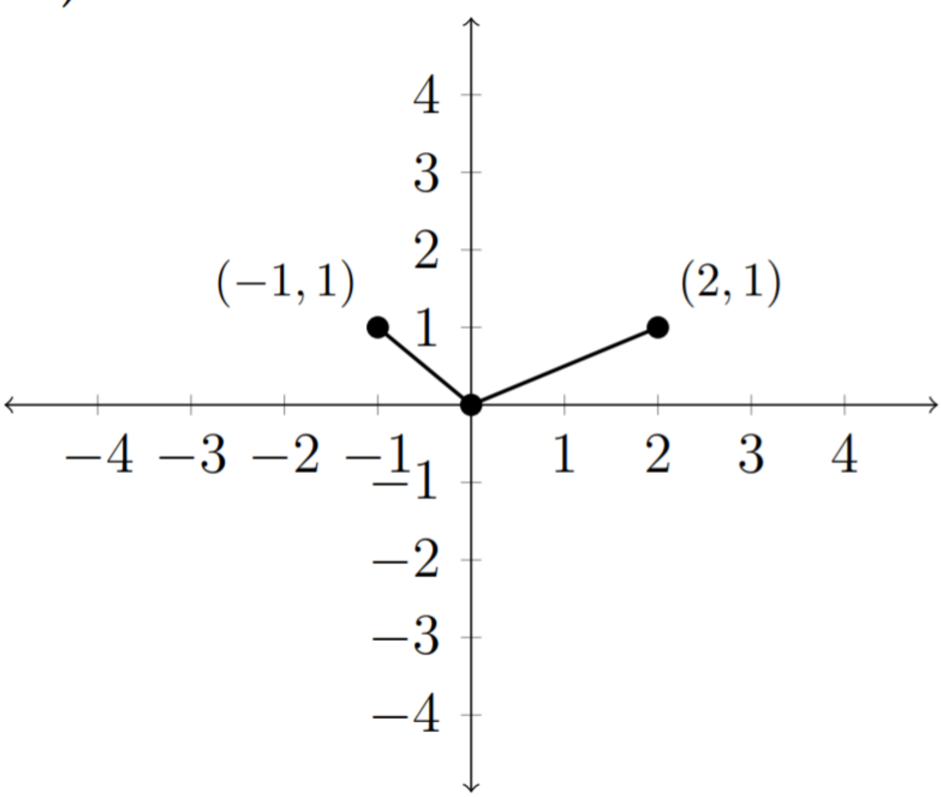
11) $$f(x) = \begin{cases} 2 & x \leq -1 \\ x^2 & x > -1 \end{cases}$$

12) $$f(x) = \begin{cases} 1 - x^2 & x \leq 2 \\ x & x > 2 \end{cases}$$

13) $$f(x) = \begin{cases} 4 & x < -2 \\ x^2 & -2 \leq x \leq 2 \\ -x+6 & x > 2 \end{cases}$$

14) $$f(x) = \begin{cases} 3x & x < 0 \\ x+1 & 0 \leq x \leq 2 \\ (x-2)^2 & x > 2 \end{cases}$$

15)



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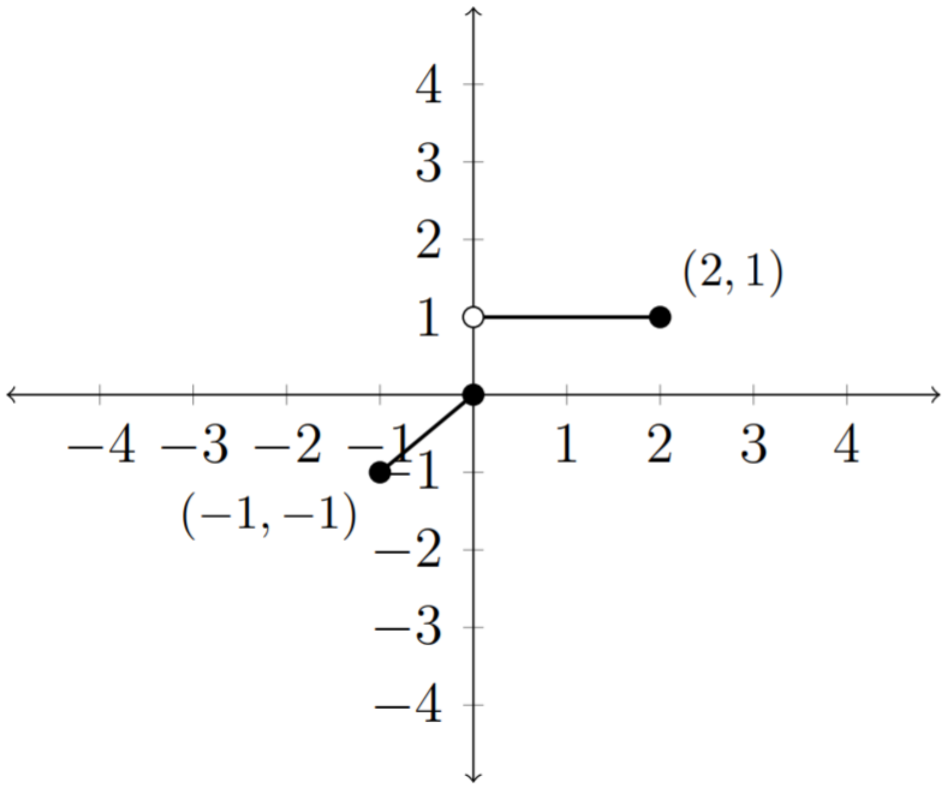

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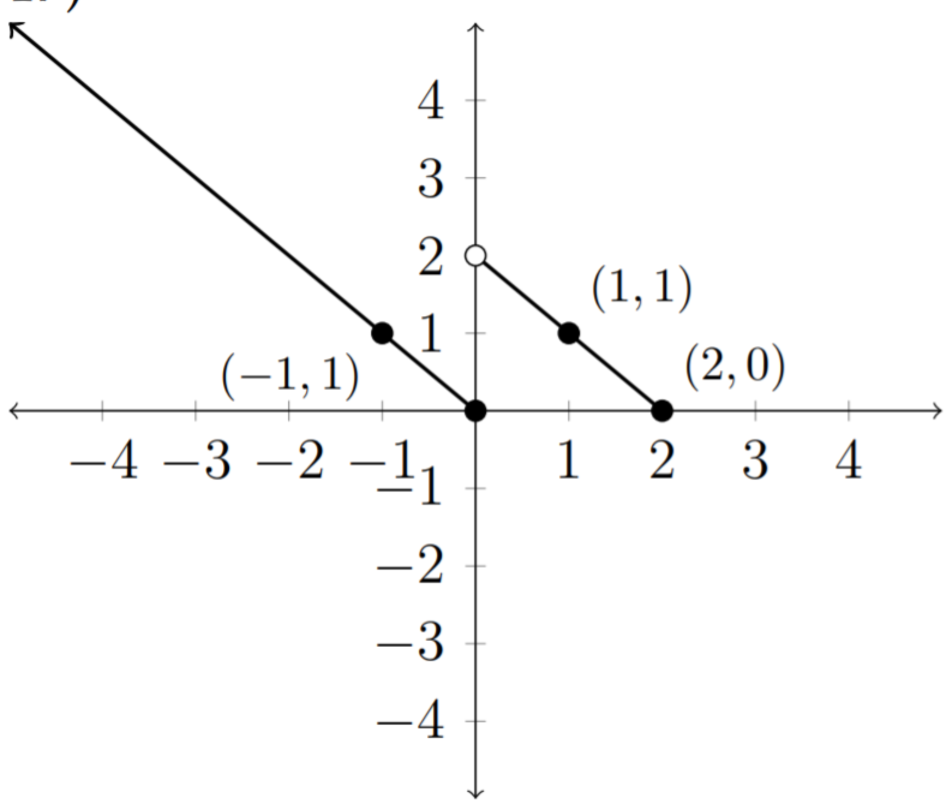

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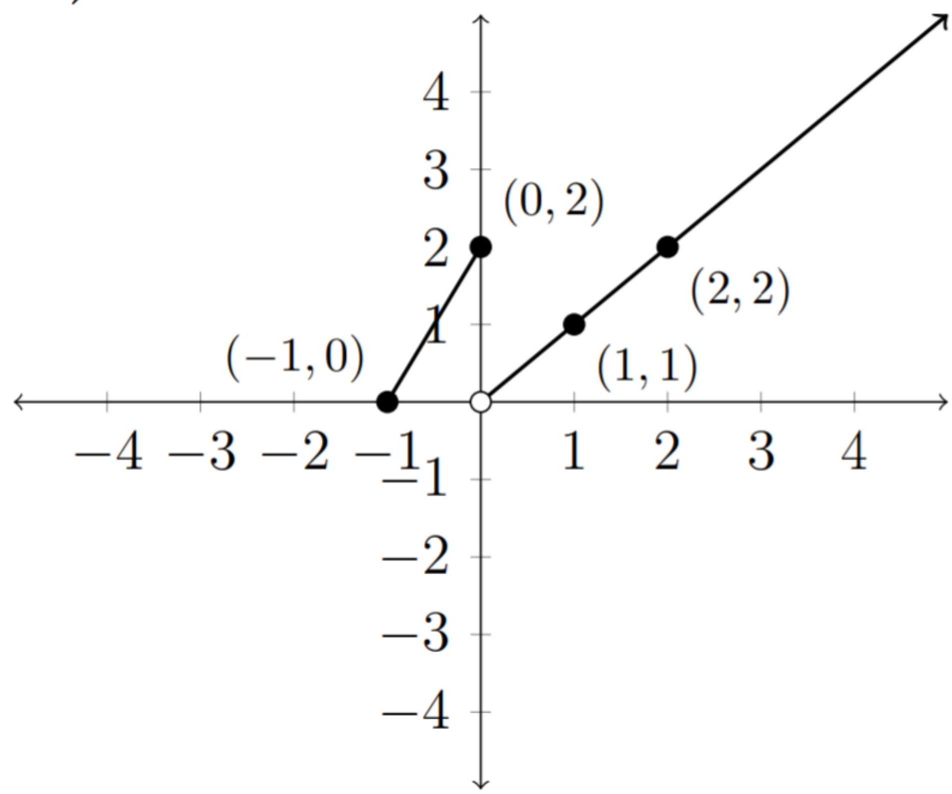


4.7: Composite functions

Similar to the way in which we used transformations to analyze the equation of a function, it is sometimes helpful to consider a given function as being several functions of the variable combined together.

For example, instead of thinking of the function $f(x)=(2x-7)^3$ as being a single function, we can think of it as being two functions:

$$\begin{gathered} g(x)=2x-7 \\ \text{and} \\ h(x)=x^3 \end{gathered} \tag{4.7.1}$$

Then $f(x)$ is the combination or "composition" of these two functions together. The first function multiplies the variable by 2, and subtracts 7 from the result. The second function takes this answer and raises it to the third power. The notation for the composition of functions is an open circle: 0

In the example above we would say that the function $f(x)=(2x-7)^3$ is equivalent to the composition $h \circ g(x)$ or $h(g(x))$. The order of function composition is important. The function $g \circ h(x)$ would be equivalent to $g(h(x))$, which would be

$$\text{equal to } g\left(x^3\right)=2\left(x^3\right)-7=2x^3-7 \tag{4.7.2}$$

Exercises 4.7

Find $f \circ g(x)$ and $g \circ f(x)$ for each of the following problems.

1) $f(x)=x^2$
$g(x)=x-1$

2) $f(x)=|x-3|$
$g(x)=2x+3$

3) $f(x)=\frac{x}{x-2}$
$g(x)=\frac{x+3}{x}$

4) \(\quad f(x)=x^{3}-1 \\ g(x)=\frac{1}{x^{3}+1}\)

5) $f(x)=\sqrt{x+1}$
$g(x)=x^4-1$

6) \(\quad f(x)=2 x^{3}-1 \\ g(x)=\sqrt[3]{\frac{x+1}{2}}\)

Find functions $f(x)$ and $g(x)$ so that the given function $h(x)=f \circ g(x)$

7) $h(x)=(3x+1)^2$

8) $h(x)=\left(x^2-2x\right)^3$

9) $h(x)=\sqrt{1-4x}$

10) $h(x)=\sqrt[3]{x^2-1}$

11) $h(x)=\left(\frac{x+1}{x-1}\right)^2$

12) $h(x)=\left(\frac{1-2x}{1+2x}\right)^3$

13) $h(x)=\left(3x^2-1\right)^{-3}$

14) $h(x)=\left(1+\frac{1}{x}\right)^{-2}$

15) $h(x)=\sqrt{\frac{x}{x-1}}$

16) $h(x)=\sqrt[3]{\frac{x-1}{x}}$

17) $h(x)=\sqrt{\left(x^2-x-1\right)^3}$

18) $h(x)=\sqrt[3]{\left(1-x^4\right)^2}$

19) $h(x)=\frac{2}{\sqrt{4-x^2}}$

20) $h(x)=-\left(\frac{3}{x-1}\right)^5$

21) A spherical weather balloon is inflated so that the radius at time t is given by the equation:

$$r=f(t)=\frac{1}{2}t+2 \tag{4.7.3}$$

Assume that r is in meters and t is in seconds, with $t=0$ corresponding to the time the balloon begins to be inflated. If the volume of a sphere is given by the formula:

$$v(r)=\frac{4}{3}\pi r^3 \tag{4.7.4}$$

Find $V(f(t))$ and use this to compute the time at which the volume of the balloon is $36\pi \mathrm{m}^3$

4.8: Inverse Functions

An inverse function undoes the action of the original function. So the inverse of a function that squared a number would be a function that square rooted a number. In general, an inverse function will take a y value from the original function and return the x value that produced it.

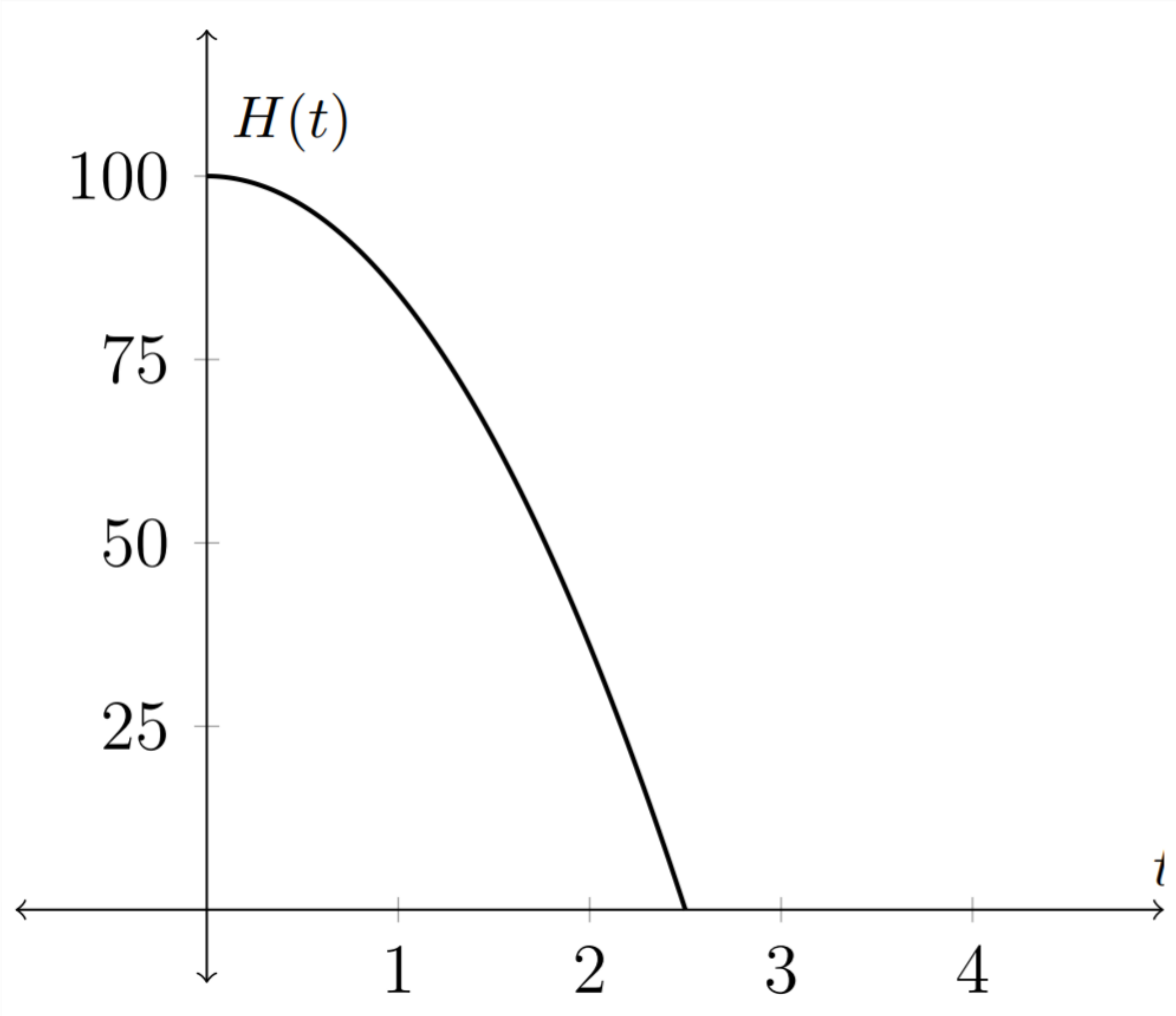
We can see this in an application. Given an object with little or no air resistance that is dropped from 100ft, the function that describes its height as a function of time would be:

$$H(t) = 100 - 16t^2 \tag{4.8.1}$$

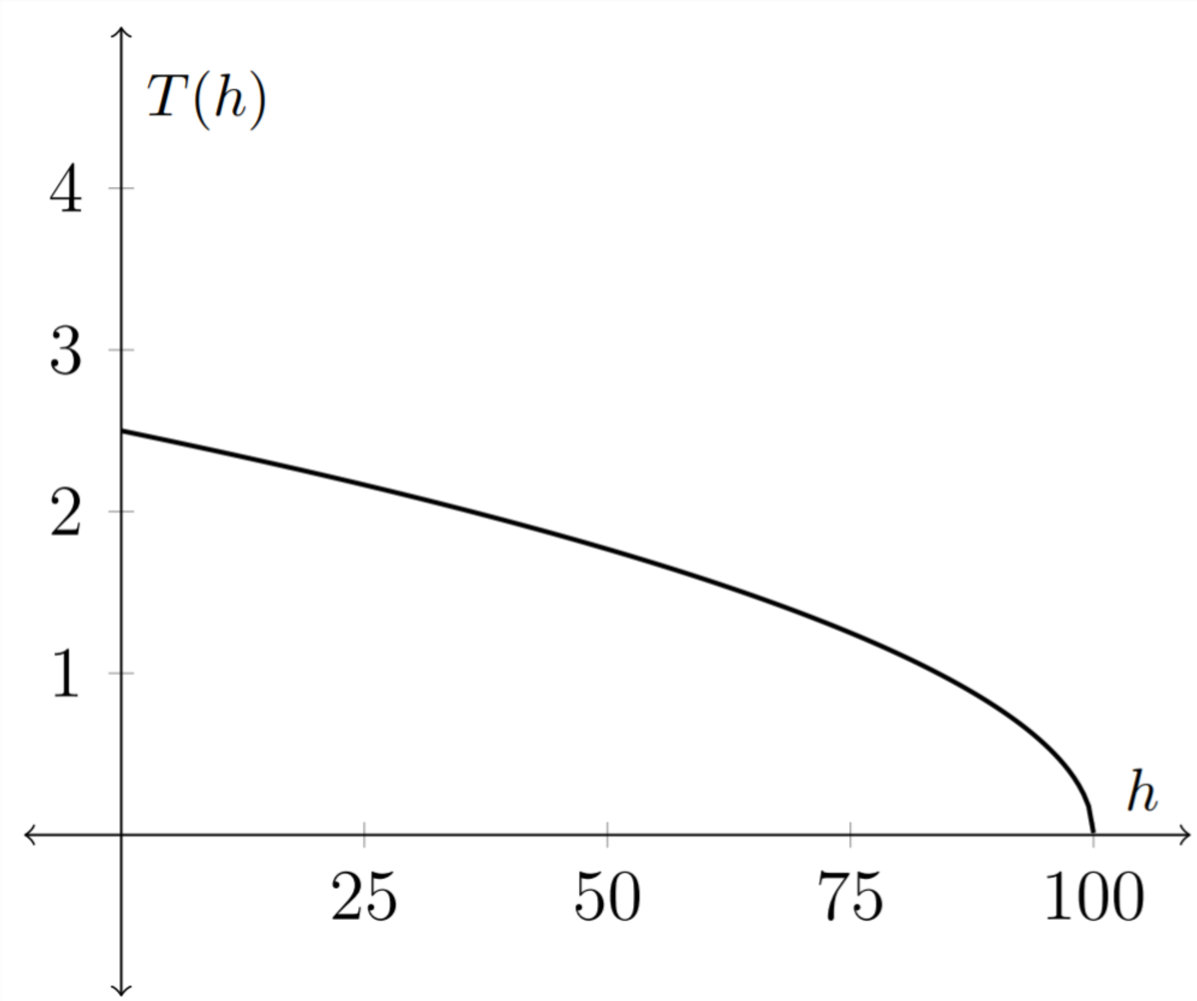
In this function, $H(t)$ is the height of the object at time t. If we wanted to turn this around so that it described the time for a given height, then we would want to isolate the t variable. In this example, the graph of the function would change in that the original independent variable - t, becomes the dependent variable in the inverse function.

$$\begin{aligned} h &= 100 - 16t^2 \\ 16t^2 &= 100 - h \\ t^2 &= \frac{100-h}{16} \\ t &= \sqrt{\frac{100-h}{16}} \\ T(h) &= \frac{\sqrt{100-h}}{4} \end{aligned}$$

Original function: $H(t) = 100 - 16t^2$



Inverse function: $T(h) = \frac{\sqrt{100-h}}{4}$



Notice that the graph of the function inverse is the original function reflected over the line $y = x$, because an inverse function interchanges the independent and dependent variables.

Finding a formula for an inverse function can be more confusing when we consider a standard function $y = f(x)$. In our standard notation, x is always considered to be the independent variable and y is always considered to be the dependent variable.

Notice in the example above that when we graphed the function and its inverse, the label on the x axis changed from t to h. In a standard function, the x axis will always be the x axis and the y axis will always be the y axis. To compensate for this, when we find a function inverse for a function stated in terms of x and y we generally interchange the x and y terms so that x remains the independent variable.
In our example, we had

$$H(t) = 100 - 16t^2 \tag{4.8.2}$$

and found the inverse to be

$$T(h) = \frac{\sqrt{100 - h}}{4} \tag{4.8.3}$$

If the original function had been stated in terms of x and y, then the process would have looked like this:

$$\begin{aligned} f(x) &= 100-16x^2 \\ y &= 100-16x^2 \\ 16x^2 &= 100-y \\ x^2 &= \frac{100-y}{16} \\ x &= \sqrt{\frac{100-y}{16}} \\ x &= \frac{\sqrt{100-y}}{4} \end{aligned}$$

Then we switch the x and y variables to keep the x as the independent variable:

$$y=f^{-1}(x)=\frac{\sqrt{100-x}}{4} \tag{4.8.4}$$

So

$$f(x)=100-16x^2 \tag{4.8.5}$$

and

$$f^{-1}(x)=\frac{\sqrt{100-x}}{4} \tag{4.8.6}$$

Exercises 4.8

Given the function $f(x)$, find the inverse function $f^{-1}(x)$

1) $f(x)=3x$
2) $f(x)=-4x$
3) $f(x)=4x+2$
4) $f(x)=1-3x$
5) $f(x)=x^3-1$
6) $f(x)=x^3+1$
7) $f(x)=x^2+4(x\geq 4)$
8) $f(x)=x^2+9(x\geq 9)$
9) $f(x)=\frac{4}{x}$
10) $f(x)=-\frac{3}{x}$
11) $f(x)=\frac{1}{x-2}$
12) $f(x)=\frac{4}{x+2}$
13) $f(x)=\frac{2}{x+3}$
14) $f(x)=\frac{4}{2-x}$
15) $f(x)=\frac{3x}{x+2}$
16) $f(x)=-\frac{2x}{x-1}$
17) $f(x)=\frac{2x}{3x-1}$
18) $f(x)=-\frac{3x+1}{x}$
19) $f(x)=\frac{3x+4}{2x-3}$
20) $f(x)=\frac{2x-3}{x+4}$
21) $f(x)=\frac{2x+3}{x+2}$
22) $f(x)=-\frac{3x+4}{x-2}$

Find the inverse function for each of the following applications.

23) The volume of water left in a 1000 gallon tank that drains in 40 minutes is modeled by the equation:

$$V(t) = 1000\left(1 - \frac{t}{40}\right)^2 \tag{4.8.7}$$

Find $T(v)$ - the function that tells you how long the water has been draining given a particular volume left in the tank. Time is measured in minutes and the volume is measured in gallons.

24) The speed of a vehicle in miles per hour that leaves skid marks d feet long is modeled by the equation:

$$R(d) = 2\sqrt{5d} \tag{4.8.8}$$

Find $D(r)$ - the function that tells you the stopping distance for a vehicle traveling r miles per hour.

25) The period of a pendulum of length ℓ can be expressed by the relationship:

$$T(\ell) = 2\pi\sqrt{\frac{\ell}{980}} \tag{4.8.9}$$

Find the function $L(t)$ that determines the length of a pendulum given its period. Here the time is measured in seconds and the length is measured in centimeters.

26) The volume of a sphere of radius r is given by the formula:

$$V(r) = \frac{4}{3}\pi r^3 \tag{4.8.10}$$

Find $R(v)$ - the function that determines the radius of a sphere given its volume.

4.9: Optimization

One of the major applications of differential calculus is optimization. This is the process of finding maximum or minimum function values for a given relationship. There are four typical types of problems that we will examine in this section.
a) Analytic Optimization - these problems typically use the distance formula to determine the closest point to a particular curve.

b) Geometry/cost Optimization - these problems generally give a box or container of a particular shape and ask either to determine the cheapest manufacturing cost given a particular volume or to determine the greatest volume given a particular cost.

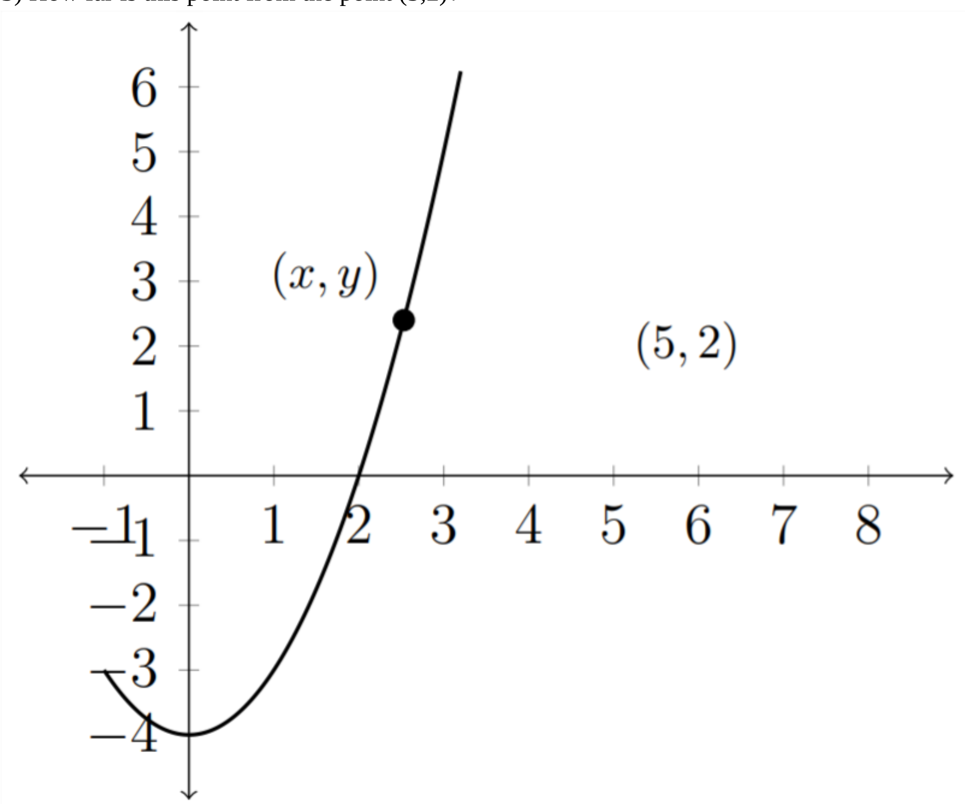
c) Distance Optimization - these problems generally use two objects travelling at right angles to each other and determine the maximum or minimum distance between the objects.

d) Distance/cost Optimization - these problems are usually focused on a situation in which two distances at right angles can be cut with a diagonal at a certain point to minimize cost or time.

Analytic Optimization

$$d=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}} \tag{4.9.1}$$

1) Express the distance of a point (x, y) (in the first quadrant) on the graph of the parabola $y=x^{2}-4$ from the point (5,2) as a function of x
2) Use the graph of the distance function $d(x)$ from Part I to determine the point of the graph of the parabola $y=x^{2}-4$ that is closest to the point (5,2)
3) How far is this point from the point (5,2)?



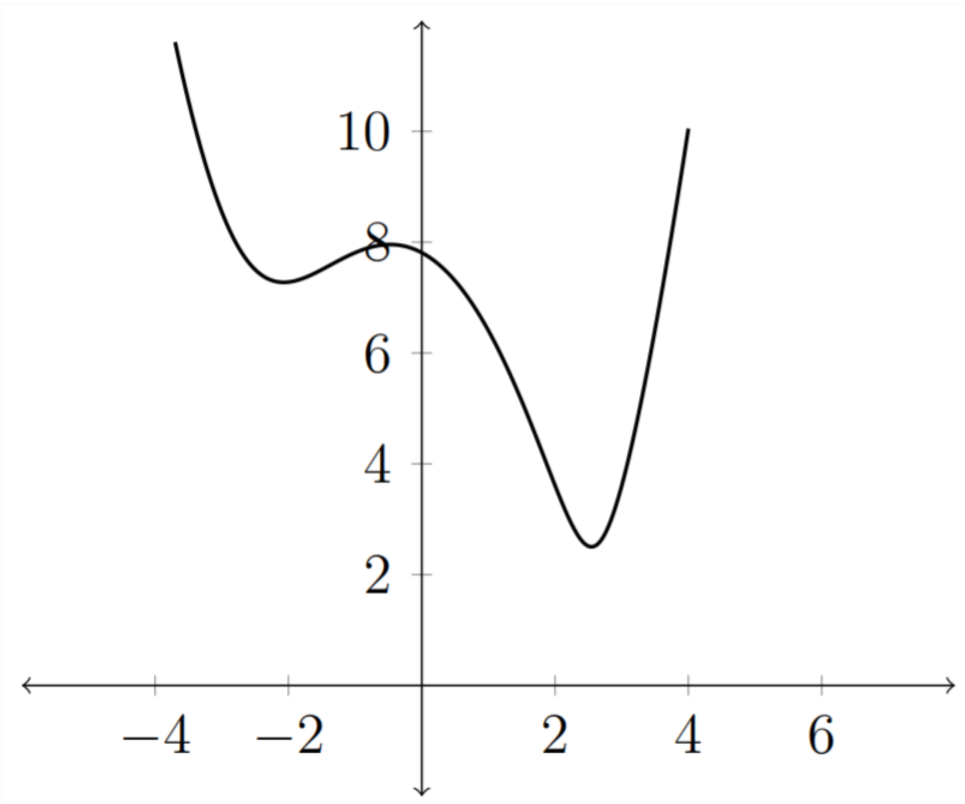
1) Using the distance formula $d=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}$, we can say that any point on the curve $y=x^{2}-4$ is a distance of:

$$d=\sqrt{(x-5)^2+(y-2)^2} \tag{4.9.2}$$

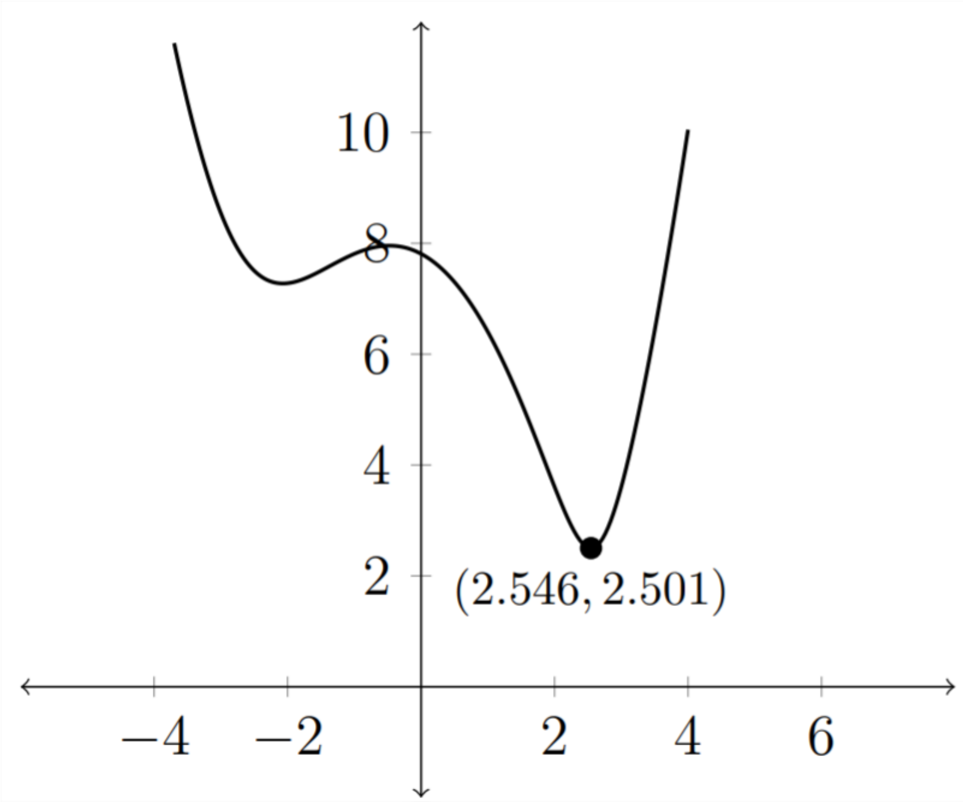
from the point $(5,2)$. Because the question asks for this to be expressed as a function of x, we should replace the y variable with an equivalent expression involving x

$$\begin{gathered} d=\sqrt{(x-5)^2+\left(x^2-4-2\right)^2} \\ \text{or} \\ d=\sqrt{(x-5)^2+\left(x^2-6\right)^2} \end{gathered} \tag{4.9.3}$$

2) The graph of this distance function is below:



Note that in this graph, the x axis represents the x values from the original graph, but the y axis in this graph is the distance of a point on the original curve from the point $(5,2)$. We can see that this distance function has a clear minimum value that occurs between $x=2$ and $x=4$. We can use the graphing calculator to find this value - in Calculus we would use algebra to find this value.

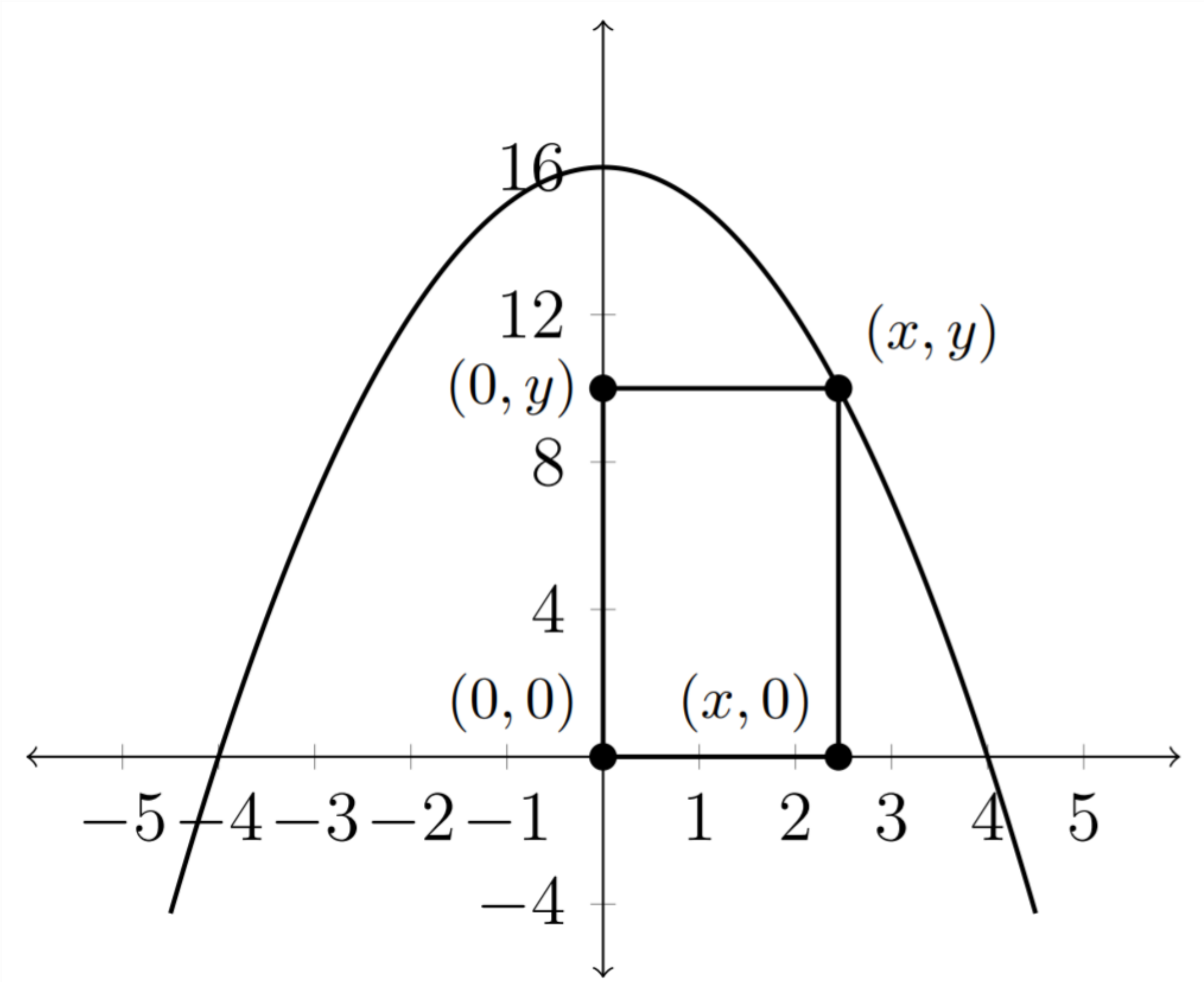


Both values indicated in the graph are approximate. The value $x \approx 2.546$ indicates the x value of the point on the original curve that is closest to the point (5,2) The $y \approx 2.501$ indicates the distance of the point from the point (5,2)
since the x value of the point on the original curve that is closest to the point (5,2) is about 2.546, that means that the y value is:

$$y = x^2 - 4 \approx 2.546^2 - 4 \approx 6.482 - 4 \approx 2.482 \tag{4.9.4}$$

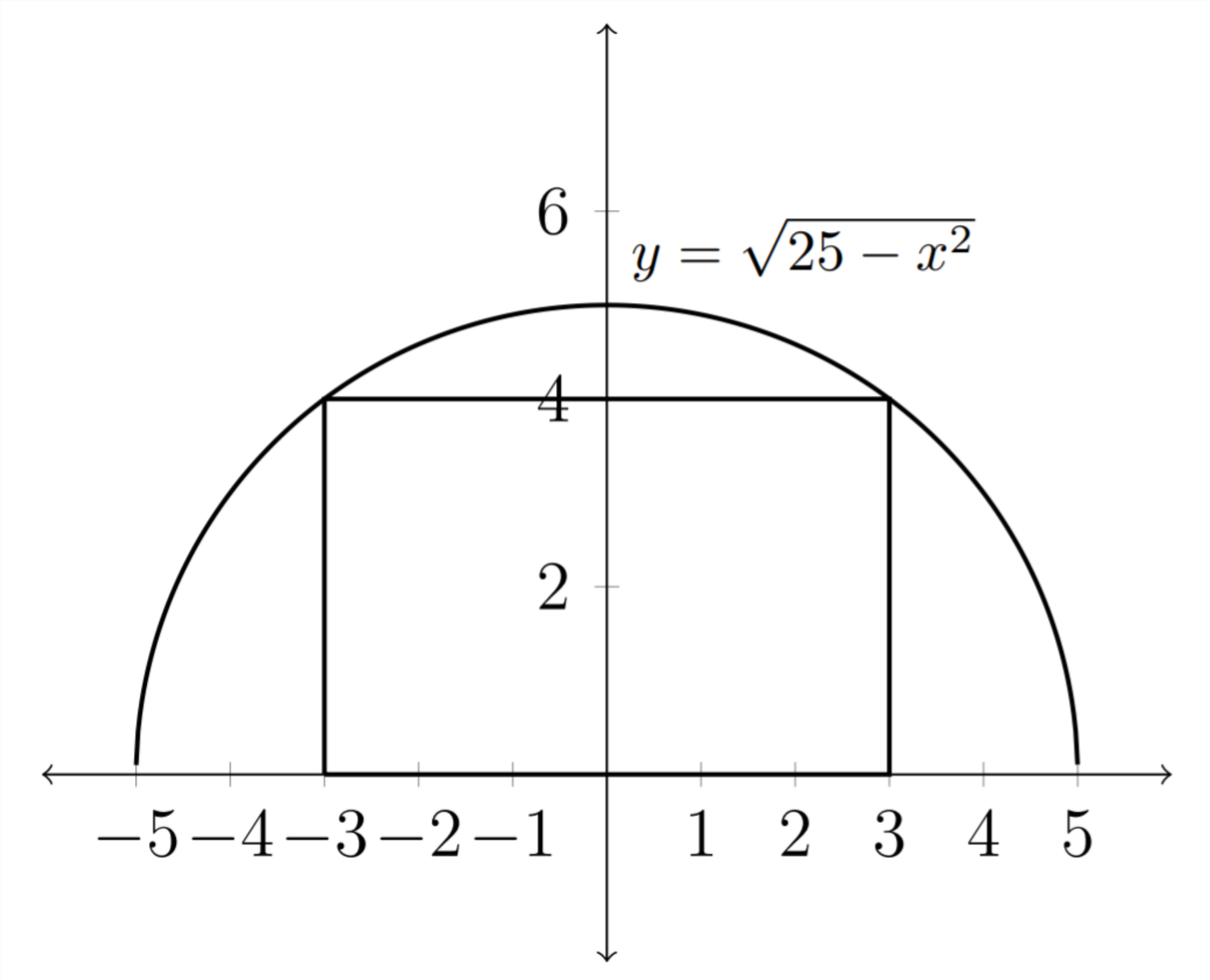
So the answer for Part 2 is that (2.546,2.482) is closest point on the curve $y = x^2 - 4$ to the point (5,2)
3) We already determined the distance when we found the minimum point on the distance graph. The distance is about 2.501 units.

Exercises $4.9(a)$
1) Given the function $f(x) = 4 - x^2$, find x and y coordinates of the point on the curve that is closest to the point $(7, 3)$. How far away from the point (7,3) is the point you found?
2) Given the function $f(x) = 6 - x^2$, find x and y coordinates of the point on the curve that is closest to the point $(1, 1)$. How far away from the point (1,1) is the point you found?
3) Given the function $f(x) = x^2 + 2x - 5$, find x and y coordinates of the point
on the curve that is closest to the point $(-2, 5)$. How far away from the point (-2,5) is the point you found?
4) Given the function $f(x) = x^2 - 5x + 8$, find x and y coordinates of the point on the curve that is closest to the point $(-2, 2)$. How far away from the point (-2,2) is the point you found?
5) Given the function $f(x) = x^3 - 4x^2 + 2x + 1$, find x and y coordinates of the point on the curve that is closest to the point $(2, 1)$. How far away from the point
(2,1) is the point you found?
6) Given the function $f(x) = 2x^3 - 5x - 1$, find x and y coordinates of the point
on the curve that is closest to the point $(-2, 8)$. How far away from the point (-2,8) is the point you found?
7) A rectangle has one corner in the 1 st quadrant on the graph of $y = 16 - x^2$ one corner at the origin and the other two corners on the positive x axis and the positive y axis.



a) Express the area A of the rectangle as a function of x.
b) For what value of x is the area largest?
c) What is the maximum area?

8) A rectangle is inscribed in a semi-circle of radius 5 so that two corners are on the x axis and two corners are on the semi-circle.



The formula for the graph of the semi-circle is $y=\sqrt{25-x^2}$

a) Express the area A of the rectangle as a function of x.

b) For what value of x is the area largest?

c) What is the maximum area?

Geometry/cost Optimization

A closed box with a square base is built from material that costs $\$1$ per ft^2 for the four sides and the bottom and $\$5$ per ft^2 for the top. What are the dimensions of the box of largest volume that can be constructed for $\$72$?

There are two issues we are working with in this problem - the volume of the box and the surface area of the box. The volume of the box is important because that's what we're looking to maximize. The surface area of the box is important because that's what will control the cost of the box - notice that the costs are given in terms of ft^2 or square feet which is related to the surface area.

The box has a square base, but the height is some other dimension - h. So the volume of the box will be length $*$ width $*$ height, but since the box has a square base the length and width will be the same - we'll call them x. From this we see that the volume can be expressed as:

$$V=x*x*h=x^2h \tag{4.9.5}$$

For the cost of the box, we need to know the surface area. Each side of the box should be included - the base and the top have the same area (x^2) because they are both squares that are x units on each side. The four sides of the box are all rectangles that are x

by h units. That means each one has an area of $x * h-$ so all four would be $4 * x * h$ or $4xh$
This makes the surface area of the box:

$$S = 2x^2 + 4xh \tag{4.9.6}$$

The cost of the box then will be:

$$C = x^2(\$1) + x^2(\$5) + 4xh(\$1) \tag{4.9.7}$$

Here we see the one x^2 is multiplied by $\$1$, because the bottom will cost $\$1$ per ft^2, but the other x^2 is multiplied by $\$5$, because the top costs $\$5$ per ft^2. The four sides : $4xh$ is also multiplied by \$1.

So, our final formula for the cost is:

$$C = 6x^2 + 4xh \tag{4.9.8}$$

but we already know that we want to spend $\$72$ on the box, so we can say that:

$$72 = 6x^2 + 4xh \tag{4.9.9}$$

This means that:

$$\begin{gathered} \frac{72-6x^2}{4x} = h \\ \text{or} \\ \frac{18}{x} - 1.5x = h \end{gathered} \tag{4.9.10}$$

The reason why this is important is that it will allow us to express the volume of the box in terms of x. Remember that the volume was:

$$V = x^2 h \tag{4.9.11}$$

so now we can see that:

$$\begin{gathered} V = x^2\left(\frac{18}{x} - 1.5x\right) \\ \text{or} \\ V = 18x - 1.5x^3 \end{gathered} \tag{4.9.12}$$

The graph for this function is below:

We can see in the graph that the maximum volume given a cost of $\$72$ will be when the base of the box is 2 feet by 2 feet. The volume will be $24\mathrm{ft}^3$ and the height will be 6 feet. So the answer to the question is $2\mathrm{ft} \times 2\mathrm{ft} \times 6\mathrm{ft}$

Exercises 4.9(b)

1) A rectangular storage container with an open top has a volume of $10\mathrm{m}^3$. The length of its base is twice the width. Material for the base costs $\$10$ per square meter. The sides require material that costs $\$6$ per square meter. Find the cost of materials for the cheapest such container.
2) A poster is to contain $108\mathrm{cm}^2$ of printed matter, with margins of $6\mathrm{cm}$ each at the top and bottom of the page and $2\mathrm{cm}$ on each side of the page. What is the minimum cost of the poster if it is to be made of material which costs $\$0.20$ per square centimeter?
3) If $1200\mathrm{cm}^2$ of material is available to make a box with a square base and an open top, find the largest possible volume of the box.
4) A box with a square base and open top must have a volume of $32,000\mathrm{cm}^3$ Find the dimensions of the box that minimizes the amount of material used.
5) A Norman window has the shape of a semi-circle on top of a rectangle so that the diameter of the semi-circle is the same as the

width of the rectangle. Given that the perimeter of the window should be $30\mathrm{ft}$, find the dimensions of the window that will admit the greatest amount of sunlight (maximize the area).

6) A rectangular box with no top is to be made from material costing $\$0.50$ per square foot. The width of the base is 5 ft and it will have a volume of $6\mathrm{ft}^3$. What are the dimensions of the box that minimizes the cost? What is the cost of this box?

Surface Area of a cylinder $= 2\pi r^2 + 2\pi rh$

Volume of a cylinder $= \pi r^2 h$

7) A cylindrical can with a top will be made from 200 in ^{2} of tin. What would be the height and radius of the can of maximum volume?

8) A cylindrical can with a top will have a volume of $500\mathrm{cm}^3$. If the sides and bottom cost $\$0.002$ per square cm and the top costs $\$0.0035$ per square cm, find the dimensions of the least expensive can that will have a volume of $500\mathrm{cm}^3$ How much does it cost to make this can?

Distance Optimization

One ship is 10 miles due east of a buoy and is sailing due west, towards the buoy at 12 mph. Another ship is 10 miles due south of the same buoy and sailing due north, also towards the buoy at 7 mph.

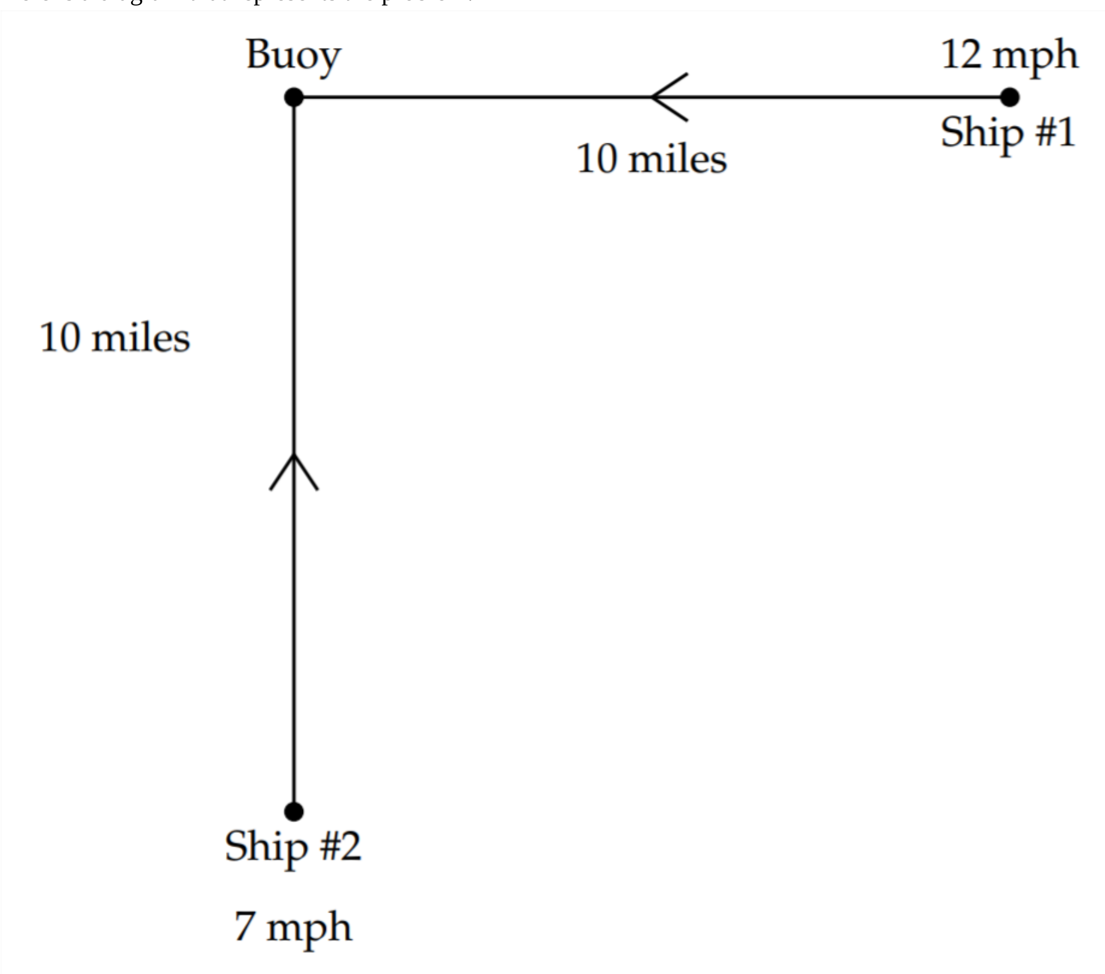
a) Write a function that represents the distance between the two ships in terms of
t, the elapsed time in hours.

b) Graph the function and determine the value of t when the ships are closest together. How far apart are the ships at this time?

Round the value of t to the nearest 100 th of an hour.

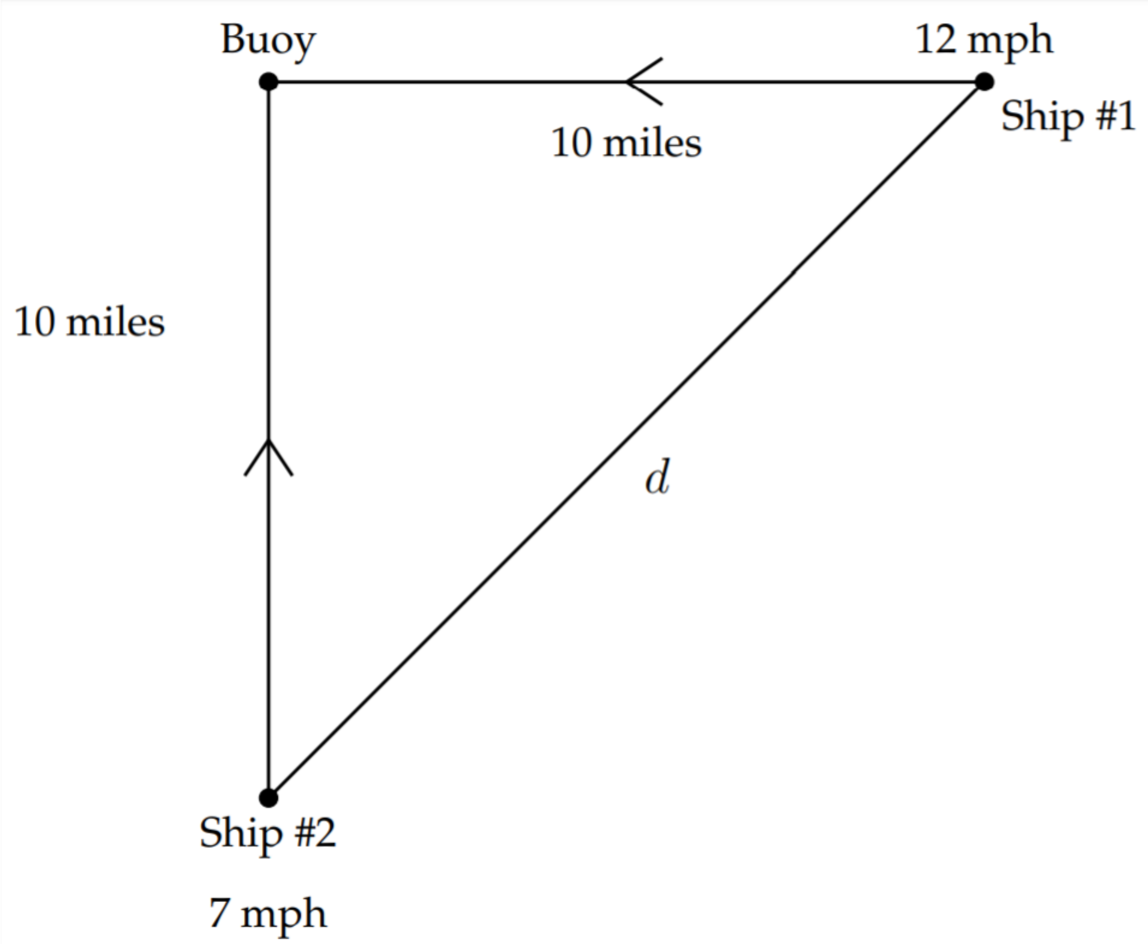
Round the distance to the nearest tenth of a mile.

Here is a diagram that represents the problem:



The distance between the two ships is measured along the diagonal. Because this creates a right triangle, we can use the

Pythagorean Theorem to represent the distance between the ships.



In this case the legs of the right triangle begin as 10 miles, but they get shorter as the ships travel closer to the buoy. For Ship #1, the distance decreases by 12 miles every hour - this means that the first ship will pass the buoy in less than one hour. The distance between Ship #1 and the buoy can be represented as $(10-12t)$ where t is the number of hours spent traveling. Similarly, the distance between Ship #2 and the buoy can be represented as $(10-7t)$

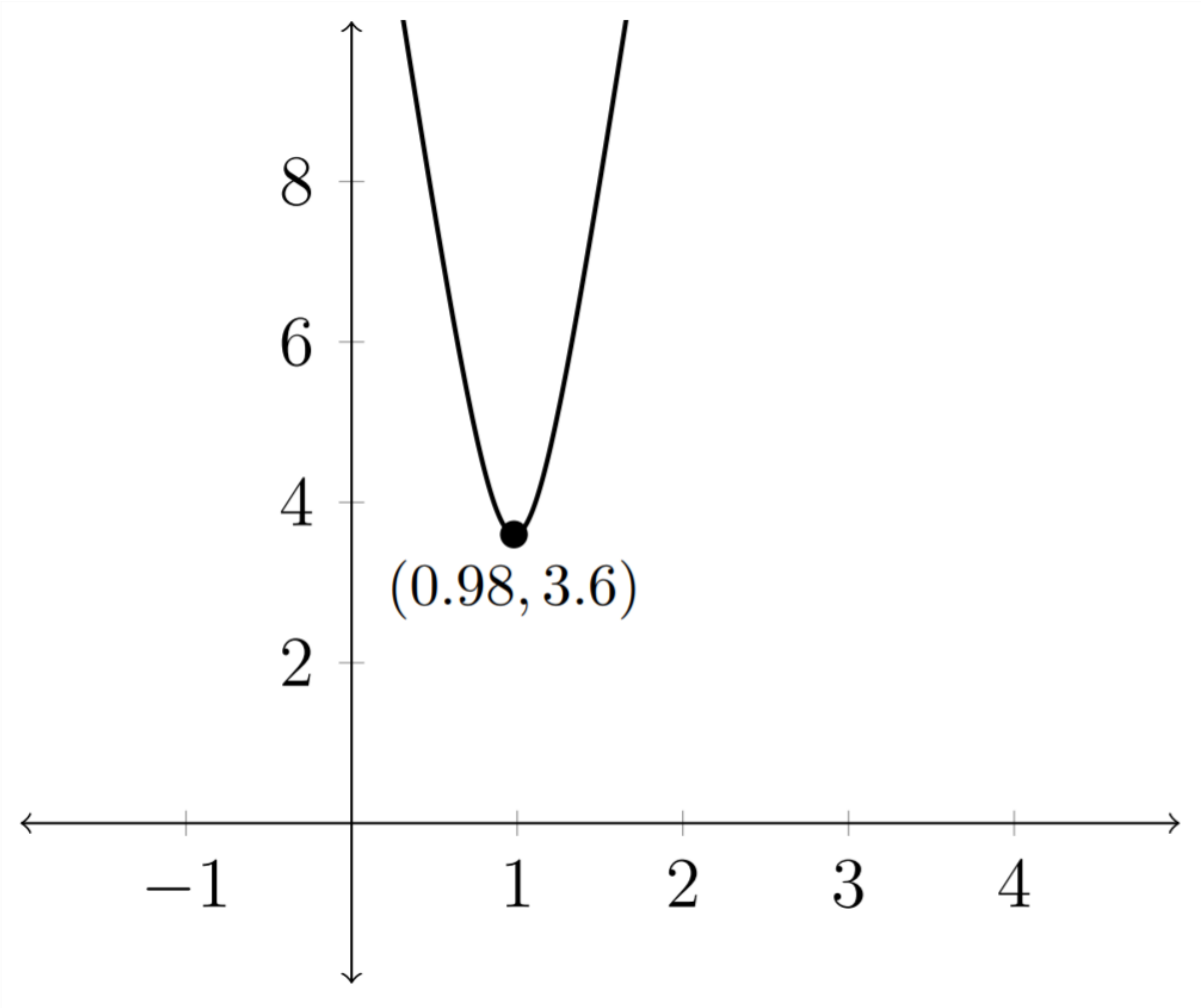
So, using the Pythagorean Theorem, we can represent the distance between the ships at any given time as:

$$d^2 = (10-7t)^2 + (10-12t)^2 \tag{4.9.13}$$

or

$$d = \sqrt{(10-7t)^2 + (10-12t)^2} \tag{4.9.14}$$

The graph of this distance function looks like this:



The minimum point on the distance graph indicates the time at which the ships are closest together. After traveling for 0.98 hours, the ships will be about 3.6 miles apart. They will then start moving farther apart.

Exercises $4.9(c)$

1) A runner starts running north from a given point running at 5 meters per second. At the same time, a second runner heads east from the same starting point, running at 8 meters per second. Represent the distance between the runners as a function of t, the elapsed time. How long will it take for the runners to be 150 meters apart?

2) A runner starts running north from a given point running at 10 meters per second. At the same time, a second runner heads east from the same starting point, running at 7 meters per second. Represent the distance between the runners as a function of t, the elapsed time. How long will it take for the runners to be 200 meters apart?

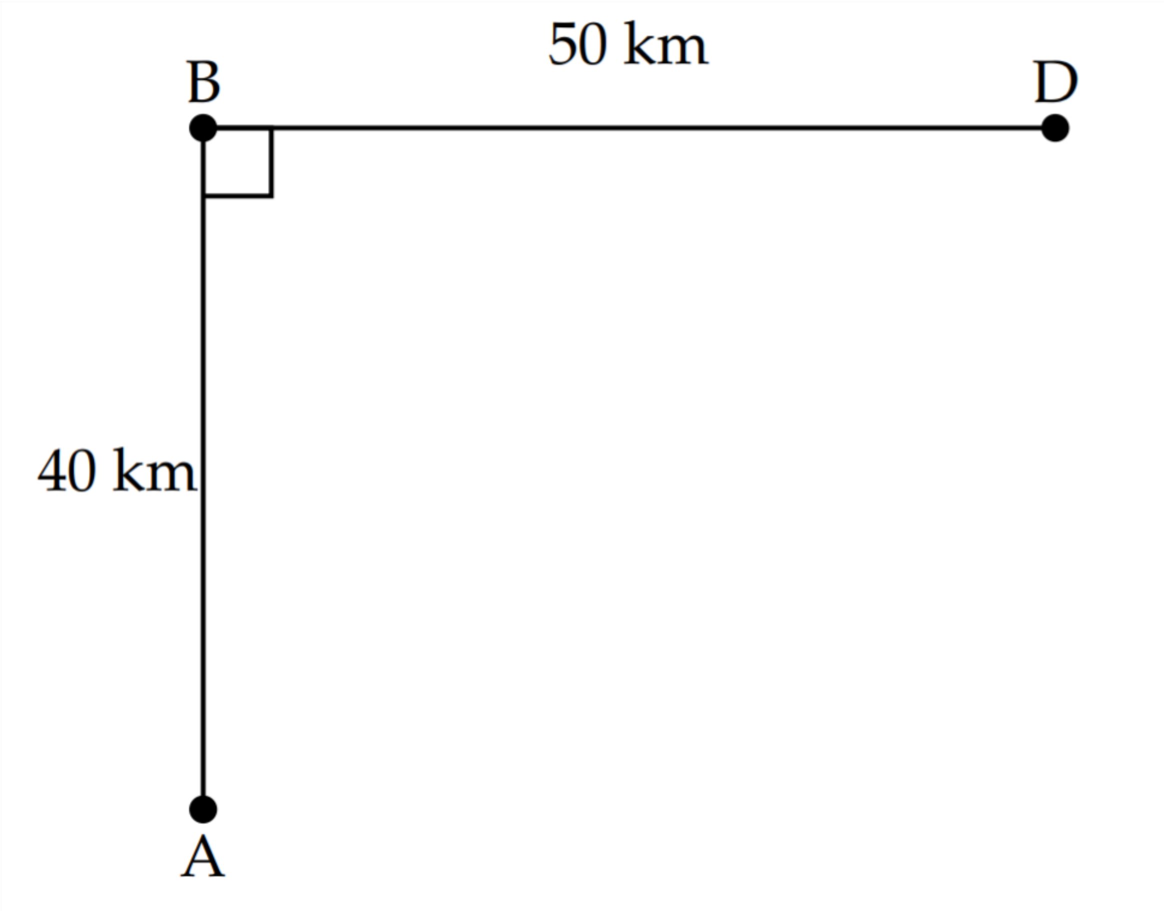
3) At 1pm a bicycle is 9 miles due north of an ice cream shop and is traveling south (towards the ice cream shop) at 15 mph. At the same time, another bike rider is 2 miles east of the ice cream shop and is traveling east (away from the ice cream shop) at 12 mph. Represent the distance between the bike riders at any given time after 1pm. At what time is the distance between the riders at a minimum? How far apart are they at this point?

4) A car leaves a four-way intersection at 9: 15 am, traveling south at 70 mph. At the same time, another car is 65 miles west of the intersection traveling east at 55 mph. Express the distance between the two cars as a function of the time since they left the intersection. At what time are the cars closest together? How far apart from each other are they at this time?

At Noon, a dune buggy is in the desert at the point A indicated below. Point A is 40km from a paved road, which covers the 50km between points B and D

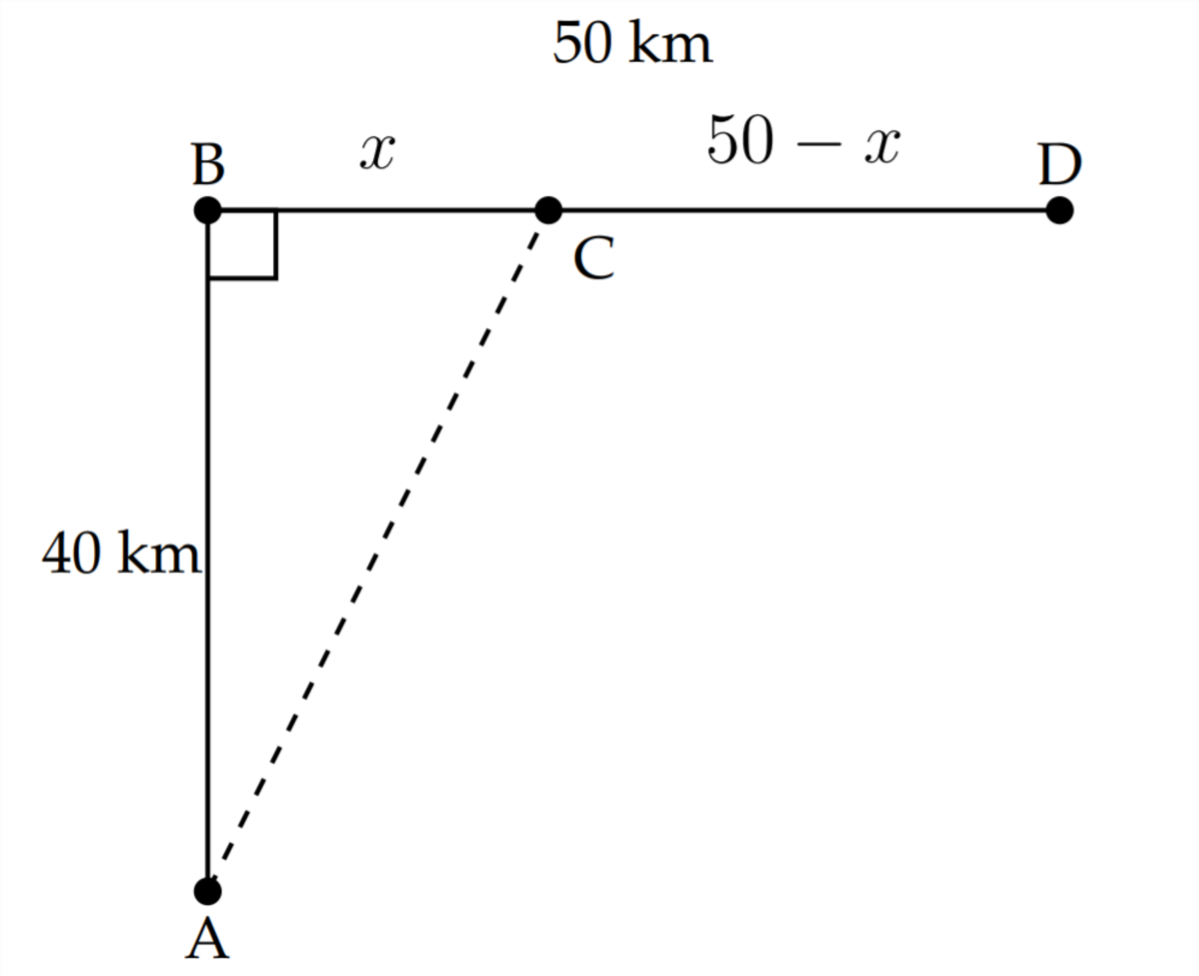
If the dune buggy can travel 45 kph in the desert and 75 kph on the road, where is the best place for the driver to leave the desert and turn onto the road in order to minimize the travel time?

What time does the driver arrive at Point D assuming they follow the path for minimal travel time?



One option is that the dune buggy can travel in a straight line from A to B (at 45 kph) and then travel on the paved road from B to D (at $75\mathrm{kph}$). This would take $1.5\mathrm{hrs}$
Another option is for the dune buggy to travel straight from A to D at 45 kph. The distance here is about $64\mathrm{km}$ (found using the Pythagorean Theorem). That means the travel time would be about 1.423 hrs.

The optimization here says that, if the dune buggy were to travel part of the way in the desert (along a diagonal) and the rest of the way on the paved road, then there might be a minimal time that is shorter than the 1.423 hrs. it would take going straight from A to D



If we express the distance between B and C as x, then the distance between C and D will be $50-x$. The distance travelled in the desert is the distance from A to C and this is found using the Pythagorean Theorem.

$$\overline{AC}=\sqrt{40^2+x^2}=\sqrt{x^2+1600} \tag{4.9.15}$$

since $d=r*t,$ then the time it takes travel a given distance will be $\frac{d}{r}=t$
In the case of this problem the time to travel from A to C is:

$$t_1=\frac{\sqrt{x^2+1600}}{45} \tag{4.9.16}$$

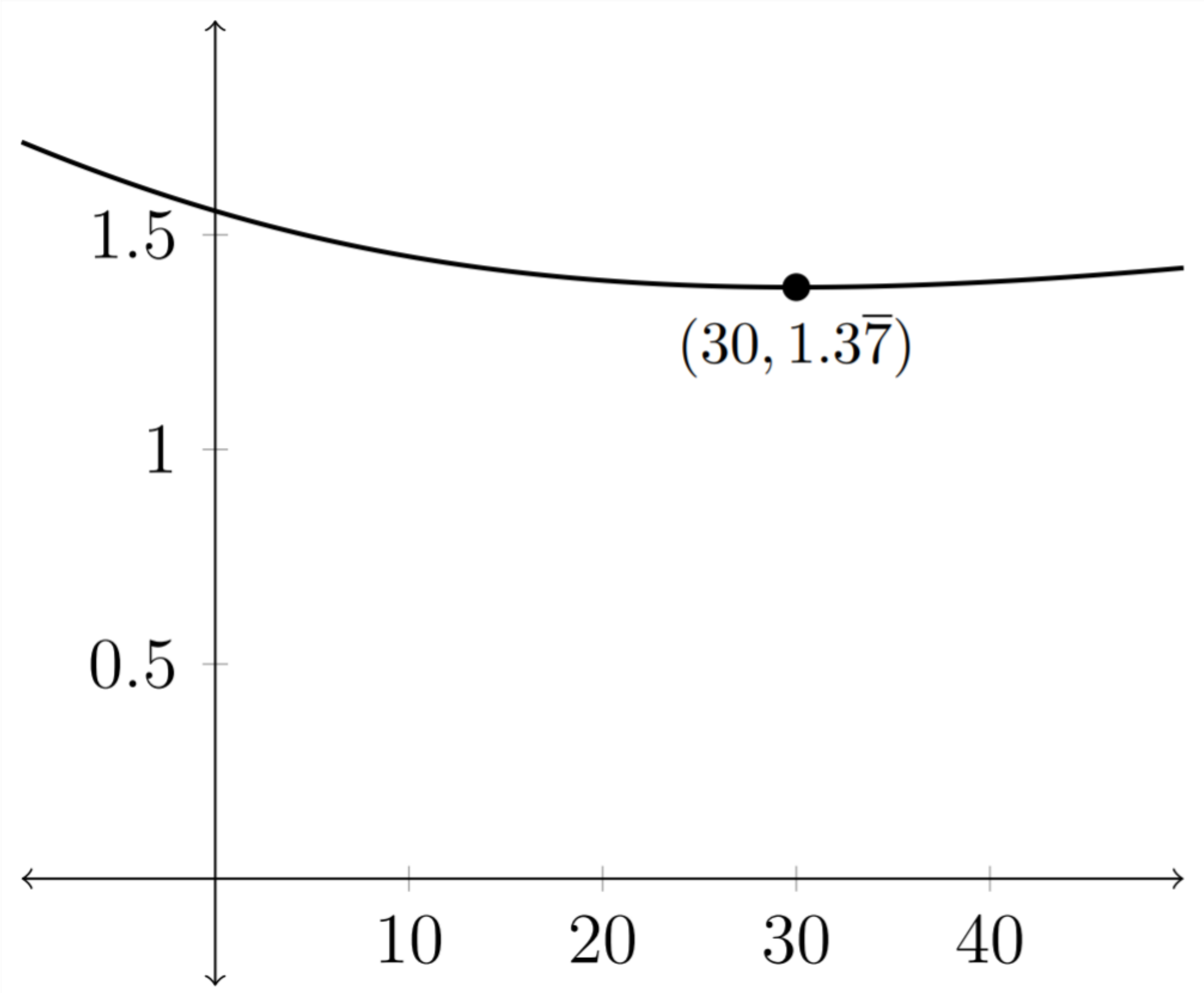
The time to travel from C to D is:

$$t_2=\frac{50-x}{75} \tag{4.9.17}$$

So, the total travel time will be:

$$t=t_1+t_2=\frac{x^2+1600}{45}+\frac{50-x}{75} \tag{4.9.18}$$

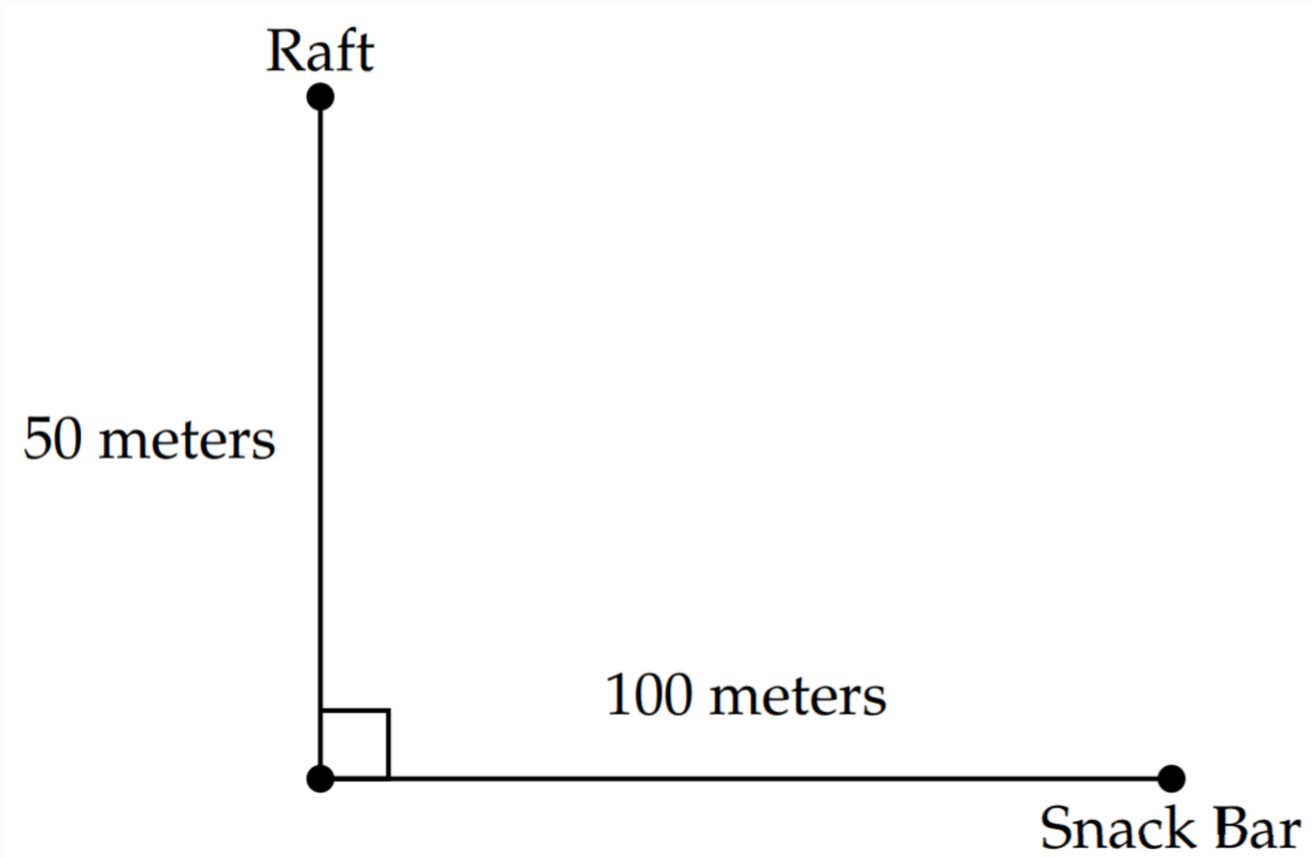
The graph of this function is below:



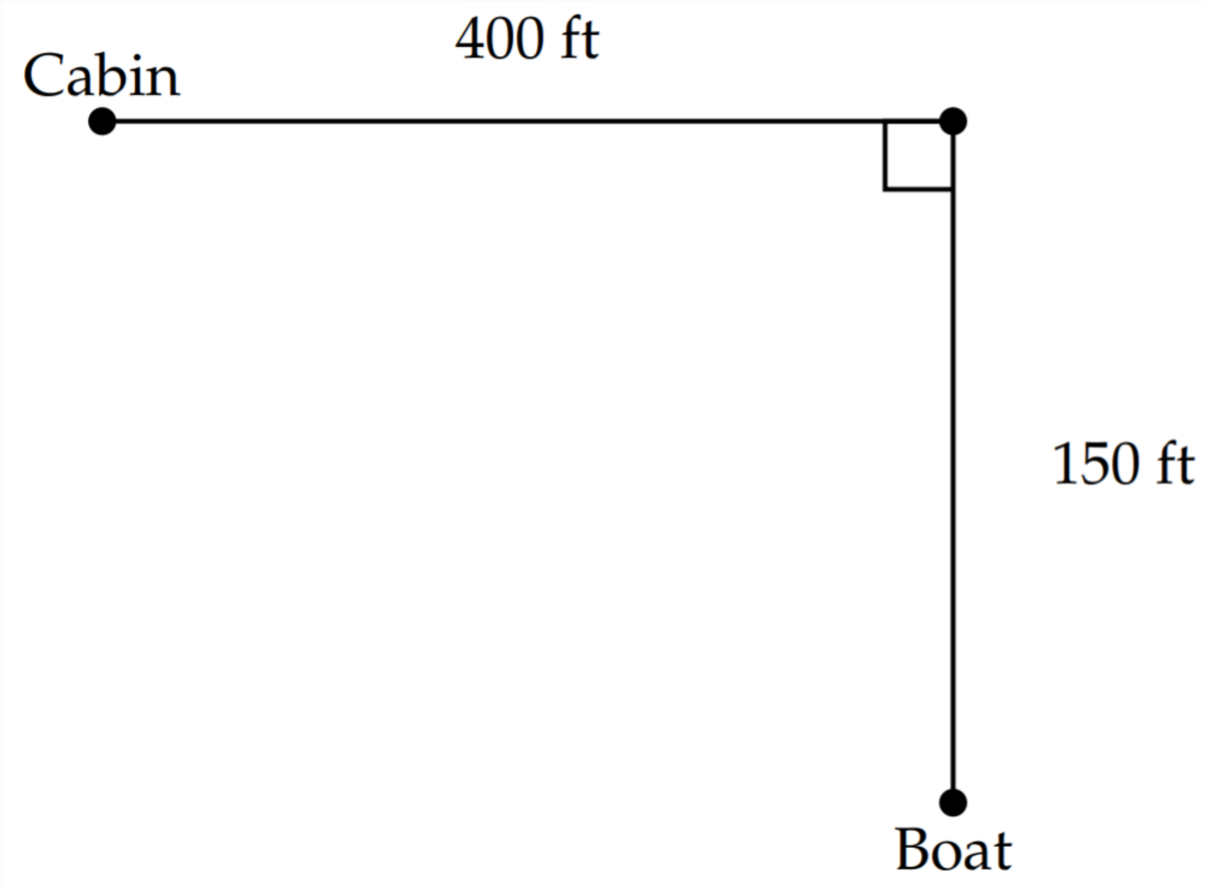
It is hard to see, but there is a minimum for the graph where $x = 30$ and $t = 1.3\bar{7}$

Exercises $4.9(d)$

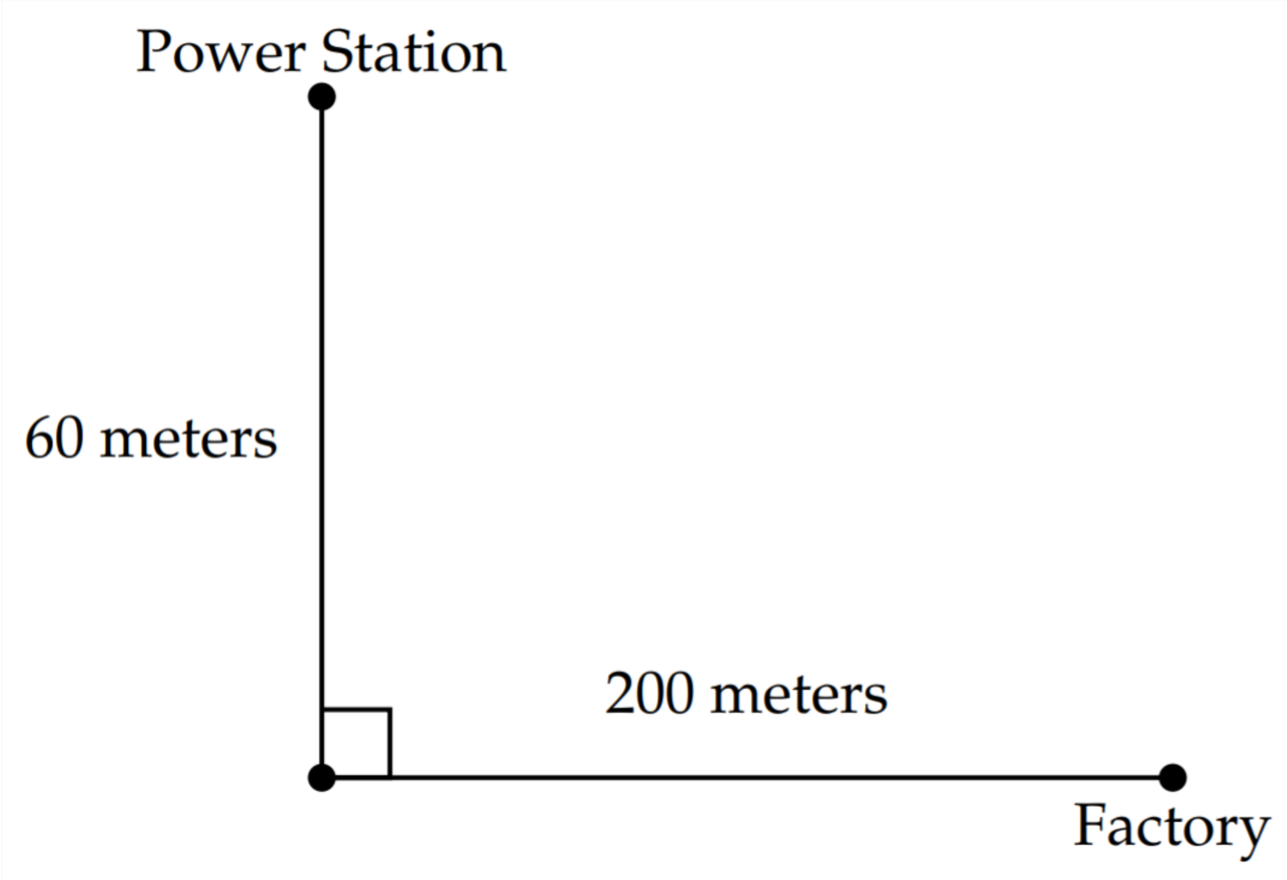
1) A woman wants to swim to shore from a raft that is 50 meters offshore and then run to the snack bar located 100 meters down the shoreline. The woman swims 1 meter per second and runs at 5 meters per second. Where on the shoreline should she swim to minimize her travel time?



2) A man is in a boat on a lake 150ft from shore. He wants to return to his cabin. He is 400ft down the shoreline from the cabin. He plans to row to shore, leave the boat where he lands and then run the rest of the way to the cabin. If he rows 7 ft per second and runs 15 ft per second, where should he beach the rowboat to minimize the travel time?

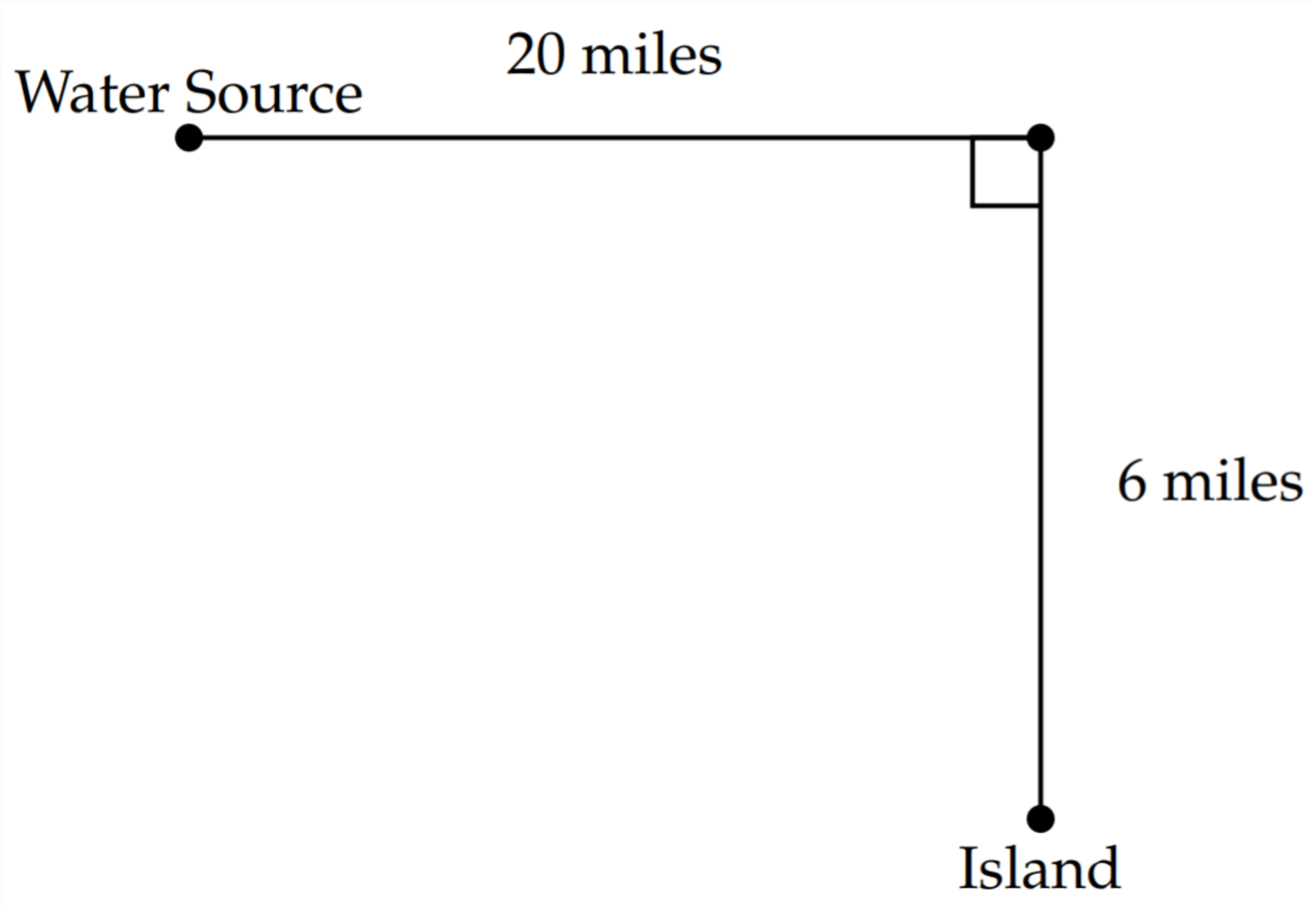


3) A power station and a factory are located on opposite sides of a river that is
60 meters wide. The factory is 200 meters from the point on the shore directly opposite the power station. A power line must be laid between the power station and the factory. It costs $\$25$ per meter to run the cable in the river and $\$20$ per meter to run the cable on the land. How much of the cable should be laid in the river and how much should be laid on land to minimize the cost? Where should the cable come ashore? What is the minimum cost?



4) A freshwater pipeline is being constructed to an island that is 6 miles from the mainland. The water source is 20 miles down the coast from the point on the shore that is closest to the island. The cost of laying the pipeline along the shore is $\$8,000$ per mile while the cost of laying the pipeline in the water is $\$12,000$ per mile. How much of the pipeline should be laid in the water and

how much should be laid on land to minimize the cost? Where should the pipeline leave the shore? What is the minimum cost?



CHAPTER OVERVIEW

5: Conic Sections - Circle and Parabola

The conic sections as originally conceived in ancient Greece were "slices" of two cones tip-to-tip. Slicing this figure in different ways produces each of the four conic sections - Circle, Ellipse, Parabola and Hyperbola. This chapter will examine the Circle and the Parabola.

5.1: The Equation of the Circle

The equation for a circle is typically given as:

$$(x-h)^2+(y-k)^2=r^2 \tag{5.1.1}$$

In this equation, the point (h, k) represents the center of the circle and r represents the radius of the circle. This equation is derived from the distance formula. The definition of a circle is the locus (or collection) of points that are equidistant from a given point (the center of the circle).

So, taking this definition, we can say that any point (x, y) that is on the circle should be a distance of r from the center. Using the distance formula:

$$d=\sqrt{\left(x_1-x_2\right)^2+\left(y_1-y_2\right)^2} \tag{5.1.2}$$

Squaring both sides:

$$d^2=\left(x_1-x_2\right)^2+\left(y_1-y_2\right)^2 \tag{5.1.3}$$

In this situation, the distance d is the radius of the circle, r. This distance should be the same for all points on the circle. So any point on the edge of the circle, (x, y) should be a distance of r from the center of the circle, (h, k)

$$(x-h)^2+(y-k)^2=r^2 \tag{5.1.4}$$

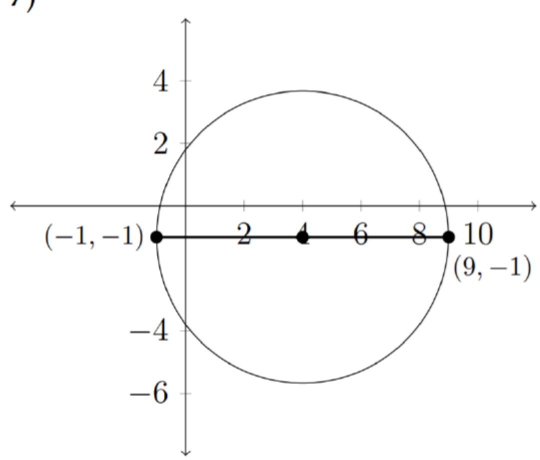
Exercises $5.1.1$

Determine the center and radius for each circle and sketch the graph of the circle.

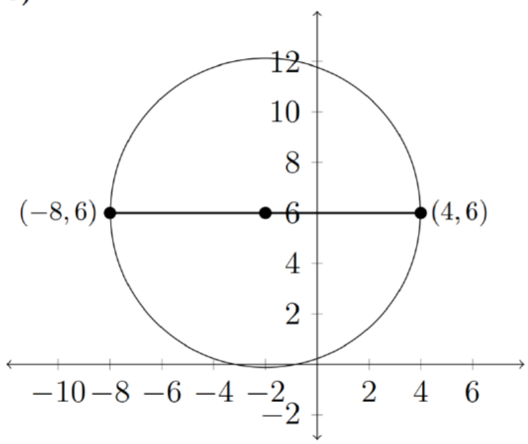
1) $(x-3)^2+(y+1)^2=9$
2) $(x+4)^2+(y+2)^2=16$
3) $(x-1)^2+(y-6)^2=20$
4) $(x-2)^2+(y+5)^2=27$
5) $(x+7)^2+(y-4)^2=33$
6) $(x-5)^2+(y-3)^2=50$

Determine the equation for each circle. Each line represents a diameter

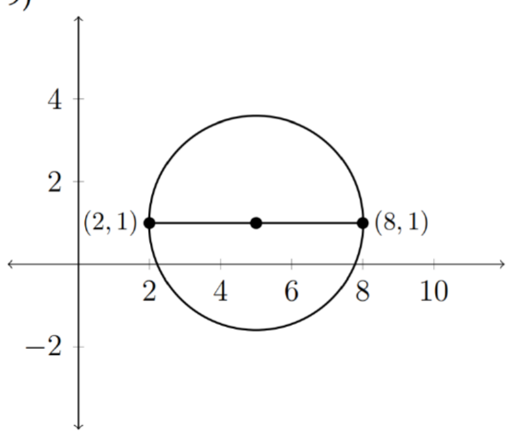
7)



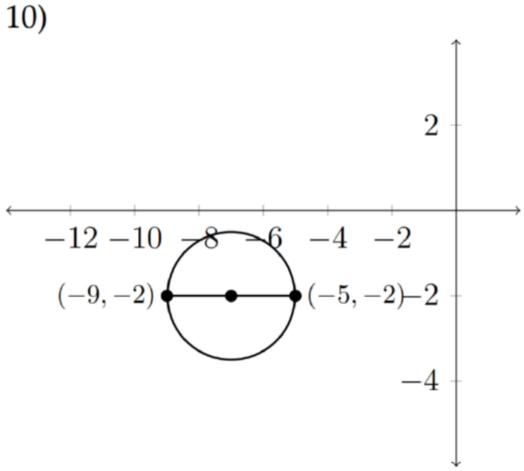
8)



9)



10)



Sometimes the equation for a circle is not given in the standard form. In this situation, you need to put the equation into standard form and then determine the center and radius. In order to put the equation into standard form you will need to complete the square. Completing the square is a mathematical technique that is often useful and is the basis for how the quadratic formula is derived.

Suppose that we are given the equation of a circle that is not in standard form:

$$x^2+6x+y^2+10y+25=0 \tag{5.1.5}$$

We need to restate this relationship so that the center and radius can be easily determined from the equation. In order to do this, we need to complete the square for both the x and the y variables. There are a variety of methods for completing the square I will demonstrate one of these below.

Take the original equation and move any term that doesn't have a variable to the other side:

$$x^2+6x+y^2+10y=-25 \tag{5.1.6}$$

Then open a space after the $6x$ and the $10y$:

$$x^2+6x+y^2+10y=-25 \tag{5.1.7}$$

The idea is that we want trinomial expressions for both x and y that are perfect squares.
If we look at squaring binomial expressions, we can see that there is a pattern:

$$\begin{gathered}(x+3)^2=(x+3)(x+3)=x^2+6x+9\\(x+4)^2=(x+4)(x+4)=x^2+8x+16\\(x+5)^2=(x+5)(x+5)=x^2+10x+25\\(x+6)^2=(x+6)(x+6)=x^2+12x+36\end{gathered} \tag{5.1.8}$$

We can see that if we have x^2+6x, then this would correspond to $(x+3)(x+3)$ or $(x+3)^2$. But there's a problem: $(x+3)^2$ is not equal to x^2+6x. It's equal to x^2+6x+9. We can't just add a 9 to the x^2+6x, but we can add a 9 to both sides.

$$x^2+6x+9+y^2+10y=-25+9 \tag{5.1.9}$$

Similarly, to complete the square on the y, we see that y^2+10y corresponds to $(x+5)(x+5)$ or $(x+5)^2$. Here, we would need to add 25 to both sides to create a perfect square.

$$\begin{aligned}x^2+6x+9+y^2+10y+25&=-25+9+25\\x^2+6x+9+y^2+10y+25&=9\\(x+3)^2+(y+5)^2&=9\end{aligned}$$

So, the center of the circle is (-3,-5) and the radius is 3
Sometimes it is not so obvious what the values of h and k should be in completing the square. Consider the equation below:

$$x^2+20x+y^2+30y+15=0 \tag{5.1.10}$$

If we look back at the examples for squaring binomials, we can see the pattern that relates the coefficient of the linear term to the values for h and k

$$\begin{gathered}(x+3)^2=(x+3)(x+3)=x^2+6x+9\\(x+4)^2=(x+4)(x+4)=x^2+8x+16\\(x+5)^2=(x+5)(x+5)=x^2+10x+25\\(x+6)^2=(x+6)(x+6)=x^2+12x+36\end{gathered} \tag{5.1.11}$$

Notice that the coefficient of the linear term is always double the value of the numeral in the parentheses and the constant term is always that number squared.

$$(x+n)^2=(x+n)(x+n)=x^2+2nx+n^2 \tag{5.1.12}$$

So, given the problem above, $x^2+20x+y^2+30y+15=0,$ first we can move the
15 to the right hand side of the equation and then complete the squares on the x and y variables.

$$\begin{gathered}x^2+20x+y^2+30y=-15\\(x+?)^2+(y+?)^2\\x^2+20x+100+y^2+30y+225=-15+100+225\\(x+10)^2+(y+15)^2=310\end{gathered} \tag{5.1.13}$$

We can now see that the center of the circle is at the point (-10,-15) and the radius of the circle is $\sqrt{310}\approx 17.6$

Exercises $5.1.2$

Find the center and radius for each circle.

If the equation is not written in standard form, complete the square to rewrite the equation in standard form: $(x-h)^2+(y-k)^2=r^2$

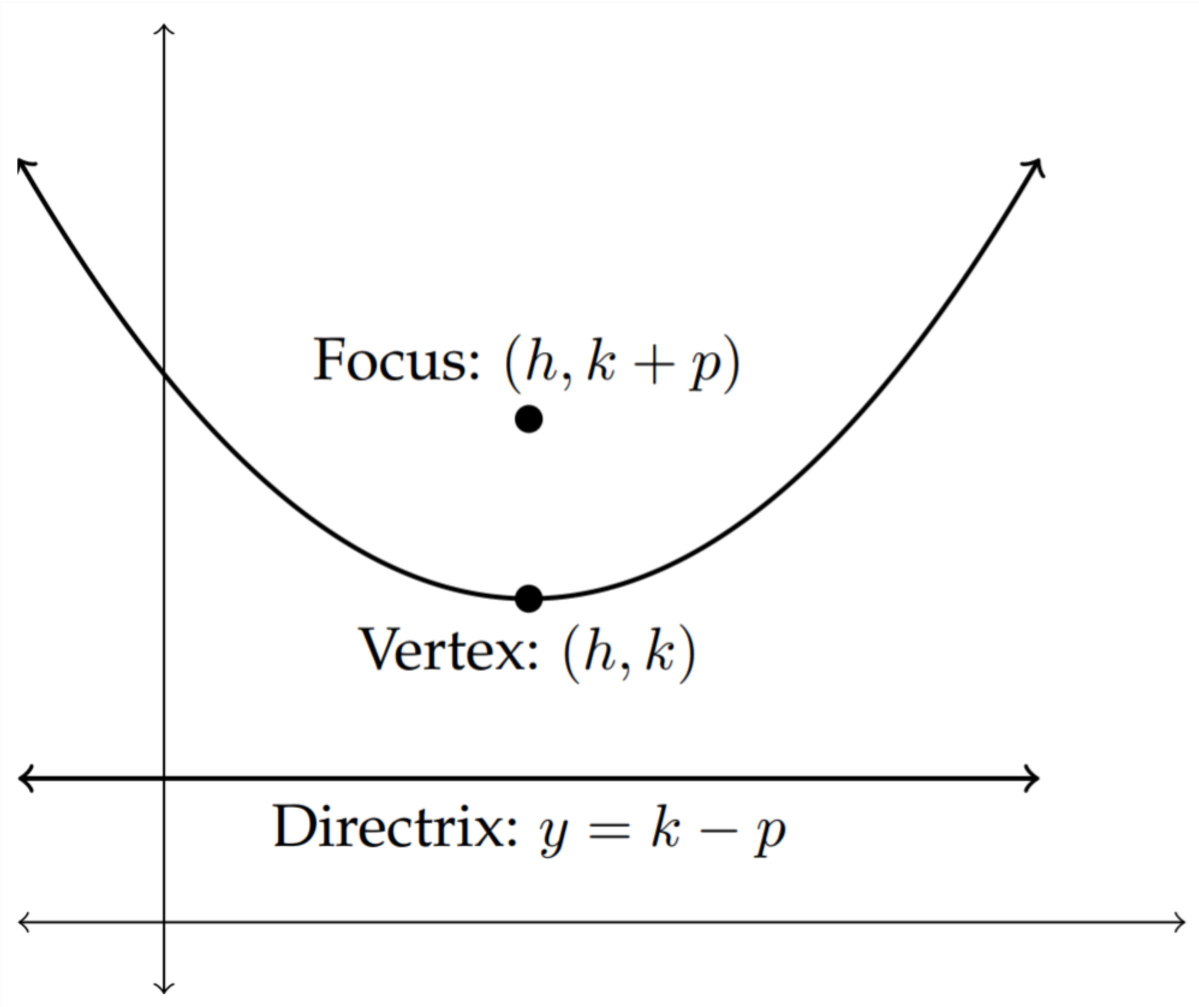
1. $x^2+y^2=4$
2. 2) $x^2+(y-1)^2=1$
3. 3) $2(x-3)^2+2y^2=8$
4. 4) $3(x+1)^2+3(y-1)^2=6$
5. 5) $x^2+y^2-2x-4y-4=0$
6. 6) $x^2+y^2+4x+2y-20=0$
7. 7) $x^2+y^2+4x-4y-1=0$
8. 8) $x^2+y^2-6x+2y+9=0$
9. 9) $2x^2+2y^2+8x+7=0$
10. 10) $3x^2+3y^2-12y=0$
11. 11) $2x^2+2y^2-12x+8y-24=0$
12. 12) $2x^2+8x+2y^2=0$
13. 13) $x^2+y^2+15x=0$
14. 14) $x^2+y^2+x+y-\frac{1}{2}=0$
15. 15) $x^2+y^2-x+2y+1=0$
16. 16) $3x^2+3y^2-6x+3y=4$
17. 17) $2x^2+2y^2-10x-18y=1$
18. 18) $2x^2+2y^2-5x+y=0$
19. 19) $9x^2+9y^2+9x-27y+1=0$
20. 20) $2x^2+2y^2-7x+y=0$

5.2: The Equation of the Parabola

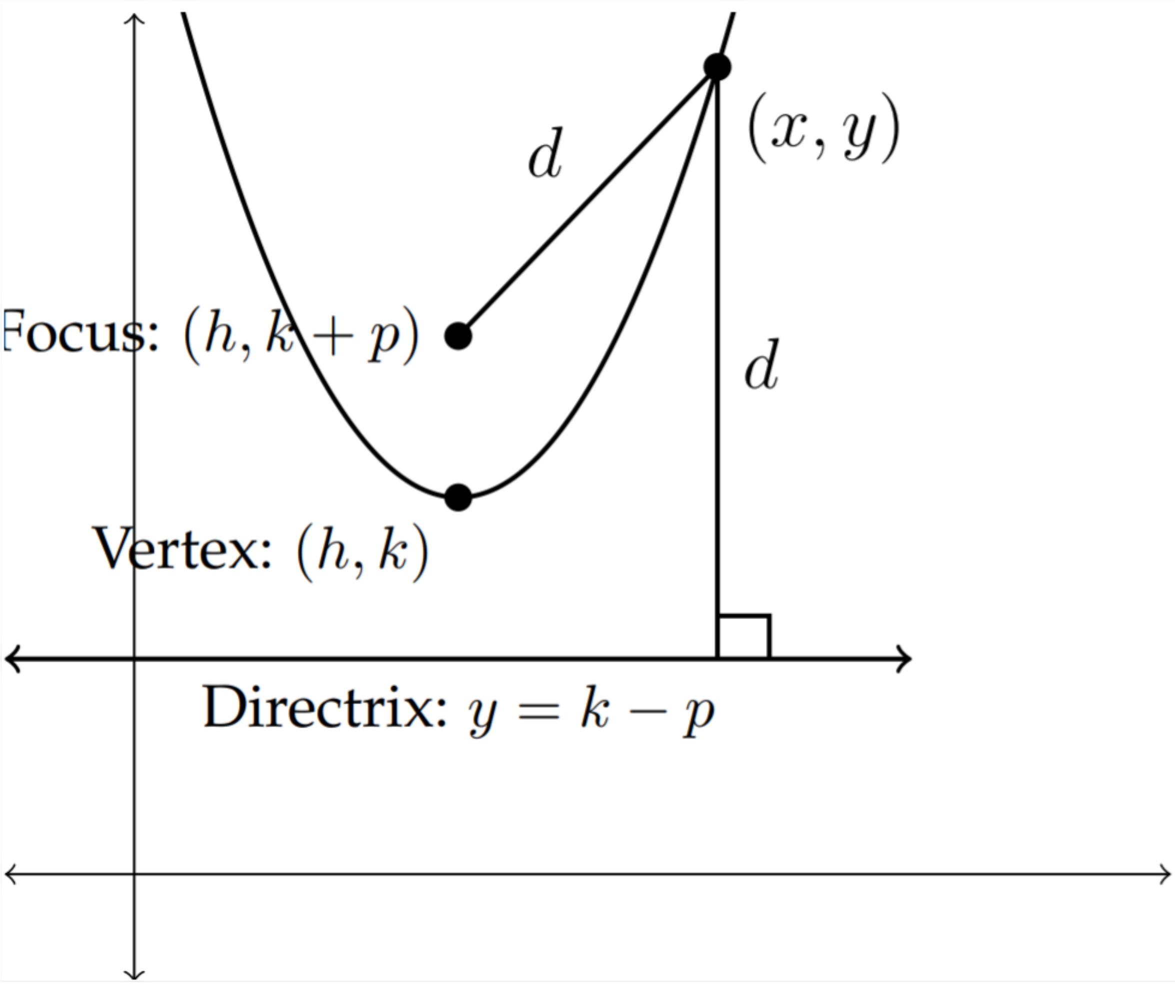
The equation of the parabola is often given in a number of different forms. One of the simplest of these forms is:

$$(x-h)^2 = 4p(y-k) \tag{5.2.1}$$

A parabola is defined as the locus (or collection) of points equidistant from a given point (the focus) and a given line (the directrix). Another important point is the vertex or turning point of the parabola. If the equation of a parabola is given in standard form then the vertex will be (h, k). The focus will be a distance of p units from the vertex within the curve of the parabola and the directrix will be a distance of p units from the vertex outside the curve of the parabola. This value (p) is called the focal distance.



Any point on the curve of the parabola is equidistant from the focus $(h, k+p)$ and the directrix $(h, k-p)$. Notice that the focus is a point and is identified with the coordinates of the point while the directrix is a line and is identified with the equation for that line.



We can derive the standard equation for a parabola using the distance formula.

In the picture above the two distances labeled " d'' should be the same distance. The vertical distance between the point (x, y) and the directrix $y = k - p$ is simply the difference between their y coordinates:

$$d = y - (k - p) \tag{5.2.2}$$

To find the distance between the point (x, y) and the focus $(h, k + p)$ we need to use the distance formula:

$$d = \sqrt{(x-h)^2 + (y-(k+p))^2} \tag{5.2.3}$$

Then we set the two distances equal to each other:

$$\sqrt{(x-h)^2 + (y-(k+p))^2} = y - (k - p) \tag{5.2.4}$$

Square both sides:

$$(x-h)^2 + (y-(k+p))^2 = (y-(k-p))^2 \tag{5.2.5}$$

We'll need to expand each side and collect like terms, but we'll leave the $(x - h)^2$ alone because it will appear in this form in the final equation.

$$(x-h)^2+y^2-2(k+p)y+(k+p)^2=y^2-2(k-p)y+(k-p)^2 \tag{5.2.6}$$

The y^2 terms cancel each other out:

$$(x-h)^2-2(k+p)y+(k+p)^2=-2(k-p)y+(k-p)^2 \tag{5.2.7}$$

Now we'll expand the k, p and y terms:

$$(x-h)^2-2ky-2py+k^2+2pk+p^2=-2ky+2py+k^2-2pk+p^2 \tag{5.2.8}$$

Here, the k^2 and p^2 terms cancel out, as do the $-2ky$ terms leaving:

$$(x-h)^2-2py+2pk=2py-2pk \tag{5.2.9}$$

If we collect everything excep the $(x-h)^2$ on the right hand side, we'll have:

$$(x-h)^2=4py-4pk \tag{5.2.10}$$

Factor out $4p$ and we have the standard equation for a parabola:

$$(x-h)^2=4p(y-k) \tag{5.2.11}$$

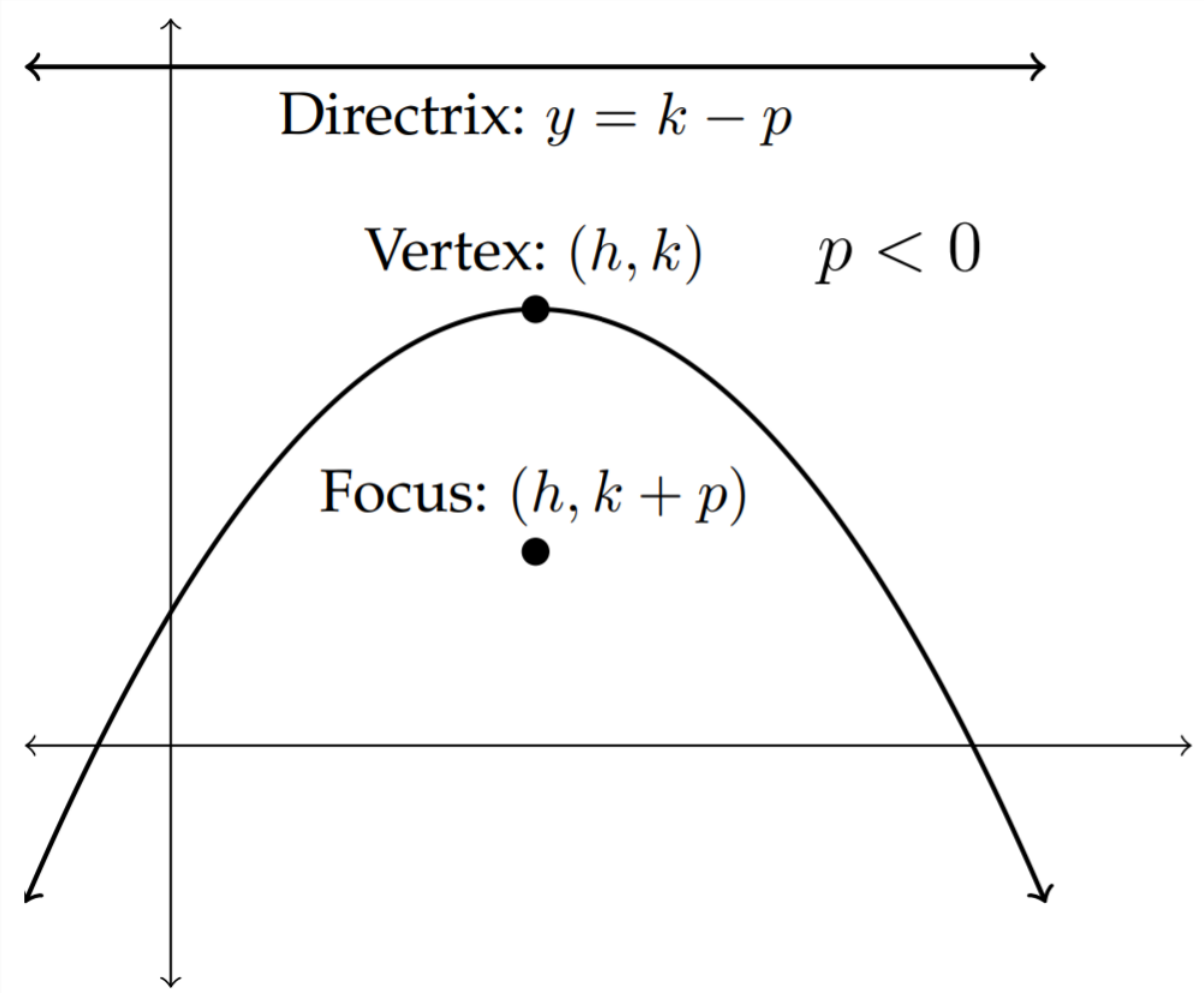
This equation will be different depending on the orientation of the parabola. An upward facing parabola will have this standard equation and both sides will have the same sign. For example, $(x-5)^2=12(y-1)$ is an upward facing parabola, as is $-(x-5)^2=-12(y-1)$. You can see that this is the same equation, but has been multiplied by -1 on both sides.
A parabola in which one side is positive and one side negative, like

$$(x+1)^2=-8(y-10) \tag{5.2.12}$$

is a downward facing parabola. This form might also appear as

$$-(x+1)^2=8(y-10) \tag{5.2.13}$$

The picture below illustrates this situation:

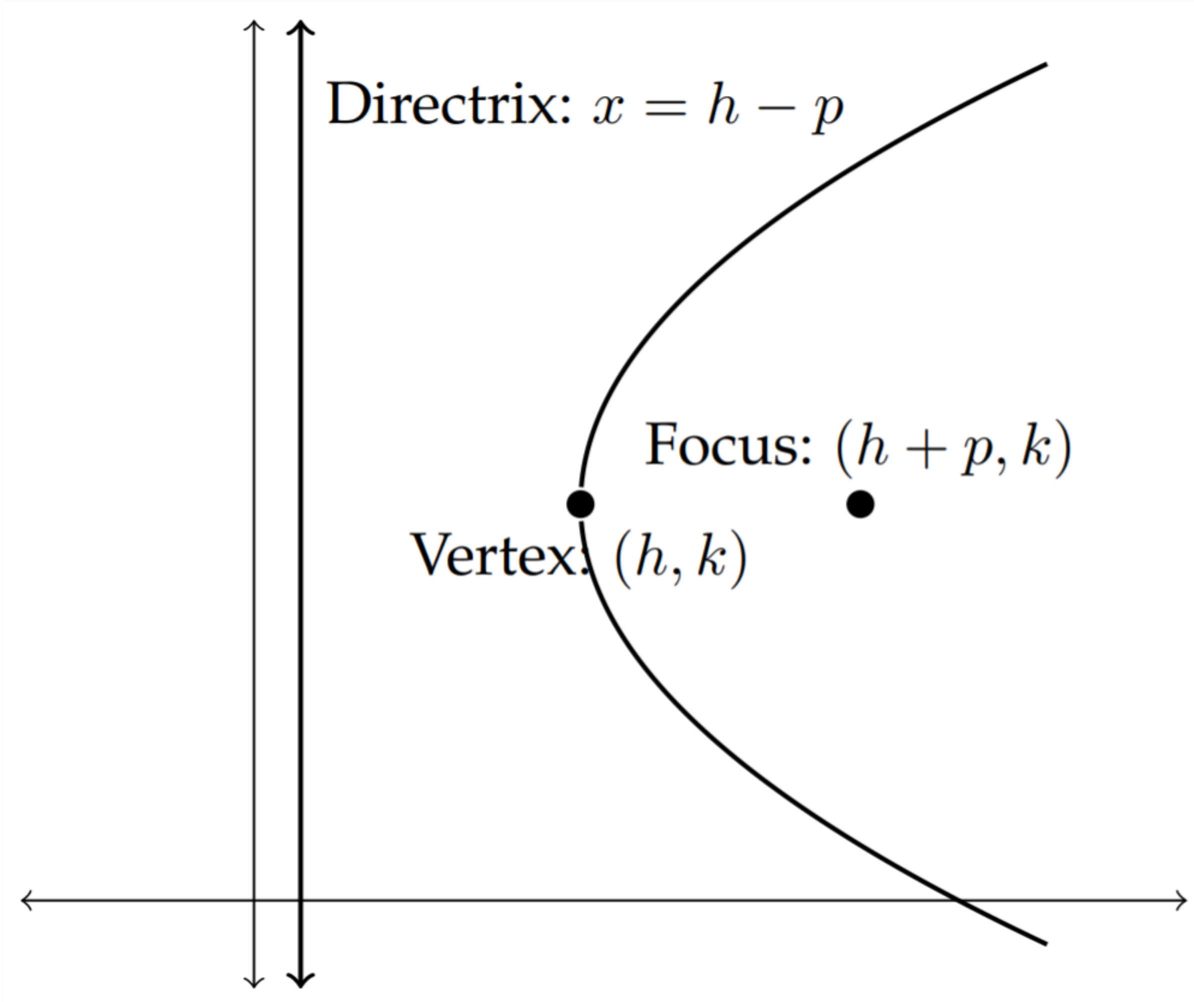


Notice that, in this case, the focus is below the vertex and the directrix is above it. Remember that the focus is always in the interior of the curve, while the directrix is always outside of the curve.

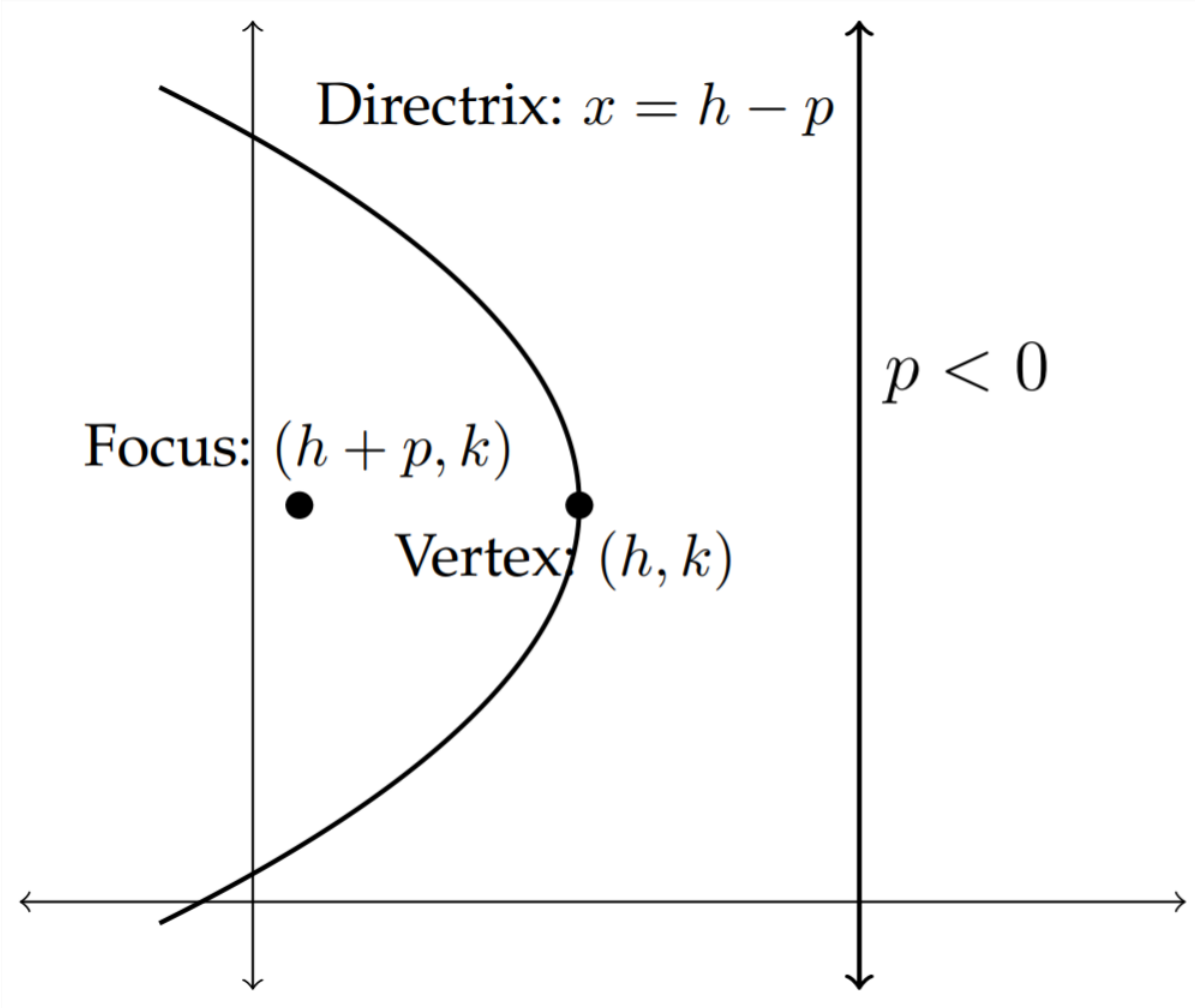
Parabolas can also open to the right or left. In these cases, it is the y variable that is squared in the standard equation:

$$(y-k)^2 = 4p(x-h) \tag{5.2.14}$$

In these situations, the focus is to the left or right of the vertex. If both sides of the equation have the same sign, such as $(y-5)^2 = 12(x-1)$ or $-(y-5)^2 = -12(x-1)$ then this would be a parabola opening to the right.



If the two sides have opposite signs, then the parabola will open to the left



The orientation of the vertex, focus and directrix allows us to determine the equation of a parabola if we are given certain pieces of information about the vertex, focus and directrix. The vertex always falls halfway between the focus and directrix. The key pieces of information in determining the equation of a parabola are:

1) the vertex: this gives us the values for h and k for the equation.

2) the orientation: this allows us to determine the appropriate form of the equation

a) $(x-h)^2 = 4p(y-k)$

b) $(x-h)^2 = -4p(y-k)$

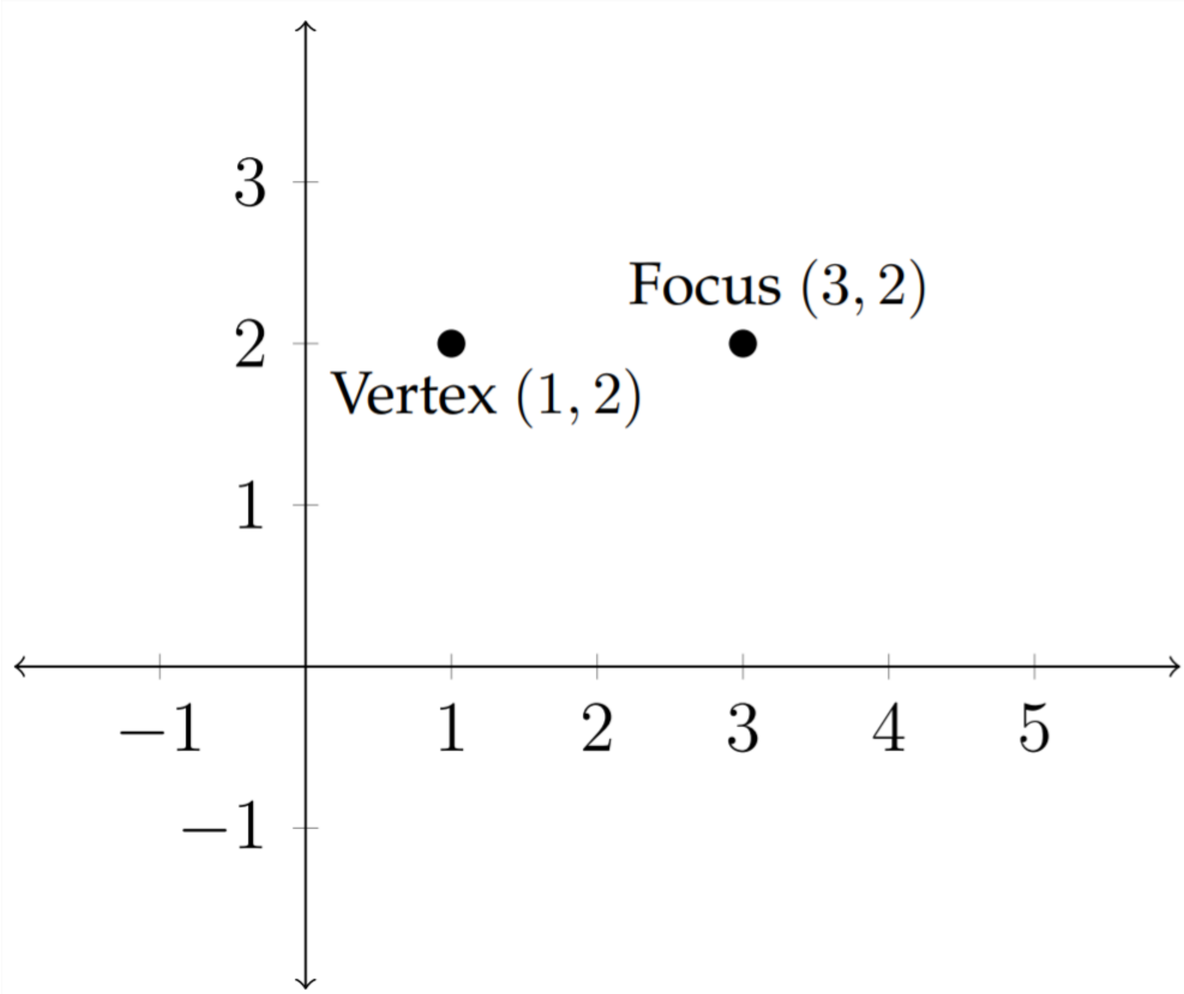
c) $(y-k)^2 = 4p(x-h)$

d) $(y-k)^2 = -4p(x-h)$

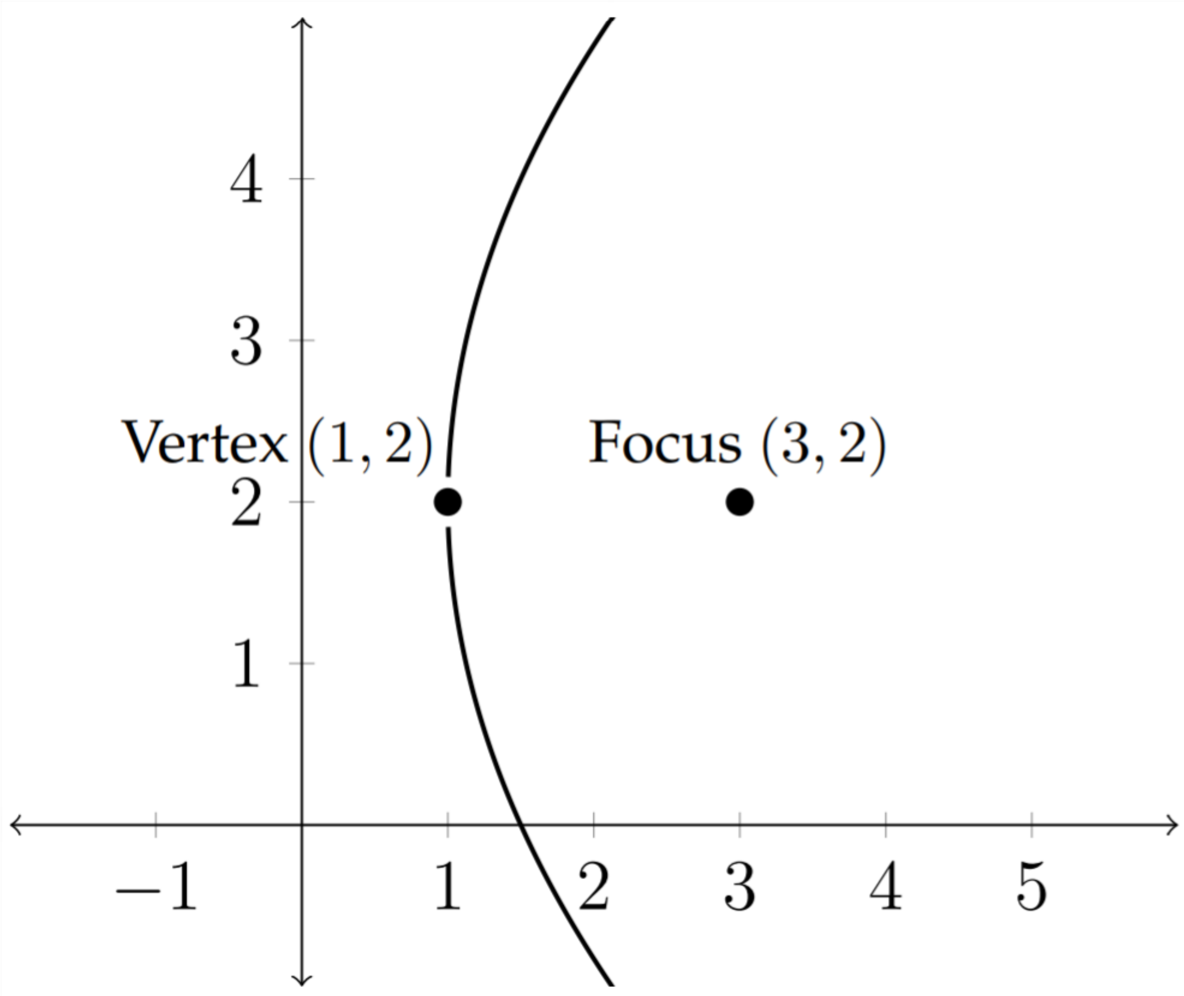
Given the following information determine an equation for the parabola described.

Focus at (3,2)   Vertex at (1,2)

First draw a little sketch of the problem:



since the focus always falls within the interior of the parabola's curve, this parabola is facing to the right. Its directrix is the line $x = -1$



The equation for this parabola would be:

$$(y-2)^2 = 8(x-1) \tag{5.2.15}$$

Exercises $5.2(a)$

Determine the equation in standard form for each of the parabolas described below.

1) Focus: (2,5) & Vertex: (2,6)
2) Focus: (-4,3) & Vertex: (-3,3)
3) Focus: (1.5,0) & Vetrex: (0,0)
4) Focus: (-9,0) & Vertex: (-9,-0.5)
5) Focus: (5,1) & Directrix: x=12
6) Focus: (0,3) & Directrix: y=1
7) Focus: (1,-3) & Directrix: y=2
8) Focus: (-4,8) & Directrix: x=-6
9) Vertex: (1,3), axis of symmetry parallel to y -axis
contains the point (4,7.5)
10) Vertex: (4,2), axis of symmetry parallel to y -axis
contains the point (5,0)
11) Focus: (0,0) & Vertex: (0,-1.25)
12) Focus: (0,0) & Vertex: (0,2.5)
13) Focus: (2,7) & Directrix: x=0.5
14) Focus: (-3,0) & Directrix: x=-2
15) Vertex: (3,-2), axis of symmetry parallel to x -axis,

contains the point (7,3)
16) Vertex: (-6,-6), axis of symmetry parallel to y -axis,
contains the point (0,0)

If we are given the equation of a parabola and need to find the vertex, focus and directrix, it is often helpful to put the equation in standard form. This usually requires completing the square.
If we're given the equation:

$$x^2+6x-4y+1=0 \tag{5.2.16}$$

then we know that this will be either an upward or downward facing parabola. Let's move everything away from the x terms but leave a space to complete the
square:

$$x^2+6x=4y-1 \tag{5.2.17}$$

To complete the square on this, we need add 9 to both sides so that we can rewrite the left side as $(x+3)^2$

$$\begin{aligned} x^2+6x+9 &= 4y-1+9 \\ (x+3)^2 &= 4y+8 \end{aligned}$$

Then we can factor out the coefficient of the y variable (even if it doesn't factor evenly).

$$(x+3)^2=4(y+2) \tag{5.2.18}$$

So, this is an upward facing parabola with the vertex at the point (-3,-2) . To find the focus and directrix, we need to know the vlaue of p. since $4p=4$, then we know that $p=1$. This means that the focus will be 1 unit above the vertex at the point (-3,-1) and the directrix will be one unit below the vertex at the line y=-3.

$$\begin{aligned} \text{Vertex:} &\quad (-3,-2) \\ \text{Focus:} &\quad (-3,-1) \\ \text{Directrix:} &\quad (y=-3) \end{aligned} \tag{5.2.19}$$

Here's another example:
Express the equation in standard form and determine the vertex, focus and directrix of the following parabola.

$$y^2-2y-1+8x=0 \tag{5.2.20}$$

We know that this parabola will be either right or left facing.
First, let's move all the terms not containing y to the right-hand side but leave space to complete the square.

$$y^2-2y=-8x+1 \tag{5.2.21}$$

Add 1 to both sides to complete the square:

$$\begin{aligned} y^2-2y+1 &= -8x+1+1 \\ (y-1)^2 &= -8x+2 \end{aligned}$$

Factor out the coefficient of the x variable:

$$(y-1)^2=-8(x-0.25) \tag{5.2.22}$$

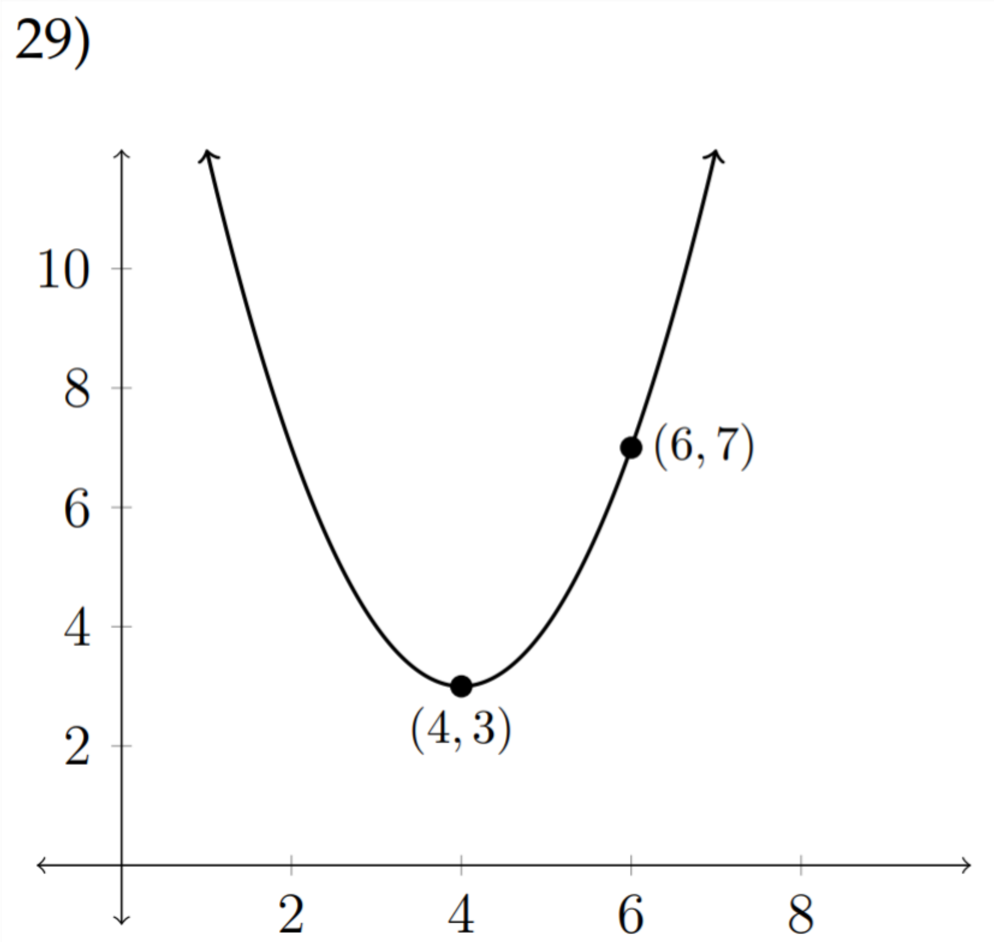
So, this is a left-facing parabola with a vertex at the point $(0.25, 1)$. To find the focal distance, we say that $4p = -8$, so $p = -2$ since it is left-facing, the focus will be a distance of 2 units to the left of the vertex at the point (-1.75,1) and the directrix will be a distance of 2 units to the right of the vertex at the line $x = 2.25$

Exercises $5.2(b)$
Express each equation in standard form and determine the vertex, focus and directrix of each parabola.
Standard form:

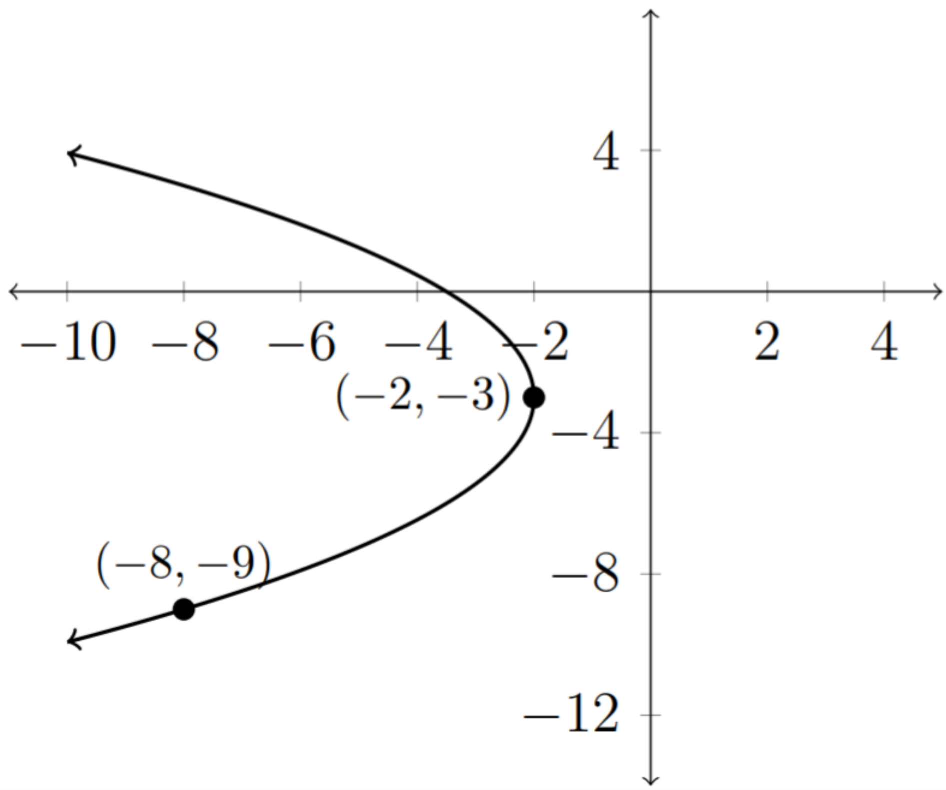
$$\begin{aligned} 4p(y-k) &= (x-h)^2 \\ 4p(x-h) &= (y-k)^2 \end{aligned} \tag{5.2.23}$$

1) $(x-2)^2 = 8(y+1)$
2) $(x+5)^2 = 12(y-3)$
3) $(y+1)^2 = 6(x-2)$
4) $(y+4)^2 = 10(x+1)$
5) $(x+3)^2 = -5(y+2)$
6) $(x-4)^2 = -7(y-6)$
7) $(y+8)^2 = -6(x+4)$
8) $(y-3)^2 = -9(x+4)$
9) $(x^2 + 8x + y + 6 = 0$
10) $x^2 + 6x + y - 3 = 0$
11) $y^2 + 6y + 8x + 1 = 0$
12) $y^2 + 8y - 4x + 8 = 0$
13) $x^2 + 4x - 3y + 7 = 0$
14) $x^2 + 2x - 6y - 11 = 0$
15) $y^2 + 6y - 4x + 4 = 0$
16) $y^2 - 4y + 3x + 9 = 0$
17) $x + y^2 - 3y + 1 = 0$
18) $10 + x + y^2 + 5y = 0$
19) $x + y^2 - 3y + 4 = 0$
20) $3x + y^2 + 8y + 4 = 0$
21) $x^2 + 3x + 3y - 1 = 0$
22) $x^2 + 5x - 4y - 1 = 0$
23) $x^2 - 8x - 4y + 3 = 0$
24) $6x - y^2 - 12y + 4 = 0$
25) $2x + 4y^2 + 8y - 5 = 0$
26) $4x^2 - 12x + 12y + 7 = 0$
27) $3x^2 - 6x - 9y + 4 = 0$
28) $2x - 3y^2 + 9y + 5 = 0$

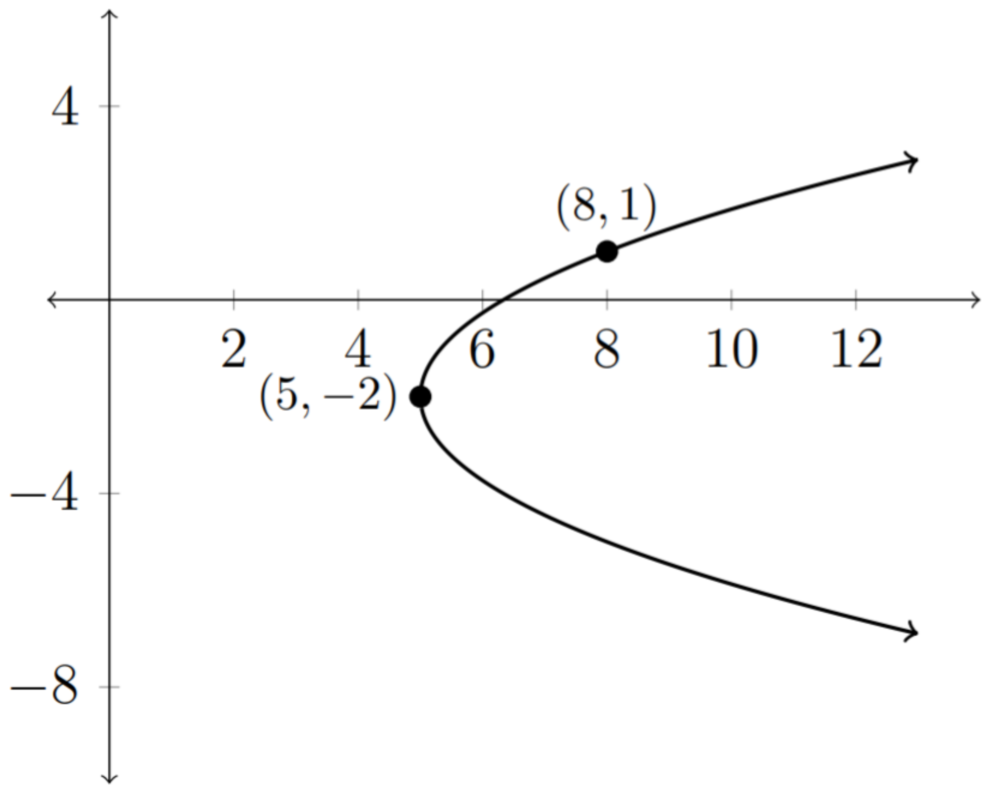
Write an equation in standard form for each parabola pictured below.



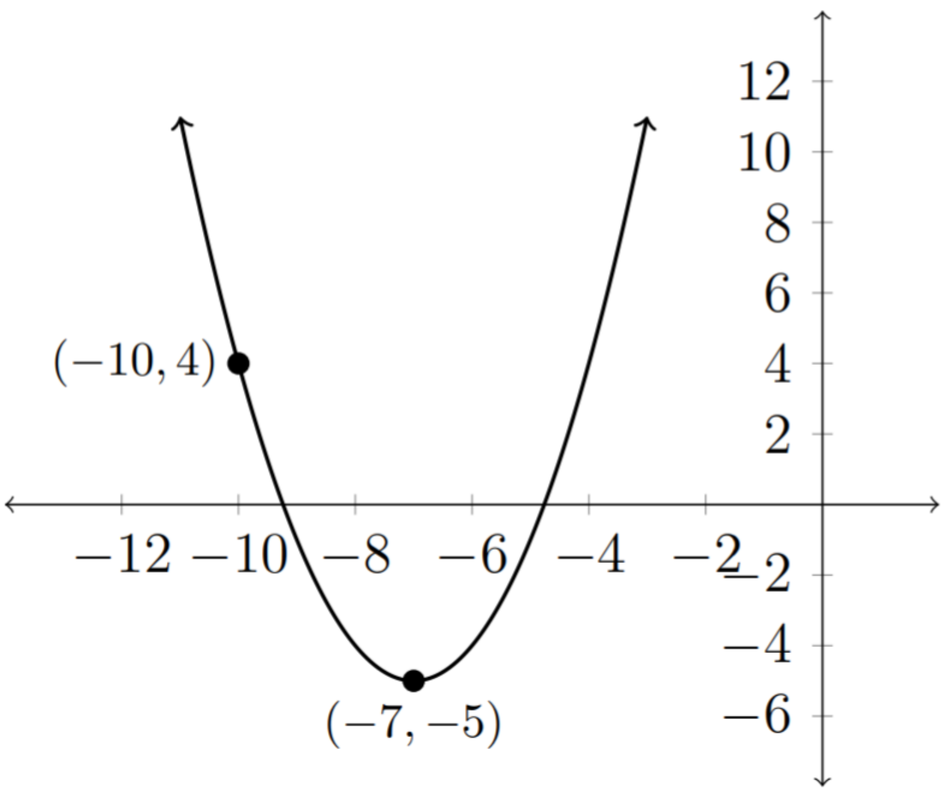
30)



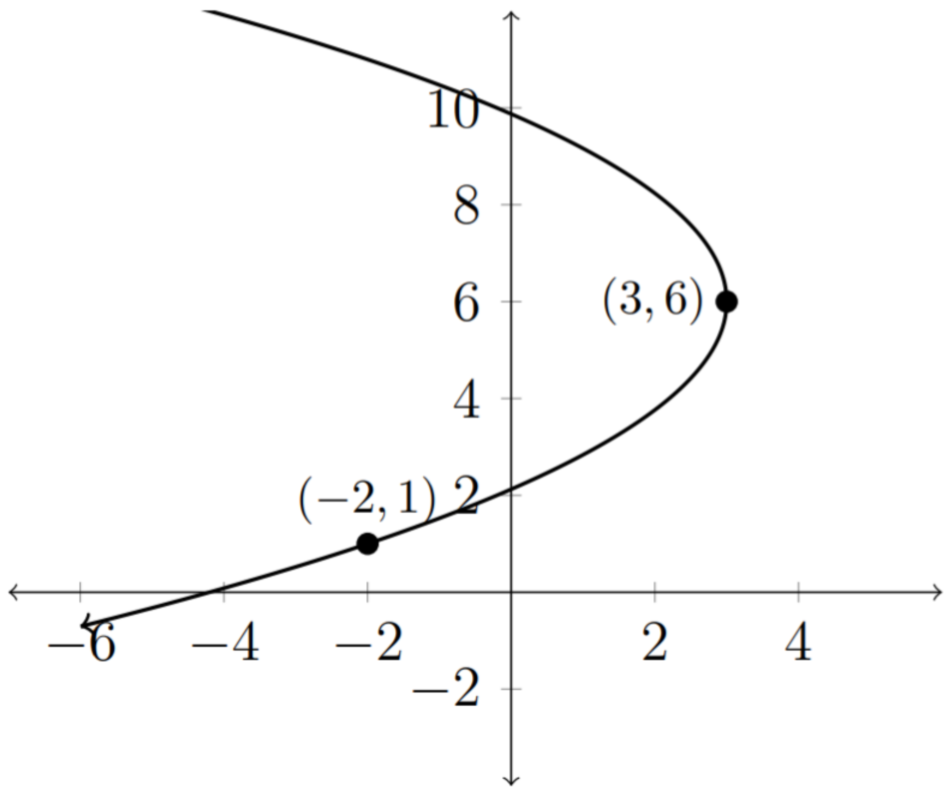
31)



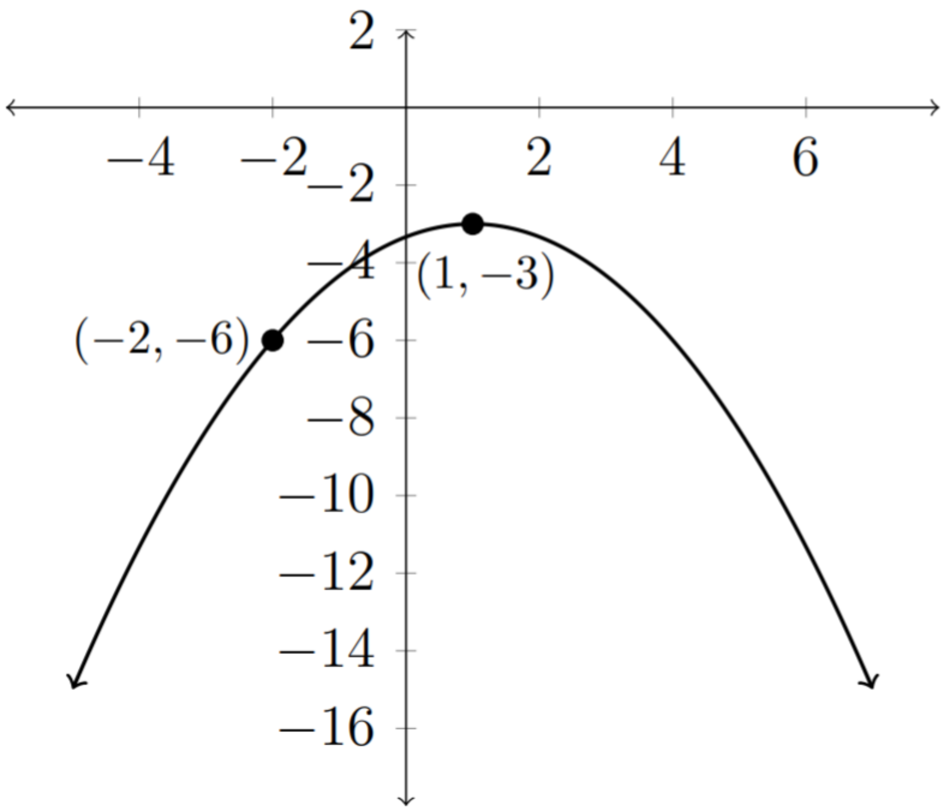
32)



33)



34)



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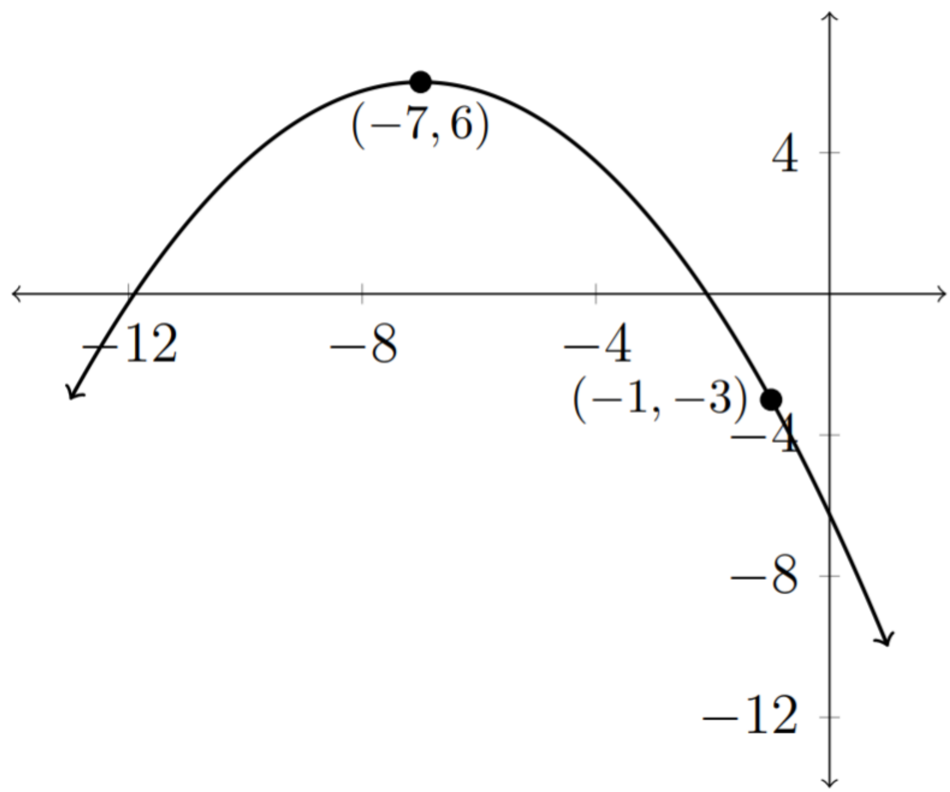
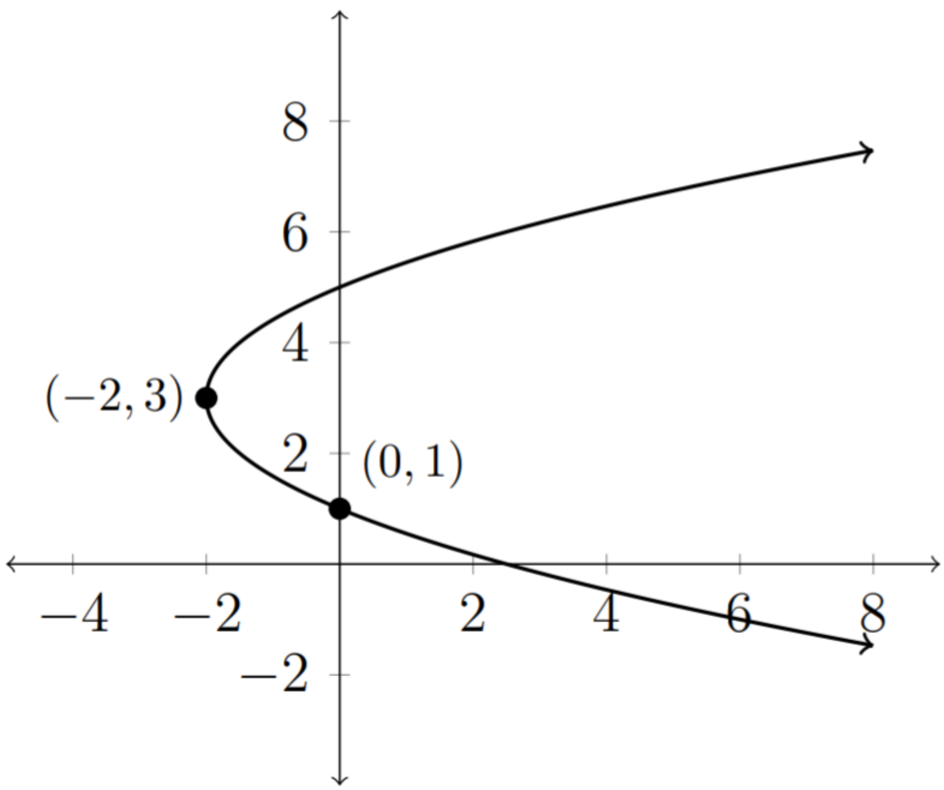


36)



5.3: Applications of the Parabola

A parabola that is rotated around its axis of symmetry to create a three dimensional object is called a paraboloid. One of the special properties of a parabola is that any light (or other electromagnetic wave) striking the interior of the parabola is reflected to the focus. The proof of this is somewhat complex, but this property makes the paraboloid a very useful shape in application.

The paraboloid is used to make satellite dishes so that the signal from the satellite is reflected to the center of the dish. This strengthens the signal. Parabolic microphones are often used at sporting events so that noises on the field can be heard more clearly on the sidelines. Flashlights and headlights also use this property in reverse. With the light at the focus of the paraboloid, all the light is reflected straight ahead, thus concentrating the beam of light.

One last application which has become used more frequently over the last five or ten years is the use of the paraboloid in solar power. One use of these properties sets up a tower at the focus of the paraboloid with mirrors banked around the tower in the paraboloid shape. This focuses all of the sun's power on the tower in the center. Often, salt is used in the tower since it has very high melting point. The salt is melted by the reflected sunlight and flows into a steam turbine. This has been found to be more efficient than just using standard solar panels.

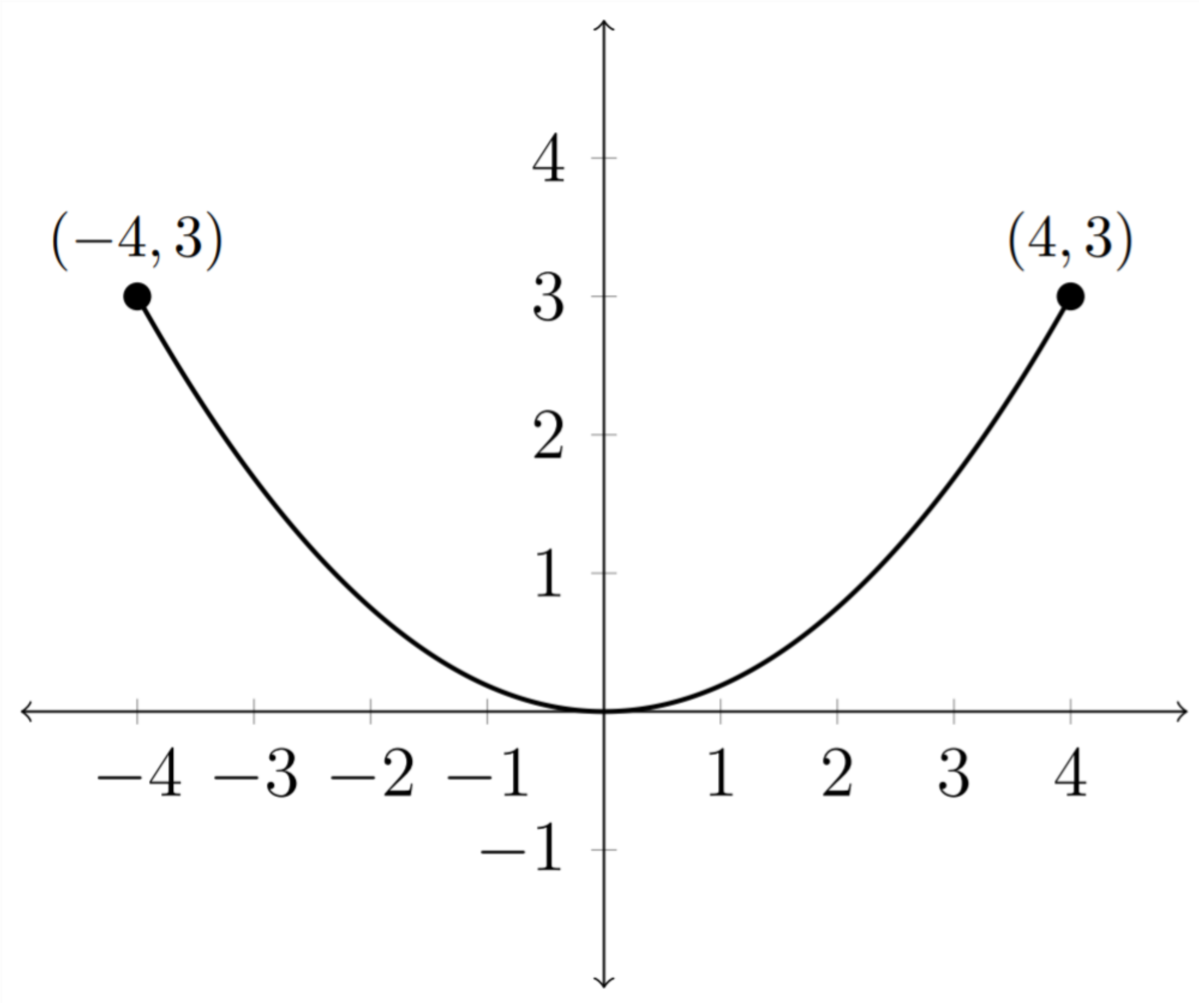
Example

If a satellite dish is 8 feet across and $3\mathrm{ft}$. deep, how far from the bottom of the dish should the receiver be placed so that it is at the focus of the paraboloid?

First let's consider what this looks like in two dimensions. Because we are setting up the graph of this parabola, we can choose to place the vertex at the origin. This makes things a little easier.

since the dish was a total of 8 feet across, we split this between the two sides of the graph, creating the points (4,3) and $(-4,3)$. since the vertex for this parabola is at the origin, the standard equation is somewhat simplified.

Here's what this cross-section looks like:

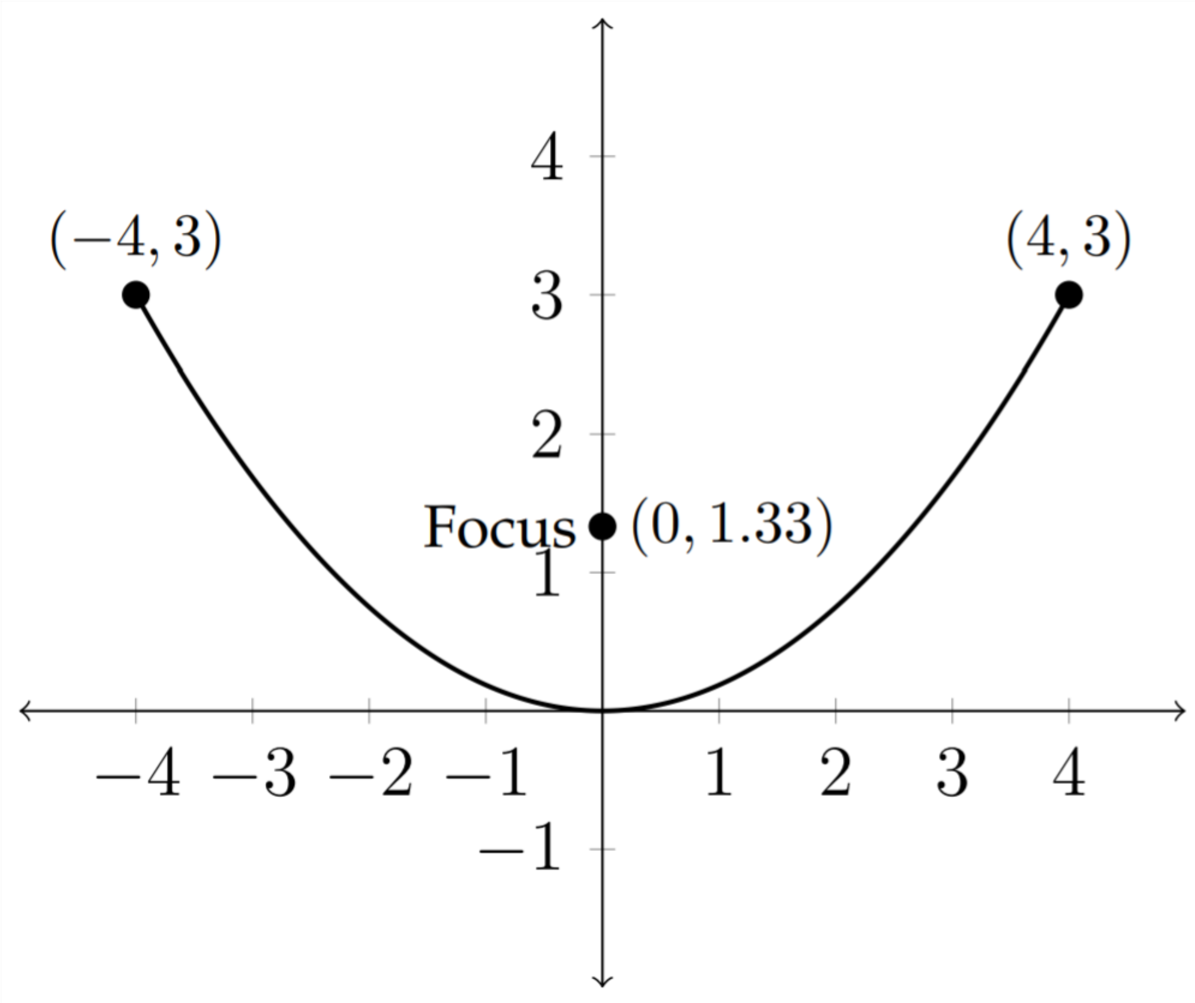


Taking the standard equation for an upward facing parabola: $4p(y-k)=(x-h)^2$ and using the point (0,0) for the vertex leaves us with $4p(y-0)=(x-0)^2$ or $4py=x^2$

The main thing we need to do is find the value of p for this situation. This will tell us where the receiver should be on the satellite dish. Plugging in the points on the graph that we know will allow us to solve for p

$$\begin{aligned} 4py &= x^2 \\ 4p(3) &= (4)^2 \\ 12p &= 16 \\ p &= \frac{4}{3} \end{aligned}$$

Thus the receiver should be $\frac{4}{3}$ feet or 1 foot 4 inches from the bottom of the dish.



Exercises 5.3

1) A satellite dish in the shape of a paraboloid is 10ft. across and 3 ft. deep. How far from the vertex at the bottom of the dish should the receiver be placed?

2) A satellite dish in the shape of a paraboloid is 6 ft. across and 2 ft. deep. How far from the vertex at the bottom of the dish should the receiver be placed?

3) The reflector in a flashlight in the shape of a paraboloid is 2 inches across and 1 inch deep. How far from the vertex at the bottom of the reflector should the light source be placed so that the light is reflected straight ahead?

4) The reflector in a flashlight in the shape of a paraboloid is 5 inches across and 3 inches deep. How far from the vertex at the bottom of the reflector should the light source be placed so that the light is reflected straight ahead?

5) A car headlight in the shape of a paraboloid has a bulb that is placed 1.25 inches from the vertex at the bottom of the headlight. If the headlight is 6 inches in diameter, how deep should the headlight be for the bulb to be at the focus of the paraboloid?

6) A spotlight in the shape of a paraboloid has a bulb that is placed 3 inches from the vertex at the bottom of the light housing. If the spotlight is 18 inches in diameter, how deep should the light be for the bulb to be at the focus of the paraboloid?

When a cable, chain or string hangs between two ends, it makes the shape of a catenary, which is generally described by an equation of the form:

$$f(x)=\frac{e^{x}+e^{-x}}{2} \tag{5.3.1}$$

However, when the cables of a suspension bridge support the weight of the roadbed on the bridge, the cables are pulled into the shape of a parabola.

7) The cables of a suspension bridge are in the shape of a parabola. The towers of the bridge that support the cable are 400ft. apart

and 600ft. tall. If the cables touch the roadway at the center of the bridge mid-way between the towers, how high is the cable 75ft. from the center of the bridge?

8) The cables of a suspension bridge are in the shape of a parabola. The towers of the bridge that support the cable are 800ft. apart and 160ft. tall. If the cables touch the roadway at the center of the bridge mid-way between the towers, how high is the cable 120ft. from the center of the bridge?

9) A spotlight in the shape of a paraboloid has the light source 2.5ft. from the vertex at the bottom of the light. If the depth of the spotlight housing is 4ft, how wide should the face of the spotlight be?

10) A flashlight reflector in the shape of a paraboloid has a light source that is
0.5 inches from the vertex at the bottom of the reflector. How deep should the reflector be if the face of the flashlight is 3 inches in diameter?

11) The parabolic mirror for the Mount Palomar observatory telescope is 200 inches in diameter and the mirror is 3.75 inches deep at the center. How far from the center of the mirror is the focal point where the light is reflected?

12) A reflecting telescope contains a mirror in the shape of a paraboloid. If the mirror is 5 inches across at the opening and 2 inches deep, how far from the vertex at the bottom of the mirror is the focal point?

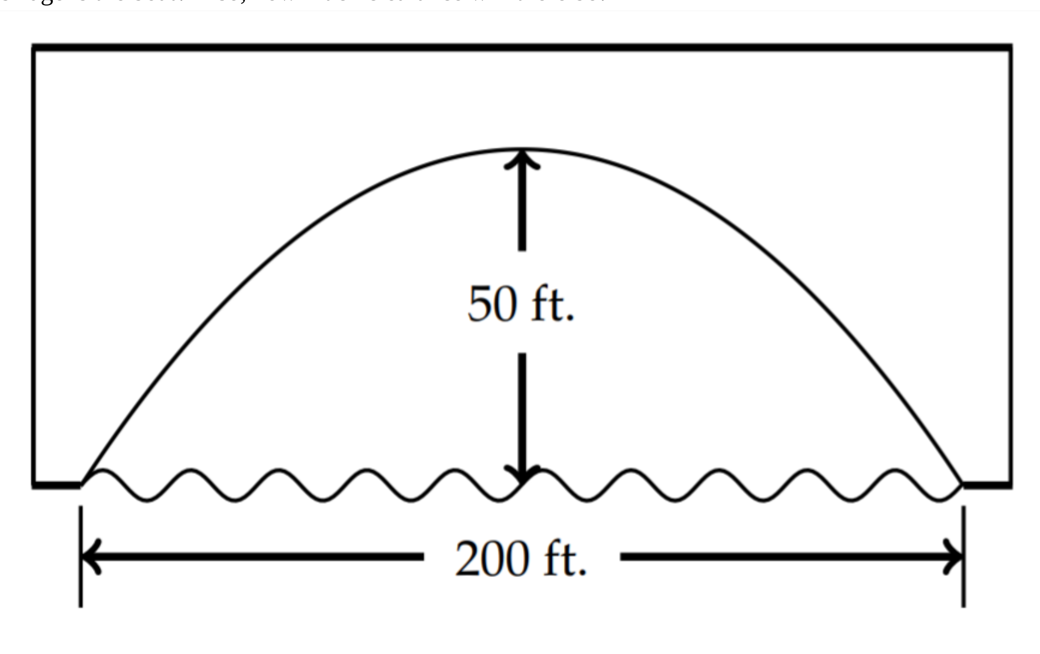
13) The cables of a suspension bridge are in the shape of a parabola. The towers of the bridge that support the cable are 500ft. apart and 80ft. tall. If the cables are 15 ft. above the roadway at the center of the bridge mid-way between the towers, how high is the cable 100ft. from the center of the bridge?

14) The cables of a suspension bridge are in the shape of a parabola. The towers of the bridge that support the cable are 2000ft. apart and 400ft. tall. If the cables are 25ft. above the roadway at the center of the bridge mid-way between the towers, how high is the cable 500ft. from the center of the bridge?

15) A flashlight reflector housing in the shape of a paraboloid is 4 inches deep and 4 inches across at its widest point. How far from the vertex at the bottom of the housing should the light source be placed?

16) A flashlight reflector housing in the shape of a paraboloid is 8 inches in diameter with the light source placed 1 inch from the vertex at the bottom of the housing. How deep should the housing be for the light source to be placed at the focal point?

17) A bridge is built in the shape of a parabolic arch (see figure below). If the arch is 50ft. above the water at the center and 200ft. wide at the water's surface, will a boat that is 35 ft. tall clear the arch 30 ft. from the center? If not, how much taller than the bridge is the boat? If so, how much clearance will there be?



18) A bridge is built in the shape of a parabolic arch. If the arch is 30 ft. above the water at the center and 150ft. wide at the water's surface, find the height of the arch above the water at distances of $10, 25, 40,$and 50ft. from the center.

CHAPTER OVERVIEW

6: Sequences and Series

6.1: Sequences

A sequence of numbers in a one-to-one correspondence with the natural numbers $\{1,2,3,4,\ldots\}$ can be defined in several ways. The terms of the sequence may simply be listed:

$$\{2,4,8,16,32,\ldots\} \tag{6.1.1}$$

A general expression for the sequence may be identified:

$$a_n = 2^n \tag{6.1.2}$$

In this situation, n is generally understood to be drawn from the ordered set of natural numbers. In addition, a sequence may be defined recursively. That is to say that each successive term will be defined in relation to the preceding term.
For the example that we're using above, a recursive definition would be as follows:

$$\begin{gathered} a_1 = 2 \\ a_n = 2 * a_{n-1} \\ \text{or} \\ a_{n+1} = 2 * a_n \end{gathered} \tag{6.1.3}$$

A sequence may be thought of as a function or relation in which the domain is restricted to positive whole numbers.

Examples
Find the first four terms of the given sequence and the 10 th term of the sequence.
1) $a_n = n^2 + 3$

$$a_1 = 4, a_2 = 7, a_3 = 12, a_4 = 19, a_{10} = 103 \tag{6.1.4}$$

2) $a_1 = 5$

$$\begin{gathered} a_n = a_{n-1} + 6 \\ a_1 = 5, a_2 = 11, a_3 = 17, a_4 = 23, a_{10} = 59 \end{gathered} \tag{6.1.5}$$

Finding a general or recursive definition for a sequence can be trickier than just writing out the terms. Common things to look for -

Is this an alternating sequence? That is, do the terms bounce back and forth between positive and negative values. If so, then you will need to include $(-1)^n$ or $(-1)^{n+1}$ in your general term.

$$\begin{gathered} \text{Example: } \{-1,2,-3,4,\ldots\} \\ a_n = (-1)^n(n) \end{gathered} \tag{6.1.6}$$

or

$$a_1 = -1 \tag{6.1.7}$$

$$a_n = (-1)(a_{n-1}) + (-1)^n \tag{6.1.8}$$

Is there a common difference between the terms? If so, then the sequence behaves much like a linear function and will have a form similar to $y = mx + b$, where m is the common difference.

$$\begin{gathered} \text{Example: } \{5,8,11,14,\ldots\} \\ a_n = 3n + 2 \end{gathered} \tag{6.1.9}$$

or

$$\begin{aligned} a_1 &= 5 \\ a_n &= a_{n-1} + 3 \end{aligned} \tag{6.1.10}$$

Is there a common multiplier? If so, then this should be a power function where a particular base is being raised to the power of n.

$$\begin{aligned} &\text{Example: } \{3, 15, 75, 375, \ldots\} \\ &a_n = 3 * 5^{n-1} \\ &\text{or} \\ &a_1 = 3 \\ &a_n = 5 * a_{n-1} \end{aligned} \tag{6.1.11}$$

Other patterns to look for are perfect squares and perfect cubes.

Exercises 6.1

Find the first four terms of the given sequence and the 10 th term of the sequence.

1) $a_n = 3n + 1$
2) $a_n = 4n - 12$
3) $a_n = -5n + 3$
4) $a_n = -2n + 7$
5) $a_n = 2n^2$
6) $a_n = 5n^2 - 1$
7) $a_n = (-1)^n (4n)$
8) $a_n = (-1)^{n+1} \left(\frac{1}{n}\right)$
9) $a_n = \frac{2^n}{3^{n-1}}$
10) $a_n = \frac{5^n}{2^{n+1}}$
11) $a_n = \frac{(-1)^{n-1}}{2n+5}$
12) $a_n = \frac{(-1)^n}{3n-2}$
13) $a_1 = -3 \text{and} a_n = a_{n-1} + 4$
14) $a_1 = 2 \text{and} a_n = a_{n-1} + 12$
15) $a_1 = 7 \text{and} a_n = 9 - a_{n-1}$
16) $a_1 = -5 \text{and} a_n = 17 - a_{n-1}$
17) $a_1 = 1 \text{and} a_n = n + a_{n-1}$
18) $a_1 = 4 \text{and} a_n = n - a_{n-1}$
19) $a_1 = \frac{1}{2} \text{and} a_n = \frac{(-1)^n}{a_{n-1}}$
20) $a_1 = \frac{2}{5} \text{and} a_n = \frac{(-1)^{n+1}}{2a_{n-1}}$

For each of the given sequences - find a general term a_n, and also find a recursive definition for the sequence.

21) $\{6, 7, 8, 9, 10, \ldots\}$
22) $\{9, 11, 13, 15, 17, \ldots\}$
23) $\{1, 4, 7, 10, 13, \ldots\}$
24) $\{-5, 4, 13, 22, 31, \ldots\}$
25) $\{-2, 6, -18, 54, \ldots\}$
26) $\{5, -10, 20, -40, 80, \ldots\}$
27) $\{1, -1, -3, -5, -7, \ldots\}$
28) $\{-8, -15, -22, -29, \ldots\}$
29) $\left\{\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}, \ldots\right\}$
30) $\left\{\frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \ldots\right\}$
31) $\left\{-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots\right\}$
32) $\left\{\frac{1}{2}, -\frac{2}{5}, \frac{3}{8}, -\frac{4}{11}, \ldots\right\}$

33) $\{5, -25, 125, -625, \ldots\}$

34) $\left\{1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots\right\}$

35) $\left\{1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots\right\}$

36) $\left\{\frac{2}{3}, \frac{9}{4}, \frac{8}{27}, \frac{81}{16}, \frac{32}{243} \ldots\right\}$

6.2: Arithmetic and Geometric Sequences

Two common types of mathematical sequences are arithmetic sequences and geometric sequences. An arithmetic sequence has a constant difference between each consecutive pair of terms. This is similar to the linear functions that have the form $y = mx + b$. A geometric sequence has a constant ratio between each pair of consecutive terms. This would create the effect of a constant multiplier.

Examples

Arithmetic Sequence:
$\{5, 11, 17, 23, 29, 35, \ldots\}$
Notice here the constant difference is 6. If we wanted to write a general term for this sequence, there are several approaches. One approach is to take the constant difference as the coefficient for the n term: $a_n = 6n+?$ Then we just need to fill in the question mark with a value that matches the sequence. We could say for the sequence:
$\{5, 11, 17, 23, 29, 35, \ldots\}$
$a_n = 6n - 1$
There is also a formula which you can memorize that says that any arithmetic sequence with a constant difference d is expressed as:
$a_n = a_1 + (n-1)d$
Notice that if we plug in the values from our example, we get the same answer as before:
$a_n = a_1 + (n-1)d$
$a_1 = 5, d = 6$
So, $a_1 + (n-1)d = 5 + (n-1) * 6 = 5 + 6n - 6 = 6n - 1$
or $a_n = 6n - 1$
If the terms of an arithmetic sequence are getting smaller, then the constant difference is a negative number.
$\{24, 19, 14, 9, 4, -1, -6, \ldots\}$
$a_n = -5n + 29$

Geometric Sequence
In a geometric sequence there is always a constant multiplier. If the multiplier is greater than 1, then the terms will get larger. If the multiplier is less than 1, then the terms will get smaller.
$\{2, 6, 18, 54, 162, \ldots\}$
Notice in this sequence that there is a constant multiplier of 3. This means that 3 should be raised to the power of n in the general expression for the sequence. The fact that these are not multiples of 3 means that we must have a coefficient before the 3^n
$\{2, 6, 18, 54, 162, \ldots\}$
$a_n = 2 * 3^{n-1}$
If the terms are getting smaller, then the multiplier would be in the denominator:
$\{50, 10, 2, 0.4, 0.08, \ldots\}$
Notice here that each term is begin divided by 5 (or multiplied by $\frac{1}{5}$).
$\{50, 10, 2, 0.4, 0.08, \ldots.\}$
$a_n = \frac{50}{5^{n-1}}$ or $a_n = \frac{250}{5^n}$ or $a_n = 50 * \left(\frac{1}{5}\right)^{n-1}$ and so on

Exercises 6.2
Determine whether each sequence is arithmetic, geometric or neither.
If it is arithmetic, determine the constant difference.
If it is geometric determine the constant ratio.
1) $\{18, 22, 26, 30, 34, \ldots\}$
2) $\{9, 19, 199, 1999, \ldots\}$
3) $\{8, 12, 18, 27, \ldots\}$
4) $\{15, 7, -1, -9, -17, \ldots\}$
5) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\right\}$
6) $\{100, -50, 25, -12.5, \ldots\}$

7) $\{-8, 12, 32, 52, \ldots\}$
8) $\{1, 4, 9, 16, 25, \ldots\}$
9) $\{11, 101, 1001, 10001, \ldots\}$
10) $\{12, 15, 18, 21, 24, \ldots\}$
11) $\{80, 20, 5, 1.25, \ldots\}$
12) $\{5, 15, 45, 135, 405, \ldots\}$
13) $\{1, 3, 6, 10, 15, \ldots\}$
14) $\{2, 4, 6, 8, 10, \ldots\}$
15) $\{-1, -2, -4, -8, -16, \ldots\}$
16) $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$

6.3: Series

Learning about mathematical sequences is usually a precursor to learning about mathematical series. A mathematical series is a sequence of numbers that is being added together. The importance of mathematical series cannot be understated. Many equations in the sciences cannot be solved by algebraic methods and must resort to series solutions. The notation for a mathematical series is typically the Greek capital letter sigma: Σ. The sigma notation is used as a short-hand method of representing a mathematical series with a particular form.

For example, if we are given the mathematical series:

$1+5+9+13+17+21$

This can be represented as follows:

$\sum_{k=0}^{5} 4k+1$

We could also express the same series as:

$\sum_{k=1}^{6} 4k-3$

Both expressions represent the terms being added together. This first example is an example of a finite series because it has a last term. Many mathematical series are infinite series. For example:

$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\dots$

is an example of an infinite series.

Working with infinite series can be quite useful, but also somewhat confusing. The behavior of an infinite series can be contradictory depending on how you analyze it.

Exercises 6.3

Write out each series in expanded notation.

1) $\sum_{k=1}^{10} 2k-5$
2) $\sum_{k=3}^{7} 6k-3$
3) $\sum_{k=2}^{9} (-1)^k \left(\frac{1}{k}\right)$
4) $\sum_{k=0}^{10} (-1)^{k+1}(k-4)$
5) $\sum_{k=0}^{4} \frac{k^2}{2}$
6) $\sum_{k=1}^{8} \frac{k}{3^k}$

Write each series using sigma notation.

7) $8+12+16+20+24+28$
8) $5+10+15+20+25+30$
9) $2+9+16+23+\cdots+65$
10) $5+8+11+14+\cdots+95$
11) $1+4+9+16+\cdots+256$
12) $1+8+27+64+\cdots+1331$
13) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}$
14) $27-9+3-1+\frac{1}{3}-\frac{1}{9}$

6.4: Sum of a Series

To find approximate solutions to problems in the sciences, it is often necessary to calculate the sum of a finite or infinite series. There are a variety of formulas that are used to accomplish this. Some of these formulas will be presented with proofs, but others will not. If you are interested in the proofs that are not included, please let me know.

General Formulas

Constant Series - notice that there is no k in the summation, the c is a constant that does not depend on the value of k

$\sum_{k=1}^{n} c = c + c + c + \cdots + c = n * c$

Sum of the first n integers:

$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Sum of the first n perfect squares:

$\sum_{k=1}^{n} k^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Sum of the first n perfect cubes:

$\sum_{k=1}^{n} k^3 = 1 + 8 + 27 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

The first formula should be obvious. The other three formulas are usually proved using mathematical induction, which we won't cover in this course. If you're interested in these proofs and how mathematical induction works, please let me know.

Formulas for the sum of arithmetic and geometric series:

Arithmetic Series: like an arithmetic sequence, an arithmetic series has a constant difference d. If we write out the terms of the series:

$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$

we can rewrite this in terms of the first term (a_1) and the constant difference d

$\sum_{k=1}^{n} a_k = a_1 + (a_1 + d) + (a_2 + 2d) + \cdots + (a_1 + (n-1)d)$

This expression is equivalent to:

$\sum_{k=1}^{n} a_k = (a_1 + a_1 + a_1 + \cdots + a_1) + (d + 2d + 3d + \cdots (n-1)d)$

$\sum_{k=1}^{n} a_k = na_1 + d(1 + 2 + 3 + \cdots (n-1))$

Using the previous formula for the sum $1 + 2 + 3 + \cdots + (n-1)$ gives us:

$\sum_{k=1}^{n} a_k = na_1 + d\left(\frac{(n-1)n}{2}\right)$

This formula is often stated in various forms:

$\sum_{k=1}^{n} a_k = \frac{n}{2}(2a_1 + (n-1)d)$

or

$\sum_{k=1}^{n} a_k = \frac{n}{2}(a_1 + a_n)$

since $a_1 + (n-1)d = a_n$

Geometric Series:

Given a geometric series, whose first term is a and with a constant ratio of r $\sum_{k=1}^{n} a * r^{k-1}$, we can write out the terms of the series in a similar way that we did for the arithmetic series.

$\sum_{k=1}^{n} a * r^{k-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$

The trick to finding a formula for the sum of this type of series is to multiply both sides of the previous equation by r

For simplicity's sake let's rename the sum of the series $\sum_{k=1}^{n} a * r^{k-1}$ as S_n

So,

$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$

and

$r * S_n = r\left(a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}\right) = ar + ar^2 + ar^3 + \cdots + ar^n$

If we subtract these two equations, we will have:

$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$

$-\left(r * S_n = ar + ar^2 + ar^3 + \cdots + ar^n\right)$

Then we'll have:

$S_n - rS_n = a - ar^n$

Factor out S_n on the left-hand side:

$S_n(1 - r) = a - ar^n$

and divide on both sides to isolate S_n :

$\frac{S_n(1-r)}{(1-r)} = \frac{a-ar^n}{(1-r)}$

$S_n = \frac{a-ar^n}{1-r}$

So for a finite geometric series, we can use this formula to find the sum. This formula can also be used to help find the sum of an infinite geometric series, if the series converges. Typically this will be when the value of r is between -1 and 1. In other words, $|r| < 1$ or $-1 < r < 1.$ This is important because it causes the ar^n term in the above formula to approach 0 as n becomes infinite. So, if $-1 < r < 1,$ then the sum of an infinite geometric series wil be:

$S_n = \frac{a}{1-r}$

Exercises 6.4

Find the sum for each of the following finite geometric series.

1) $\sum_{k=1}^{7} 3\left(\frac{1}{4}\right)^{k-1}$

2) $\sum_{k=1}^{7} 16\left(\frac{1}{3}\right)^{k-1}$

3) $\sum_{k=1}^{7} 3^k$

4) $\sum_{k=1}^{10} 2^{k-1}$

5) $\sum_{k=1}^{5} 4^{k-1}$

6) $\sum_{k=1}^{4} 6^{k-1}$

7) $\sum_{k=1}^{7} 2^k$

8) $\sum_{k=1}^{8} 3^k$

9) $\sum_{k=1}^{5} 2^{k+2}$

10) $\sum_{1}^{6} 3^{k-4}$

Determine whether each of the following geometric series has a sum. If it does, use the formula $S_n = \frac{a}{1-r}$ to find the sum.

11) $\sum_{k=1}^{\infty} 5 * \left(\frac{2}{3}\right)^{k-1}$

12) $\sum_{k=1}^{\infty} 12 * \left(\frac{1}{2}\right)^{k-1}$

13) $\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k-1}$

14) $\sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^{k-1}$

15) $\sum_{k=1}^{\infty} \left(\frac{1}{5}\right)^{k+1}$

16) $\sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^{k+1}$

17) $\sum_{k=1}^{\infty} \left(\frac{4}{3}\right)^{k-1}$

18) $\sum_{k=1}^{\infty} \left(\frac{3}{2}\right)^{k-1}$

19) $\sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^{k+2}$

20) $\sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^{k+4}$

CHAPTER OVERVIEW

7: Combinatorics

7.1: The Fundamental Principle of Counting

Combinatorics is a major area of what is generally known as "Discrete Mathematics." The word discrete refers to quantities that are individual, separate or distinct. This creates a major division in mathematics between "continuous mathematics" and "discrete mathematics." The difference between these two areas is that continuous mathematics considers and uses all parts of the number line whole numbers, rational numbers (fractions), irrational numbers and so forth. Discrete mathematics typically uses only the whole numbers.

Rather than limiting the possibilities in mathematics, this restriction actually opens amazing new areas of consideration. The binary codes that computers use are generally controlled and kept (mostly) error free through the use of discrete mathematics. Computer security for the simplest (checking your on-line bank balance) and the most complex (high level classified data) digital information is handled through encryption that relies on the concepts of discrete mathematics.

Any type of application in the sciences that involves choices and possibilities often uses the concepts of combinatorics. Combinatorial chemistry explores the results when a series of different chemical groups are added to the same basic chemical structure to investigate the qualities of the resulting compound. In addition, combinatorics is very important to the study of probability. In order to calculate the probability of an event, it is often necessary to calculate how many

different ways something can happen.

The first major idea of combinatorics is the fundamental principle of counting. This is the idea that if two events occur in succession and there are m ways to do the first one and n ways to do the second (after the first has occurred), then there are $m * n$ ways to complete the two tasks in succession.

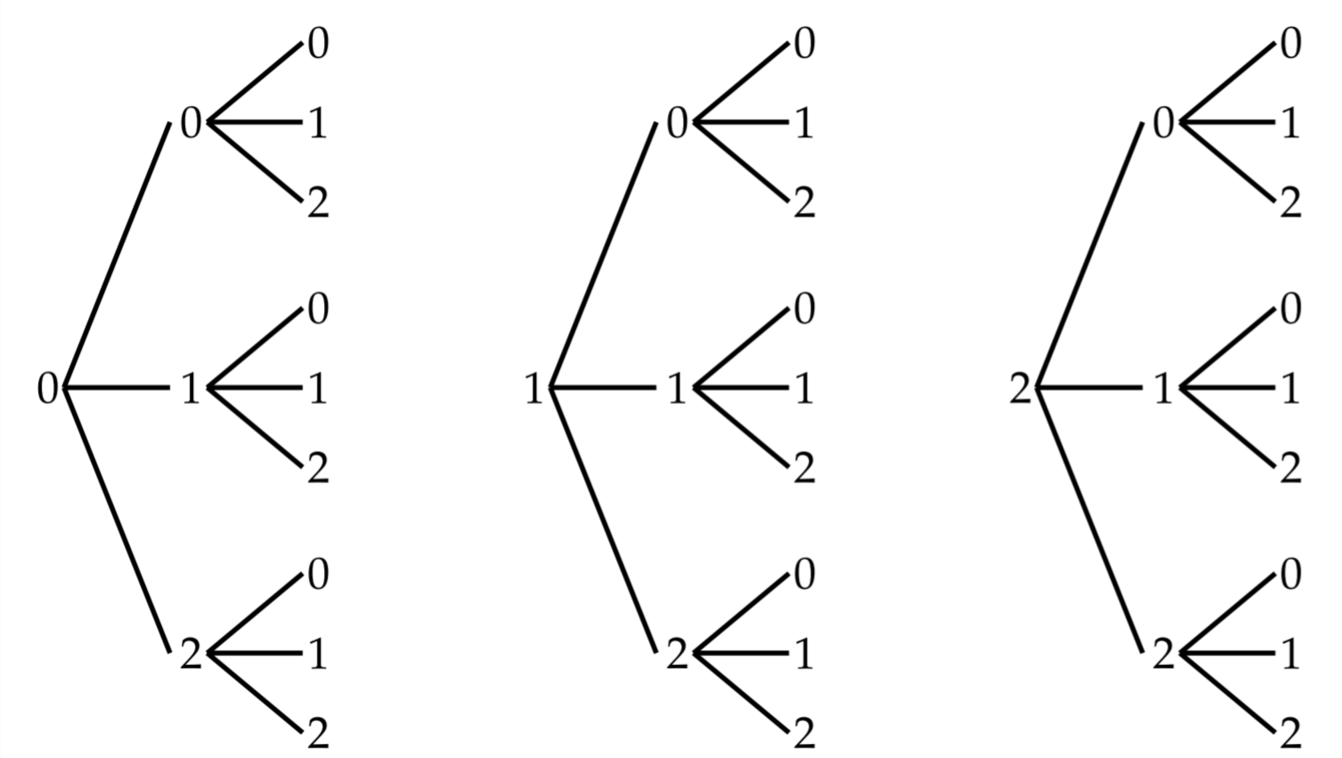
For example, in throwing two six-sided dice, there are 36 possibilities - six possibilities from the first die and six from the second. These possibilities are listed below:

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$
$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$
$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$	$(4,5)$	$(4,6)$
$(5,1)$	$(5,2)$	$(5,3)$	$(5,4)$	$(5,5)$	$(5,6)$
$(6,1)$	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$	$(6,6)$

If we were to throw a six-sided die and an eight sided-sided die, then there would be $6 * 8 = 48$ different possibilities.

Often the creation of a tree diagram can help us to visualize the possibilities. Suppose that a three-digit code using the numbers 0,1 and 2 is created so that repetition of the numbers is allowed. There would 3 possibilities for the first number, three for the second number and three for the third number meaning that there would be a total of $3 * 3 * 3 = 27$ different possible codes.

The tree diagram to illustrate this is shown on the next page.



All of the possibilities are listed out and can be constructed from the tree diagram.
Here are some sample problems for using the fundamental principle of counting (also known as the multiplication principle).

Example:
An ice cream shop offers Vanilla, Chocolate, Strawberry, Boysenberry and Rocky Road ice cream. The ice cream comes with either a waffle cone, a sugar cone or a wafer cone and can be plain or with sprinkles. How many different ways are there to order a single scoop of ice cream?

With 5 different types of ice cream, 3 different cones and 2 choices for sprinkles, there would be $5 * 3 * 2 = 30$ different possibilities.

Example:
In many states the license plates for automobiles consist of three letters followed three numbers.
How many different possibilities are there:
if repetition of letters and numbers is allowed?
if repetition of letters and numbers is not allowed?
There are six positions for letters and numbers that make up the license plate.
The first three are to be letters and the second three numbers.
There are 26 choices for the first three and 10 choices for the second three, so there are $26 * 26 * 26 * 10 * 10 * 10 = 17,576,000$ possible license plates if repetition is allowed.

If repetition is not allowed then we have to discount whichever letter or number is chosen for a particular position. So, for the first letter there are 26 possibilities, but then only 25 for the second letter and 24 for the third. Likewise, with the numbers there are 10 choices initially, then 9 choices and then 8
So, the solution without repetition would be $26 * 25 * 24 * 10 * 9 * 8 = 11,232,000$

4) A coin is flipped five times and the result each time is recorded. How many different possible outcomes are there?
5) A coin is flipped and a six-sided die is rolled and the results are recorded. If this is done three times, how many possible outcomes are there?
6) Two cards are chosen from a deck of 52 cards. If the first card is not replaced before the second card is chosen, how many

ways are there to choose:

a) A spade first and a heart second?

b) Two spades?

7) A company has 3000 employees. They plan to implement an employee ID numbering system that would consist of a letter followed by two digits. Is it possible to give each employee a different ID code under this plan?

8) A baseball team has 7 pitchers and 3 catchers. How many different batteries (pitcher - catcher combinations) are possible?

9) A string of five letters is created using the letters $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ and E. How many of these letter strings are possible if:

a) No conditions are imposed

b) Repetition of the letter A is not allowed

c) Each letter string must begin with C

d) B must be the middle letter

e) A, B and C must be the middle letters in any order with no repetition

For Part (e) please list all possibilities.

10) (\quad\) A combination lock is numbered from 0 to 30. Each combination consists of three numbers in succession. Successive numbers must be different, but the first and third can be the same. How many different combinations are possible?

7.2: Factorial Notation and Permutations

In considering the number of possibilities of various events, particular scenarios typically emerge in different problems. One of these scenarios is the multiplication of consecutive whole numbers. For example, given the question of how many ways there are to seat a given number of people in a row of chairs, there will obviously not be repetition of the individuals. So, if we wanted to know how many different ways there are to seat 5 people in a row of five chairs, there would be 5 choices for the first seat, 4 choices for the second seat, 3 choices for the third seat and so on.

$$\underline{5} * \underline{4} * \underline{3} * \underline{2} * \underline{1} = 120 \text{ choices} \tag{7.2.1}$$

In these situations the 1 is sometimes omitted because it doesn't change the value of the answer. This process of multiplying consecutive decreasing whole numbers is called a "factorial." The notation for a factorial is an exclamation point. So the problem above could be answered: $5! = 120$. By definition, $0! = 1$. Although this may not seem logical intuitively, the definition is based on its application in permutation problems.

A "permutation" uses factorials for solving situations in which not all of the possibilities will be selected.

So, for example, if we wanted to know how many ways can first, second and third place finishes occur in a race with 7 contestants, there would be seven possibilities for first place, then six choices for second place, then five choices for third place.
So, there are $\underline{7} * \underline{6} * \underline{5} = 210$ possible ways to accomplish this.
The standard notation for this type of permutation is generally ${}_nP_r$ or $P(n, r)$
This notation represents the number of ways of allocating r distinct elements into separate positions from a group of n possibilities.

In the example above the expression $\underline{7} * \underline{6} * \underline{5}$ would be represented as ${}_7P_3$ or

$$P(7, 3) \tag{7.2.2}$$

The standard definition of this notation is:

$${}_nP_r = \frac{n!}{(n-r)!} \tag{7.2.3}$$

You can see that, in the example, we were interested in ${}_7P_3$, which would be calculated as:

$${}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7*6*5*4*3*2*1}{4*3*2*1} \tag{7.2.4}$$

The $4*3*2*1$ in the numerator and denominator cancel each other out, so we are just left with the expression we foiund intuitively:

$${}_7P_3 = 7*6*5 = 210 \tag{7.2.5}$$

Although the formal notation may seem cumbersome when compared to the intuitive solution, it is handy when working with more complex problems, problems that involve large numbers, or problems that involve variables.

Note that, in this example, the order of finishing the race is important. That is to say that the same three contestants might comprise different finish orders.
1st place: Alice 1st place: Bob 2nd place: Bob 2nd place: Charlie 3rd place: Charlie 3rd place: Alice
The two finishes listed above are distinct choices and are counted separately in the 210 possibilities. If we were only concerned with selecting 3 people from a group of 7, then the order of the people wouldn't be important - this is generally referred to a "combination" rather than a permutation and will be discussed in the next section.

Returning to the original example in this section - how many different ways are there to seat 5 people in a row of 5 chairs? If we use the standard definition of permutations, then this would be ${}_5P_5$

$$_5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{120}{1} = 120 \tag{7.2.6}$$

This is the reason why $0!$ is defined as 1

EXERCISES 7.2

1) $4 * 5!$
2) $3! * 4!$
3) $5! * 3!$
4) $\frac{8!}{6!}$
5) $\frac{10!}{7!}$
6) $\frac{9! * 6!}{3! * 7!}$
7) $\frac{12! * 3!}{8! * 6!}$
8) $_{10}P_4$
9) $_4P_3$
10) $_7P_5$
11) $_9P_2$
12) $_8P_4$
13) so P_3
14) n_1
15) $_{10}P_r$
16) List all the permutations of the letters $\{a, b, c\}$
17) List all the permutations of the letters $\{a, b, c\}$ taken two at a time.
18) How many permutations are there of the group of letters $\{a, b, c, d, e\}$?
19) How many permutations are there of the group of letters $\{a, b, c, d\}$?

List these permutations.

20) How many ways can a president, vice president and secretary be chosen from a group of 20 students?
21) How many ways can a president, vice president, secretary and treasurer be chosen from a group of 50 students?
22) How many ways can 5 boys and 5 girls be seated in a row containing ten seats:
    a) with no restrictions?
    b) if boys and girls must alternate seats?
23) How many ways can 5 boys and 4 girls be seated in a row containing nine seats:
    a) with no restrictions?
    b) if boys and girls must alternate seats?
24) How many ways can 6 people be seated if there are 10 chairs to choose from?
25) How many ways can 4 people be seated if there are 9 chairs to choose from?
26) How many ways can a group of 8 people be seated in a row of 8 seats if two people insist on sitting together?
27) How many ways can a group of 10 people be seated in a row of 10 seats if three people insist on sitting together?

7.3: Permutations and Combinations

We saw in the last section that, when working with permutations, the order is always important. If we were choosing 3 people from a group of 7 to serve on a committee with no assigned roles, the nature of the problem would change.
For example, if we were choosing 3 people from a group of 7 to serve on a committee as president, vice-president and treasurer, the answer would be ${}_7P_3 = 210$ But - if we wanted to choose 3 people from a group of 7 with no assigned roles, then some of the choices in the permutation would be the same.
In a permutation:
1st place: Alice 1st place: Bob 2nd place: Bob 2nd place: Charlie 3rd place: Charlie 3rd place: Alice
the two choices listed above would be considered as being different and would be counted separately. In a "combination" in which the order of selection is not important and there are no assigned roles, we must compensate for these extra choices.

If we are choosing 3 people from a group of 7 to serve on a committee with no assigned roles then we should consider that any selection from a permutation that includes the same three people should only be counted once.
So, when we select the three people, we should consider how many different ways there are to group them and then remove those extra choices. In this example, we are choosing three people. Each group of three can be arranged in six different ways $3! = 3 * 2 = 6,$ so each distinct group of three is counted six times.
In order to find the actual number of choices we take the number of possible permutations and divide by 6 to arrive at the actual answer:

$$ {}_7C_3 = \frac{7P_3}{3!} = \frac{7!}{4! * 3!} \tag{7.3.1} $$

In a combination in which the order is not important and there are no assigned roles the number of possibilities is defined as:

$$ {}_nC_r = \frac{n!}{(n-r)! * r!} \tag{7.3.2} $$

One way to remember the difference between a permutation and a combination is that on a combination pizza it doesn't make any difference whether the sausage goes on before the pepperoni or whether the onions are put on first-so in a combination, order is not important!

EXERCISES 7.3

Find the value of the following expressions.
1) ${}_{10}C_4$
2) ${}_8C_3$
3) ${}_{10}C_6$
4) ${}_8C_5$
5) ${}_{15}C_{12}$
6) ${}_{18}C_2$
7) ${}_nC_4$
8) ${}_9C_r$
9) How many three-topping pizzas can be made if there are twelve toppings to choose from?
10) How many bridge hands of 13 cards are possible from a deck of 52 cards?
11) How many poker hands of 5 cards are possible from a deck of 52 cards?
12) How many different bridge hands of 13 cards are possible if none of the cards is higher than 10 (i.e. no face cards)?
13) How many different poker hands of 5 cards are possible if none of the cards is higher than 8?
14) If a person has 10 different t-shirts, how many ways are there to choose 4 to take on a trip?
15) If a band has practiced 15 songs, how many ways are there for them to select 4 songs to play at a battle of the bands? How many different performances of four songs are possible?
16) Fifteen boys and 12 girls are on a camping trip. How many ways can a group of seven be selected to gather firewood:
   a) with no conditions
   b) the group contains four girls and three boys
   c) the group contains at least four girls

17) A class of 25 students is comprised of 15 girls and 10 boys. In how many ways can a committee of 8 students be selected if:

a) there are no restrictions

b) no males are included on the committee

c) no females are included on the committee

d) the committee must have 5 boys and 3 girls

18) From a group of 12 male and 12 female tennis players, two men and two women will be chosen to compete in a men-vs-women doubles match. How many different matches are possible?

19) In a seventh-grade dance class, there are 20 girls and 17 boys.

a) How many ways can the students be paired off to create dance couples consisting of one boy and one girl?

b) How many ways are there to create a group of 17 boy/girl couples?

c) How many ways are there to create a group of 18 couples without restrictions?

7.4: General Combinatorics Problems

As is often the case, practicing one type of problem at a time is helpful to master the techniques necessary to solve that type of problem. More difficult is deciding which type of problem you are working on and choosing which techniques to use to solve the problem. In this section, the problem types from the previous three sections are combined.

EXERCISES 4.4

SET I

1) How many strings of six lower case letters from the English alphabet contain

a) the letter a?

b) the letters a and b in consecutive positions with a preceding b, with all the letters distinct?

c) the letters a and b, where a is somewhere to the left of b in the string, with all the letters distinct?

2) Seven women and nine men are on the faculty in the mathematics department at a school.

a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?

b) How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?

3) Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with 6 members if it must have the same number of men and women?

4) The English alphabet contains 21 consonants and 5 vowels. How many strings of 6 lower case letters of the English alphabet contain:

a) exactly one vowel?

b) exactly 2 vowels?

c) at least 1 vowel?

d) at least 2 vowels?

5) How many ways are there to select 12 countries in the United Nations to serve on a council if 3 are selected from a block of 45,4 are selected from a block of 57, and the others are selected from the remaining 69 countries?

6) Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with 6 members if it must have more women than men?

7) How many license plates consisting of three letters followed by three digits contain no letter or digit twice?

8) In the women's tennis tournament at Wimbledon, two finalists, A and B, are competing for the title, which will be awarded to the first player to win two sets. In how many different ways can the match be completed?

9) In the men's tennis tournament at Wimbledon, two finalists, A and B, are competing for the title, which will be awarded to the first player to win three sets. In how many different ways can the match be completed?

10) In how many different ways can a panel of 12 jurors and 2 alternate jurors be chosen from a group of 30 prospective jurors?

SET II

11) A class has 20 students, of which 12 are female and 8 are male. In how many ways can a committee of five students be picked from the class if:

a) No restrictions are placed on the choice of students

b) No males are included on the committee

c) The committee must have three female members and 2 male members

12) A school dance committee is to be chosen from a group of students consisting of six freshman, eight sophomores, twelve juniors and ten seniors. If the committee should consist of two freshman, three sophomores, four juniors and five seniors, how many ways can this be done?

13) A theater company consists of 22 actors - 10 men and 12 women. In the next play, the director needs to cast a leading man, leading lady, supporting male role, supporting female role and eight extras (three men and five women). In how many ways can this play be cast?

14) A hockey coach has 20 players of which twelve play forward, six play defense and two are goalies. In how many ways can the coach pick a lineup consisting of three forwards, two defense players and one goalie?

15) In how many ways can ten students be arranged in a row for a class picture if John and Jane want to stand next to each other and Ed and Sally also want to stand next to each other?

16) In how many ways can the students in the previous problem be arranged if Ed and Sally want to stand next to each other, but

John and Jane refuse to stand next to each other?

17) In how many ways can four men and four women be seated in a row of eight seats if:

a) The first seat is occupied by a man

b) The first and last seats are occupied by women

SET III

18) The social security number of a person is a sequence of nine digits that are not necessarily distinct. How many social security numbers are possible?

19) A variable name in the BASIC programming language is either a letter of the alphabet or a letter followed by a digit. How many distinct variable names are there in the BASIC language?

20) a) How many even numbers are there between 0 and 100?

b) How many even numbers with distinct digits are there between 0 and 100?

21) There are six characters - three letters of the English alphabet followed by three digits - which appear on the back panel of a particular brand of printer as an identification number. Find the number of possible identification numbers if

a) characters can be repeated

b) digits cannot repeat

c) letters cannot repeat

d) characters cannot repeat

22) A sequence of characters is called a palindrome if it reads the same forwards and backwards. For example $\mathrm{K98EE89K}$ is an eight character palindrome and $\mathrm{K98E89K}$ is a seven character palindrome. A MAN A PLAN A CANAL PANAMA is also a palindrome as are WAS IT A RAT I SAW, TACO CAT, TANGY GNAT, and NEVER ODD OR EVEN. Find the number of nine character palindromes that can be formed using the letters of the alphabet such that no letter appears more than twice in each one.

23) Find the number of ways to form a four-letter sequence using the letters A, B, C, D and E if:

a) repetitions are permitted

b) repetitions are not permitted

c) the sequence contains the letter A but repetitions are not permitted

d) the sequence contains the letter A and repetitions are permitted

24) There are 10 members A, B, C, D, E, F, G, H, I and J in a fund raising committee. The first task of the committee is to choose a chairperson, a secretary and a treasurer from this group. No individual can hold more than one office. How many ways can these three positions be filled if:

a) no one has any objection for holding any of these offices

b) Cneeds to be the chairperson

c) B does not want to be the chairperson

d) A is not willing to serve as chairperson or secretary

e) either I or J must be the treasurer

f) E or F or G must hold one of the three offices

25) Find the number of ways of picking each of the following from a standard deck of cards.

a) a king and a queen

b) a king or a queen

c) a king and a red card

d) a king or a red card

26) There are three bridges connecting two towns A and B. Between towns B and C there are four bridges. A salesperson needs to travel from A to C via B. Find:

a) the number of possible choices of bridges from A to C

b) the number of choices for a round trip from A to C and back to A

c) the number of choices for a round trip if no bridge may be crossed twice

27) A sequence of digits in which each digit is 0 or 1 is called binary number. Eight digit binary numbers are often referred to as "bytes."

a) How many bytes are possible?

b) How many bytes begin with 10 and end with 01?

c) How many bytes begin with 10 but do not end with 01?

d) How many bytes begin with 10 or end with 01?

28) A group of 12 is to be seated in a row of chairs. How many ways can this be done if:

a) two people, A and B must be seated next to each other?

b) two people A and B must not be seated next to each other?

29) A variable in the FORTRAN language is a sequence that has at most six characters with the first character being a letter of the alphabet and the remaining characters (if any) being either letters or digits. How many distinct variable names are possible?

30) Four station wagons, five sedans and six vans are to be parked in a row of
15 spaces. Find the nubmer of ways to park the vehicles if:

a) the station wagons are parked at the beginning, then the sedans, then the vans

b) vehicles of the same type must be parked together

7.5: Distinguishable Permutations

If there is a collection of 15 balls of various colors, then the number of permutations in lining the balls up in a row is ${}_{15}P_{15}=15!$. If all of the balls were the same color there would only be one distinguishable permutation in lining them up in a row because the balls themselves would look the same no matter how they were arranged.

If 10 of the balls were yellow and the other 5 balls are all different colors, how many distinguishable permutations would there be?

No matter how the balls are arranged, because the 10 yellow balls are indistinguishable from each other, they could be interchanged without any perceptable change in the overall arrangement. As a result, the number of distinguishable permutations in this case would be $\frac{15!}{10!}$, since there are $10!$ rearrangements of the yellow balls for each fixed position of the other balls.

The general rule for this type of scenario is that, given n objects in which there are n_1 objects of one kind that are indistinguishable, n_2 objects of another kind that are indistinguishable and so on, then number of distinguishable permutations will be:

$$\begin{gathered}\frac{n!}{\frac{n}{n_1!*n_2!*n_3!*\cdots*n_k!}} \\ \text{with } n_1+n_2+n_3+\cdots+n_k=n\end{gathered} \tag{7.5.1}$$

Example
Find the number of ways of placing 12 balls in a row given that 5 are red, 3 are green and 4 are yellow.
This would be $\frac{12!}{5!3!4!}=\frac{12*11*10*9*8*7*6*5*4*3*2*1}{5*4*3*2*3*2*4*3*2}$

$$\begin{aligned}&=\frac{12*11*10*9*8*7*6}{3*2*4*3*2} \\ &=27,720\end{aligned} \tag{7.5.2}$$

Another way to think about this problem is to choose five of the twelve spaces in which to place the red balls - since the order of selection is not important, there are ${}_{12}C_5$ ways to do this. Then, from the remaining 7 spaces available, we need to choose three of them in which to place the green balls. There are ${}_7C_3$ ways to do that. The four yellow balls are then placed in the remaining four spaces.

The result of this process is that there are ${}_{12}C_5$ ways to choose the places for the red balls and ${}_7C_3$ ways to choose the places for the green balls, which results in:

$${}_{12}C_5 *{}_7 C_3=\frac{12!}{5!7!}*\frac{7!}{3!4!}=\frac{12!}{5!3!4!} \tag{7.5.3}$$

This results in the same answer as when we approached the problem as a permutation. Considering the problem in this way helps us to solve problems which involve the assigning of tasks to groups of individuals.
Example
Fourteen construction workers are to be assigned to three different tasks. Six workers are needed for mixing cement, five for laying bricks and three for carrying the bricks to the brick layers. In how many different ways can the workers be assigned to these tasks?

This is also a problem of distinguishable permutation. Although the order of the workers is not important here, the result is the same:

$$\frac{14!}{6!5!3!} \tag{7.5.4}$$

Another way to think about problems of this type is that they are combination problems, since the order in which the workers are assigned is not important. In that case, we need to select six of the fourteen workers to mix cement, five to lay bricks and three to carry bricks.
To select six workers to mix cement: ${}_{14}C_6=\frac{14!}{6!8!}$
To select five workers (from the remaining 8) to lay bricks: ${}_8C_5=\frac{8!}{5!3!}$

To select three workers (from the reamining three) to carry bricks: 1

If there are $\frac{14!}{6!8!}$ ways to choose the cement crew and $\frac{8!}{5!3!}$ ways to choose the bricklayers from the remaining eight workers, then there will be:

$$\frac{14!}{6!8!} * \frac{8!}{5!3!} = \frac{14!}{6!5!3!} \tag{7.5.5}$$

ways to assign the workers to these tasks.

Exercises 7.5

Find the number of distinguishable permutations of the given letters.

1) $AAABBC$

2) $AAABBBCCC$

3) $AABCD$

4) $ABCDDDEE$

5) In how many ways can two blue marbles and four red marbles be arranged in a row?

6) In how many ways can five red balls, two white balls, and seven yellow balls be arranged in a row?

7) In how many different ways can four pennies, three nickels, two dimes and three quarters be arranged in a row?

8) In how many ways can the letters of the word ELEEMOSYNARY be arranged?

9) A man bought three vanilla ice-cream cones, two chocolate cones, four strawberry cones and five butterscotch cones for 14 children. In how many ways can he distribute the cones among the children.

10) When seven students take a trip, they find a hotel with three rooms available - a room for one person, a room for two poeple and a room for three people. In how many different ways can the students be assigned to these rooms? (one student will sleep in the car)

11) Eight workers are cleaning a large house. Five are needed to clean windows, two to clean carpets and one to clean the rest of the house. In how many different ways can these tasks be assigned to the eight workers?

7.6: Probability

If a coin is tossed three times, what is the probability of getting exactly two heads? at least two heads? no heads?

Probability is defined as the number of ways the event in question can happen divided by the number of total possibilities. We can define the collection of all possible outcomes in an experiement with all outcomes equally likely as the sample space of the experiment. Then the number of ways the event in question can happen can be defined as $n(E)$ and size of the sample space can be defined as $n(S)$. In this notation, the probability of an event $P(E)$ is defined below:

$$P(E)=\frac{n(E)}{n(S)}=\frac{\text{number of elements in } E}{\text{number of elements in } S} \tag{7.6.1}$$

The question at the opening of this section addresses a situation in which a coin is tossed three times. We can use a tree diagram to examine the sample space of this situation.
The possibilities in this experiment are:
$\{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{HTT}, \mathrm{THH}, \mathrm{THT}, \mathrm{TTH}, \mathrm{TTT}\}$
since there are eight possibilities, with 3 of them having exactly two heads, the probability is $\frac{3}{8}=0.375$

The probability of getting at least two heads is $\frac{4}{8}=\frac{1}{2}$ or 0.5
The probability of getting no heads is $\frac{1}{8}=0.125$
Example
A five-card poker hand is drawn from a standard deck of 52 cards. What is the probability that all five cards are hearts?

The number of ways to draw five cards from a deck of 52 is ${}_{52}C_5$. This is a combination since the order that the cards are drawn does not affect the outcome.
The number of ways of choosing 5 hearts from the 13 in the deck is $13\mathrm{C}_5$
So the probability of drawing five hearts from a deck of 52 cards is $\frac{13C_5}{52C_5}=\frac{1287}{2598960}\approx 0.000495$
Example
A bag contains 20 tennis balls, of which four are defective. If two balls are drawn at random from the bag, what is the probability that both are defective?

The sample space is the number of ways to choose two balls from the bag: ${}_{20}C_2$
since there are 4 defective balls, the number of ways to draw two of them is ${}_4C_2$
So, the probability that both balls will be defective is $\frac{4}{20C_2}=\frac{6}{190}\approx 0.0316$
We can also calculate the probability of drawing one defective ball:
$\frac{{}_4C_1 * {}_{16}C_1}{{}_{20}C_2}=\frac{64}{190}\approx 0.3368$

And the probability of drawing no defective balls:
$\frac{{}_{16}C_2}{{}_{20}C_2}=\frac{120}{190}\approx 0.6316$
Notice that $6+64+120=190$ and that $0.0316+0.3368+0.6316=1$ That is to say that the probabilities for all the possible events should add up to 1

This allows us to compute probabilities based on the probability that an event won't happen.

If the probability of an event is $P(E)$, then the probability that the event will not happen is $1-P(E)$

Example
An urn contains 10 red balls and 15 green balls. If six of the balls are drawn at random, what is the probability that at least one of them is red?

We can calculate this probability by finding the probability that no red balls are drawn. The number of ways to draw six green balls is ${}_{15}C_6$. The number of ways to draw six balls from the urn is ${}_{25}C_6$. So the probability of drawing six green balls is:
$\frac{{}_{15}C_6}{{}_{25}C_6}=\frac{5005}{177{,}100}\approx 0.02826$
This means that the probability of drawing at least one red ball is $1-0.02826=0.97174$
We could also find this answer by adding up the possibilities of drawing one red ball, two red balls and so on, up to six red balls:

$\frac{{}_{10}C_1 * {}_{15}C_5}{{}_{25}C_6} + \frac{10C_2 * {}_{15}C_4}{25C_6} + \frac{10C_3 * {}_{15}C_3}{25C_6} + \frac{10C_4 * {}_{15}C_2}{25C_6} + \frac{10C_5 * {}_{15}C_1}{25C_6} + \frac{10C_6}{25C_6}$
$= 0.1696 + 0.3468 + 0.3083 + 0.1245 + 0.0213 + 0.0012 =$
0.9717

Example
Given 100 computer components, it is known that 10 of the 100 are defective. If someone were to choose 6 of these components at random, what is the probability that:
a) two of them are defective?
b) at least 1 of them is defective?
There are ${}_{10}C_2$ ways to choose 2 out of the 10 defective components and ${}_{90}C_4$ ways to choose 4 non-defective components. The size of the sample space (the number of ways to choose 6 components from the 100) is ${}_{100}C_6$
So, the probability for part (a) would be:
$\frac{{}_{10}C_2 * {}_{90}C_4}{{}_{100}C_6} \approx 0.096$
The probability in part (b) is most easily computed by finding the probability that no defective components are selected and then subtracting that value from 1
$1 - \frac{{}_{90}C_6}{{}_{100}C_6} \approx 1 - 0.5223 \approx 0.4777$

Exercises 7.6
SET I
1)
- a) If a coin is tossed two times, describe the sample space.
- b) Find the probability of getting exactly two heads.
- c) Find the probability of getting at least one head.
- d) Find the probability of getting exactly one head.

2)
- a) If a coin is tossed and a single six-sided die is rolled, describe the sample space.
- b) Find the probability of getting heads and an even number.
- c) Find the probability of getting heads and a number greater than 4
- d) Find the probability of getting tails and an odd number.

3) When rolling a single, six-sided die, find the probability of:
- a) rolling a six
- b) rolling an even number
- c) rolling a number greater than 5

4) When rolling a single, six-sided die, find the probability of:
- a) rolling a two or a three
- b) rolling an odd number
- c) rolling a number divisible by three

5) If a card is drawn randomly from a standard 52 card deck, find the probability of:
- a) drawing a king
- b) drawing a face card
- c) drawing a card that is not a face card

6) If a card is drawn randomly from a standard 52 card deck, find the probability of:
- a) drawing a heart
- b) drawing a heart or a spade
- c) drawing a card that is a heart, a diamond or a spade

7) A ball is drawn randomly from an urn that contains 8 balls - five red, two white and one yellow. Find the probability that:
- a) a red ball is chosen
- b) a yellow ball is not chosen
- c) a green ball is chosen

8 , A ball is drawn randomly from an urn that contains 8 balls - five red, two white and one yellow. Find the probability that:
- a) the ball drawn is neither yellow nor white
- b) the ball drawn is either white, yellow, or red
- c) the ball chosen is not white

9) A drawer contains 18 socks of which 6 are red, 4 are white and 8 are black.

a) If one sock is drawn at random from the drawer, what is the probability that it is red?

b) If one red sock is drawn with the first choice, what is the probability that the next sock drawn is also red?

A poker hand is drawn at random from a standard deck of 52 cards.

10) Find the probability of getting five hearts.

11) Find the probability of getting five cards of the same suit.

12) Find the probability of getting five face cards.

13) Find the probability of getting ace, king, queen, jack and ten of the same suit.

14) For problem #14 refer to the sample space in Section 4.1 (or create your own

If a pair of standard six-sided dice are rolled, what is the probability of:

a) rolling a 7

b) rolling a 9

c) rolling doubles (the same number on each die)

d) not rolling doubles

e) rolling 9 or higher

SET II

An unbiased coin is tossed 5 times. Find the probability that:

15) the coin lands heads five times

16) the coin lands heads exactly once

17) the coin lands heads at least once

18) the coin lands heads more than once

Two cards are drawn without replacement from a standard deck of 52 cards. Find the probability that:

19) a pair is drawn

20) a pair is not drawn

21) two black cards are drawn

22) two cards of the same suit are drawn

A jar contains three yellow balls and five red balls. If four balls are drawn at random (without replacement), find the probability that:

23) two of the balls are yellow and two are red

24) all of the balls are red

25) exactly three of the balls are red

26) two or three of the balls are yellow

Assume that the probability of a boy being born is the same as the probability of a
girl being born. Find the probability that a family with three children will have:

27) two boys and one girl

28) at least one girl

29) no boys

30) the two oldest children are girls

31) An exam consists of ten True-or-False questions. If a student guesses at every answer, what is the probability that he or she will answer exactly six questions correctly?

32) A law firm employs 14 lawyers, 8 of whom are partners in the firm. If a group of 3 lawyers is chosen at random to attend a conference, what is the probability that 3 partners will be selected?

33) In a lot of 24 computer components, there are four defective components. If two of the components are chosen at random, what is the probability that:

a) both of the components are defective?

b) at least one of the components is defective?

34) A barrel of 60 apples contains 4 rotten apples. If 3 apples are selected at random, what is the probability that 1 or more of the apples is rotten?

35) A shelf at the home improvement store contains 80 light bulbs of which 6 are defective. If a customer chooses 2 light bulbs at random, what is the probability that:

a) both are defective?

b) at least one is defective?

36) Some computer components are shipped in boxes of 24. Before they are shipped, the quality control inspector randomly chooses 8 components from each box. If any of the 8 components selected are defective, the box is not shipped. What is the probability that a lot containing exactly 2 defective components would be shipped anyway?

37) A business is preparing to choose 12 people to go on a business trip from a group of 100 employees, 60 of whom are women and 40 of whom are men. Suppose that Ed and Mary are both employees and that the 12 people for the trip will be chosen randomly.

a) What is the probability that Ed will be chosen?

b) What is the probability that Ed and Mary will both be chosen?

c) If an equal number of men and women are to go on the trip, what is the probability that Ed will be chosen?

d) If an equal number of men and women are to go on the trip, what is the probability that Ed and Mary will both be chosen?

38) A company has has 50 sales representatives on staff. There are three sales calls that must be made on the east side of town and five sales calls on the west side. If the sales reps are chosen at random for these eight sales calls:

a) What is the probability that an individual sales rep will be chosen for any of the eight sales calls?

b) What is the probability that two sales reps will be selected to make their sales calls on the same side of town.

39) A student studying for a test knows how to do 12 of the 20 problems from the study guide. If the test contains 10 problems chosen at random from the study guide, what is the probability that at least 8 of the problems on the test are problems the student knows how to do?

CHAPTER OVERVIEW

8: Right Triangle Trigonometry

The precursors to what we study today as Trigonometry had their origin in ancient Mesopotamia, Greece and India. These cultures used the concepts of angles and lengths as an aid to understanding the movements of the heavenly bodies in the night sky. Ancient trigonometry typically used angles and triangles that were embedded in circles so that many of the calculations used were based on the lengths of chords within a circle. The relationships between the lengths of the chords and other lines drawn within a circle and the measure of the corresponding central angle represent the foundation of trigonometry - the relationship between angles and distances.

The earliest values for the sine function were calculated by Indian mathematicians in the 5th century. The cosine and tangent, as well as the cotangent, secant and cosecant were developed by Islamic mathematicians by the 11th century. European navigators used these ideas extensively to help calculate distances and direction during the Middle Ages. Modern European trigonometry as we understand it was then developed throughout the Renaissance (1450-1650) and Enlightenment (1650-1800).

8.1: Measuring Angles

Measuring Angles in Degrees

The two most common units for measuring angles are degrees and radians. Degrees are based on the ancient Mesopotamian assignment of 360° to a complete circle. This has its origin in the division of the horizon of the nighttime sky as the earth takes 365 days to travel around the sun. Because degrees were originally developed by the Mesopotamians, they are often also broken out into 60 unit measures of minutes and seconds. Sixty seconds make one minute and sixty minutes makes one degree.

60 seconds $=1$ minute or $60'' = 1'$
60 minutes $=1$ degree or $60' = 1^2$

Angles measured in degrees may also be expressed using decimal portions of a degree, for example:

$72.5^{\circ} = 72^{\circ}30'$

Converting from decimal to DMS

Converting between degrees expressed with decimals and the degrees, minutes, seconds format (DMS) is relatively simple. If you're converting from degrees expressed with decimals to DMS, simply take the portion of the angle behind the decimal point and multiply by 60. In our previous example, we would take the .5 from 72.5° and multiply this by 60 : 0.5*$60 = 30$. So, the angle in DMS units would be $72^{\circ}30'$

Examples

Convert 21.85° to DMS units

$0.85 * 60 = 51$

$\text{So}, 21.85^{\circ} = 21^{\circ}51'$

Convert 143.27^{to} DMS units

$0.27 * 60 = 16.2$

So, $143.27^{\circ} = 143^{\circ}16.2'$
In order to compute the number of seconds needed to express this angle in DMS units, we take the decimal portion of the minutes and multiply by 60:
$0.2 * 60 = 12$
So, $143.27^{\circ} = 143^{\circ}16.2' = 143^{\circ}16'12''$

In the example above we ended with a whole number of seconds. If you don'tget a whole number for the seconds then you can leave the seconds with a decimal portion. For example, if you wanted to convert 22.847° to DMS units:

$22.847^{\circ} = 22^{\circ}50.82' = 22^{\circ}50'49.2''$

Converting from DMS to decimal

To convert from DMS units to decimals, simply take the seconds portion and divide by 60 to make it a decimal:

$129^{\circ}19'30'' = 129^{\circ}19.5'$

Then take the new minutes portion and divide it by 60

$\frac{19.5}{60} = 0.325$

This is the decimal portion of the angle

$129^{\circ}19'30'' = 129^{\circ}19.5' = 129.325^{\circ}$

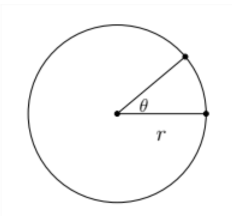
If you end up with repeating decimals in this process that's fine-just indicate the repeating portion with a bar.

Examples
Convert $42^{\circ}.27'36''$ to decimal degrees
$\frac{36}{60} = 0.6$
$42^{\circ} \quad 27'36'' = 42^{\circ} \quad 27.6'$

$\frac{27.6}{60} = 0.46$
$42^{\circ} \quad 27.6' = 42.46^{\circ}$
Convert $17^{\circ}40'18''$ to decimal degrees
$\frac{18}{60} = 0.3$
$17^{\circ}40'18'' = 17^{\circ}40.3'$
$\frac{40.3}{60} = 0.671\overline{6}$
$17^{\circ}40'18'' = 17^{\circ}40.3' = 17.671\overline{6}^{\circ}$

Measuring Angles in Radians

The other most commonly used method for measuring angles is radian measure. Radian measure is based on the central angle of a circle. A given central angle will trace out an arc of a particular length on the circle. The ratio of the arc length to the radius of the circle is the angle measure in radians. The benefit of radian measure is that it is based on a ratio of distances whereas degree measure is not. This allows radians to be used in calculus in situations in which degree measure would be inappropriate.



The length of the arc intersected by the central angle is the portion of the circumference swept out by the angle along the edge of the circle. The circumference of the circle would be $2\pi r$, so the length of the arc would be $\frac{\theta}{360^{\circ}} * 2\pi r$. The ratio of this arclength to the radius is $\frac{\frac{\theta}{1000} + 2\pi r}{r}$ or

$$\frac{2\pi}{360^{\circ}} * \theta \tag{8.1.1}$$

or in reduced form

$$\frac{\pi}{180^{\circ}} * \theta \tag{8.1.2}$$

This assumes that the angle has been expressed in degrees to begin with. If an angle is expressed in radian measure, then to convert it into degrees, simply multiply by $\frac{180^{\circ}}{\pi}$

Examples - Degrees to Radians
Convert 60° to radians
$\frac{\pi}{180^{\circ}} * 60^{\circ} = \frac{\pi}{3}$
Convert 142° to radians
$\frac{\pi}{180^{\circ}} * 142^{\circ} = \frac{71\pi}{90}$ or $0.7\overline{8}\pi$
Examples - Radians to Degrees
Convert $\frac{\pi}{10}$ to degrees
$\frac{180^{\circ}}{\pi} * \frac{\pi}{10} = 18^{\circ}$
Convert $\frac{\pi}{2}$ to degrees
$\frac{180^{\circ}}{\pi} * \frac{\pi}{2} = 90^{\circ}$
Another way to convert radians to degrees is to simply replace the π with 180° :
$\frac{\pi}{10} = \frac{180^{\circ}}{10} = 18^{\circ}$
$\frac{\pi}{2} = \frac{180^{\circ}}{2} = 90^{\circ}$

Exercises 1.1 Convert each angle measure to decimal degrees.
$3.91^{\circ}50'$
$1.\ 27 \times 40$
$4\ 34\%$

```
\begin{tabular}{llllll}
7. & \(17^{\circ}\) & \(25^{\prime}\) & \(5 .\) & \(274^{\circ}\) & \(18^{\prime}\) & \(6 .\) & \(165^{\circ}\) & \(48^{\prime}\) \\
\hline
\end{tabular}
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10. $141^{\circ}6'9''$
11. $211^{\circ}46'48''$
12. $19^{\circ}12'18''$

Convert each angle measure to DMS notation. 13. 31.425°

14. 159.84°
15. 6.78°
16. 24.56°
19. 18.9°
17. 110.25°
18. 64.16°
19. 18.9°
22. 55.17°
20. 85.14°
21. 220.43°
23.70.214°
24. 116.32°

Convert each angle measure from degrees to radians.

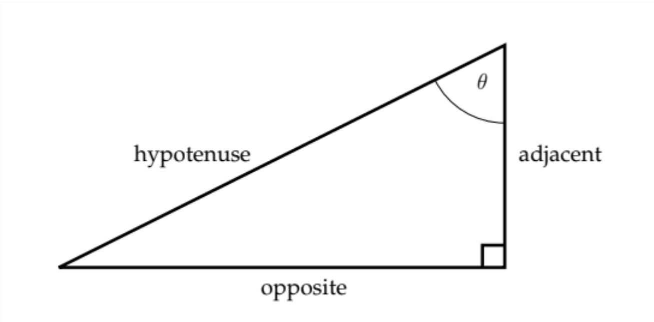
25. 30°
26. 120°
27. 45°
28. 225°
29. 60°
30. 150°
31.90°
32. 270°
33. 15°
34.36°
35. 12°
36. 104°

Convert each angle measure from radians to degrees.

37
38. $\frac{\pi}{5}$
39. $\frac{\pi}{3}$
40. $\frac{\pi}{6}$
4^7
42. $\frac{7\pi}{3}$
43. $\frac{5\pi}{2}$
44. $\frac{7\pi}{4}$
45. $\frac{5\pi}{6}$
46. $\frac{2\pi}{3}$
47. π
48. $\frac{7\pi}{2}$

8.2: The Trigonometric Ratios

There are six common trigonometric ratios that relate the sides of a right triangle to the angles within the triangle. The three standard ratios are the sine, cosine and tangent. These are often abbreviated sin, cos and tan. The other three (cosecant, secant and cotangent) are the reciprocals of the sine, cosine and tangent and are often abbreviated csc, sec, and cot.

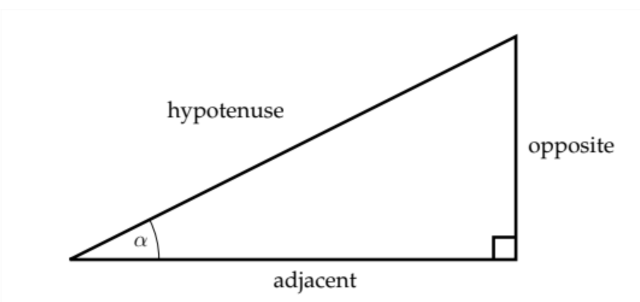


Given an angle situated in a right triangle, the sine function is defined as the ratio of the side opposite the angle to the hypotenuse, the cosine is defined as the ratio of the side adjacent to the angle to the hypotenuse and the tangent is defined as the ratio of the side opposite the angle to the side adjacent to the angle.

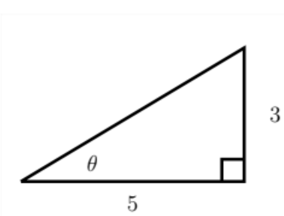
$$\sin\theta = \frac{opp}{hyp}$$
$$\cos\theta = \frac{adj}{hyp}$$
$$\tan\theta = \frac{opp}{adj}$$

A common mneumonic device to help remember these relationships is
-SOHCAHTOA-which identifies the sin as Opp over Hyp Cos as Adj over Hyp and the Tan as Opp over Adj.

An acute angle placed in the other position of a right triangle would have different oppposite and adjacent sides although the hypotenuse would remain the same.



Examples: Trigonometric Ratios
Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ for the given angle θ



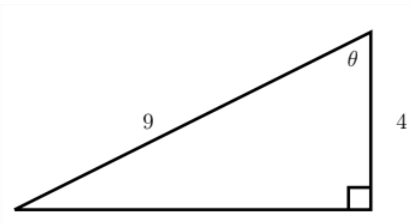
In order to find the sin and cos of the angle θ, we must first find the hypotenuse by using the Pythagorean Theorem $\left(a^2+b^2=c^2\right)$ since we know the legs of the triangle, we can substitute these values for a and b in the Pythagorean Theorem:

$$\begin{gathered}3^2+5^2=c^2 \\ 9+25=c^2 \\ 34=c^2 \\ \sqrt{34}=c\end{gathered} \tag{8.2.1}$$

Now that we know the hypotenuse $(\sqrt{34})$, we can determine the sin, cos and tan for the angle θ

$$\begin{aligned}\sin\theta&=\frac{3}{\sqrt{34}} \\ \cos\theta&=\frac{5}{\sqrt{34}} \\ \tan\theta&=\frac{3}{5}\end{aligned}$$

Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ for the given angle θ



Again, in order to find the sin, cos and tan of the angle θ, we must find the missing side of the triangle by using the Pythagorean Theorem. since, in this case, we know the hypotenuse and one of the legs, the value of the hypotenuse must be substituted for c and the length of the leg we're given can be substituted for either
a or b

$$\begin{gathered}4^2+b^2=9^2 \\ 16+b^2=81 \\ b^2=65 \\ b=\sqrt{65}\end{gathered} \tag{8.2.2}$$

Now that we know the length of the other leg of the triangle $(\sqrt{65})$, we can determine the sin, cos and tan for the angle θ

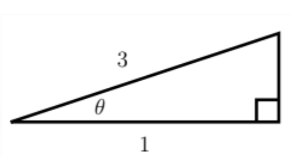
$$\begin{aligned}\sin\theta&=\frac{\sqrt{65}}{9} \\ \cos\theta&=\frac{4}{9} \\ \tan\theta&=\frac{\sqrt{65}}{4}\end{aligned}$$

In addition to the examples above, if we are given the value of one of the trigonometric ratios, we can find the value of the other two.

Example

Given that $\cos\theta=\frac{1}{3}$, find $\sin\theta$ and $\tan\theta$

Given the information about the cosine of the angle θ, we can create a triangle that will allow us to find $\sin\theta$ and $\tan\theta$

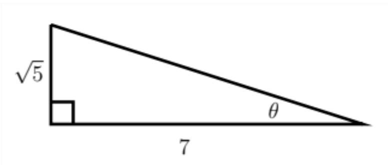


Using the Pythagorean Theorem, we can find the missing side of the triangle:

$$\begin{gathered}a^2+1^2=3^2 \\ a^2+1=9\end{gathered} \tag{8.2.3}$$

$a^2 = 8$
$a = \sqrt{8} = 2\sqrt{2}$
Then $\sin\theta = \frac{\sqrt{8}}{3}$ and $\tan\theta = \frac{\sqrt{8}}{1} = \sqrt{8}$
You might say to yourself, "Wait a minute, just because the cosine of the angle θ is $\frac{1}{3}$, that doesn't necessarily mean that the sides of the triangle are 1 and 3, they could be 2 and 6, or 3 and 9 or any values n and $3n$."

This is true, and if the sides are expressed as n and $3n$, then the missing side would be $n\sqrt{8}$, so that whenever we find a trigonometric ratio, the n' s will cancel out, so we just leave them out to begin with and call the sides 1 and 3
Example
Given that $\tan\theta = \frac{\sqrt{5}}{7}$, find $\sin\theta$ and $\cos\theta$
First we'll take the infomation about the tangent and use this to draw a triangle.



Then use the Pythagorean Theorem to find the missing side of the triangle:

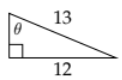
$$\begin{gathered}\sqrt{5}^2 + 7^2 = c^2 \\ 5 + 49 = c^2 \\ 54 = c^2 \\ \sqrt{54} = 3\sqrt{6} = c\end{gathered} \tag{8.2.4}$$

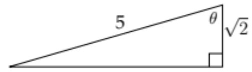
So then:

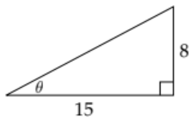
$$\sin\theta = \frac{\sqrt{5}}{\sqrt{54}} = \sqrt{\frac{5}{54}} \tag{8.2.5}$$

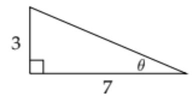
$$\cos\theta = \frac{7}{\sqrt{54}} = \frac{7}{3\sqrt{6}} \tag{8.2.6}$$

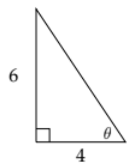
Exercises 1.2
Find $\sin\theta, \cos\theta$ and $\tan\theta$ for the given triangles.

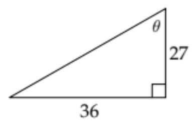
1. 

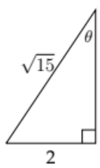
2. 

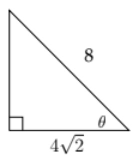
3. 

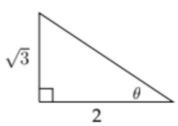
4. 

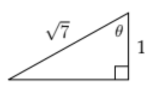
5. 

6. 

7. 

8. 

9. 

10. 

Use the information given to find the other two trigonometric ratios.

11. $\tan\theta = \frac{1}{2}$
12. $\sin\theta = \frac{3}{4}$
13. $\cos\theta = \frac{3}{\sqrt{20}}$
14. $\tan\theta = 2$
15. $\sin\theta = \frac{5}{\sqrt{40}}$
16. $\sin\theta = \frac{7}{10}$
17. $\cos\theta = \frac{9}{40}$
18. $\tan\theta = \sqrt{3}$
19. $\cos\theta = \frac{1}{2}$
20. $\cos\theta = \frac{3}{7}$
21. $\sin\theta = \frac{\sqrt{5}}{7}$
22. $\tan\theta = 1.5$

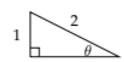
8.3: Solving Triangles

Using information about some of the sides and angles of a triangle in order to find the unknown sides and angles is called "solving the triangle." If two sides of a triangle are known, the Pythagorean Theorem can be used to find the third side. If one of the acute angles in a right triangle is known, the other angle will be its complement with their sum being 90°

Suppose that we have a right triangle in which we know the sides, but no angles. Another situation could involve knowing the angles but just one side. How could we solve for the missing measurements in these situations?

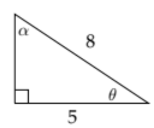
Solving problems like these uses precalculated values of the trigonometric ratios to match the lengths with the appropriate angles and vice versa. Up until the 1980's, these values were printed in tables that were included in the back of every textbook (along with tables of logarithms), but have recently been programmed into calculators using methods that are studied in Calculus.

Most calculators have a button or function designed to find the inverse sine, inverse cosine and inverse tangent $\left(\sin^{-1}, \cos^{-1}, \text{and } \tan^{-1}\right)$, these are the functions that tell you the measure of the angle that has a sine, cosine or tangent equal to a particular value.
For example, if we are given an angle θ and know that the $\sin\theta=\frac{1}{2}$



Then we can find $\sin^{-1}\left(\frac{1}{2}\right)$ on a calculator, which should return a value of 30°. If the calculator is in radian mode, it will return a value of $\approx 0.523598776.$ If you divided this number by π, you would get $0.1\overline{6}$, which means that $0.523598776 \approx \frac{\pi}{6}$ In this chapter we will work mainly in degrees. In Chapter 2, when we graph the trigonometric functions we will typically use radian measure.

Solve the triangle. Round side lengths to the nearest 100^{th} and angles to the nearest 10^{th} of a degree.



We can find the third side of the triangle by using the Pythagorean Theorem.

$$a^2+5^2=8^2$$
$$a^2+25=64$$

$a^2=39$

$$a=\sqrt{39}\approx 6.24 \tag{8.3.1}$$

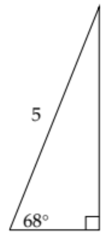
When solving problems of this type, I encourage people to use the most accurate values that are available in the problem. This way, there is less chance for rounding error to occur.

If we take the values for the sides that were given in the problem (5 and 8), then we can say that

$$\begin{aligned}\cos\theta&=\frac{5}{8}\\ \theta&=\cos^{-1}\left(\frac{5}{8}\right)\\ \theta&\approx 51.3^{\circ}\end{aligned} \tag{8.3.2}$$

Then α would be $90^{\circ}-51.3^{\circ}=38.7^{\circ}\approx\alpha$

Example 2
Solve the triangle. Round side lengths to the nearest 100^{th} and angles to the nearest 10^{th} of a degree.



First, we can find the other angle in the right triangle: $90^{\circ} - 68^{\circ} = 22^{\circ}$

Next, to find the sides, we choose a trigonometric ratio for which we know one of the sides. In this problem, we can use either the sine or the cosine.

$\sin 68^{\circ} = \frac{a}{5}$

Approximating $\sin 68^{\circ}$ on a calculator:

$0.9272 \approx \frac{a}{5}$

$$5 * 0.9272 \approx a \tag{8.3.3}$$

$4.6 \approx a$

When approximating a trigonometric value from the calculator, it is important to use at least 4 decimal places of accuracy. Again, this is to avoid rounding errors.

To solve for the remaining side we can either use the Pythagorean Theorem or use the method demonstrated above, but with the $\cos 68^{\circ}$

$\cos 68^{\circ} = \frac{b}{5}$

Approximating $\cos 68^{\circ}$ on a calculator:

$$\begin{gathered} 0.3746 \approx \frac{b}{5} \\ 5 * 0.3746 \approx b \\ 1.9 \approx b \end{gathered} \tag{8.3.4}$$

If we use the Pythagorean Theorem with two sides of the triangle to find the third, then we would say that:

$$\begin{gathered} b^2 + 4.6^2 = 5^2 \\ b^2 + 21.16 = 25 \\ b^2 = 3.84 \\ b = \sqrt{3.84} \approx 1.959 \approx 2.0 \end{gathered} \tag{8.3.5}$$

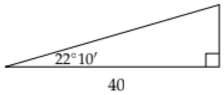
The rounding error in this example comes from the fact that the first side we found was not exactly 4.6. If we wanted a more accurate answer that matches the answer we found using the cosine ratio, we just need more accuracy in the leg of the triangle we found.

Calculating $5 * \sin 68^{\circ} \approx 4.636$ should provide enough accuracy.

$$\begin{gathered} b^2 + 4.636^2 = 5^2 \\ b^2 + 21.492496 = 25 \\ b^2 = 3.507504 \\ b = \sqrt{3.507504} \approx 1.8728 \approx 1.9 \end{gathered} \tag{8.3.6}$$

Example 3

Solve the triangle. Round side lengths to the nearest 100^{th} and angles to the nearest 10^{th} of a degree.



If we convert the angle $22^{\circ}10'$ to $22.1\overline{6}^{\circ}$, then the other acute angle in the right triangle is $90^{\circ} - 22.1\overline{6}^{\circ} = 67.8\overline{3}^{\circ}$ or $67^{\circ}50'$
Finding the remaining sides requires the use of either the cosine or tangent function.

$$\cos 22.1\overrightarrow{6}^{\circ} = \frac{40}{c} \tag{8.3.7}$$

Approximating cos $22.1\overline{6}^{\circ}$ on a calculator:

$$0.9261 \approx \frac{40}{c} \tag{8.3.8}$$

Next we need to multiply on both sides by c :

$$c * 0.9261 \approx \frac{40}{c} * c \tag{8.3.9}$$

$0.9261c \approx 40$
Then divide on both sides by 0.9261:

$$\begin{gathered} \frac{0.9261c}{0.9261} \approx \frac{40}{0.9261} \\ c \approx 43.2 \end{gathered} \tag{8.3.10}$$

To find the other leg of the triangle, we can use the tangent ratio.

$$\tan 22.1\overline{6}^{\circ} = \frac{a}{40} \tag{8.3.11}$$

Approximating tan $22.1\overline{6}^{\circ}$ on a calculator:

$$0.4074 \approx \frac{a}{40} \tag{8.3.12}$$

Then, multiply on both sides by 40:

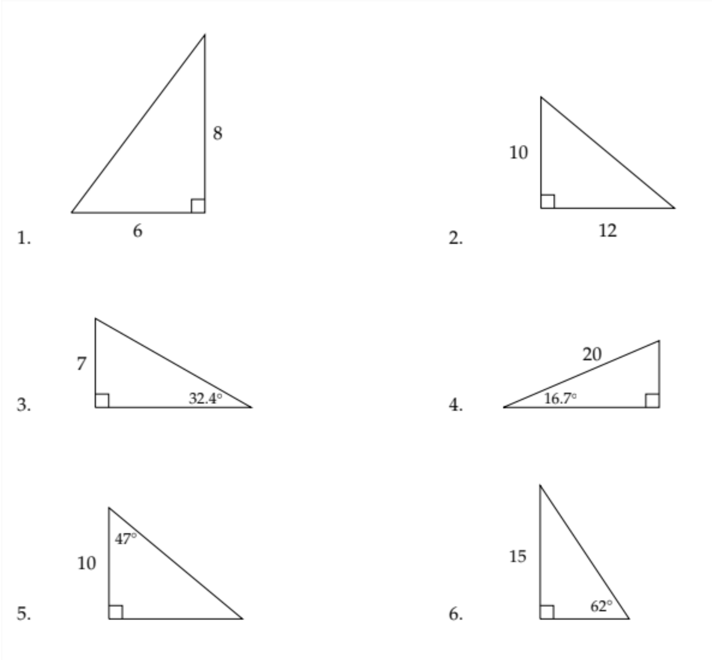
$$\begin{gathered} 40 * 0.4074 \approx a \\ 16.3 \approx a \end{gathered} \tag{8.3.13}$$

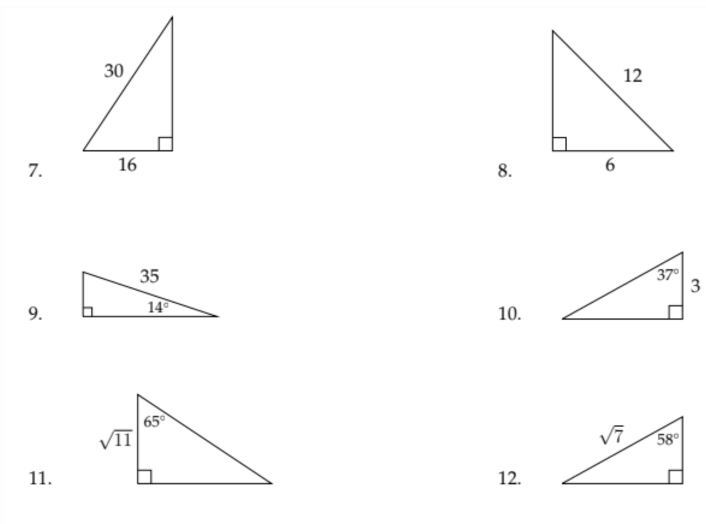
We can check this answer by the Pythagorean Theorem:

$$16.3^2 + 40^2 = 1865.69 \tag{8.3.14}$$

$$\sqrt{1865.69} \approx 43.2 \tag{8.3.15}$$

Exercises 1.3
In each problem below, solve the triangle. Round side lengths to the nearest 100^{th} and angle measures to the nearest 10^{th} of a degree.


1.
8
6
2.
10
12
3.
7
32.4°
4.
20
16.7°
5.
10
47°
6.
15
62°


7.
30
16
8.
12
6
9.
35
14°
10.
37°
3
11.
$\sqrt{11}$
65°
12.
$\sqrt{7}$
58°

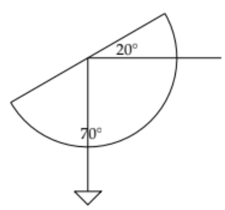
8.4: Applications

1.4 Applications

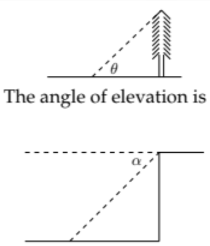
Trigonometry is often used for what is called "indirect measurement." This is a method of measuring inaccessible distances by using the relationships between lengths and angles within a triangle. Two simple examples of this process are measuring the height of a tall tree and measuring the distance across a body of water. In both cases, while it might be possible to measure the distance directly, it is often much easier to use indirect measurement.

In one example of indirect measurement, the angle of elevation of an object can be used to create a right triangle in which one angle and one side are known. The other sides of the triangle may then be solved for. In the problems in this text, the angle of elevation will typically be given in the problem. In order to actually measure the angle of elevation of an object, it is possible to use a simple protractor.

If you wanted to measure the height of a tall tree that sits on flat ground, you could use a specially modified protractor to do this. Modifying the protractor by tying a weight to the end of a string and tying the other end of the string through the hole in the protractor will help to measure the angle of elevation. Once the protractor is ready, hold it upside down and sight the top of the tree along the straight edge of the protractor. The weight hanging down will show the complement to the angle of elevation. In other words, if the angle of elevation is 20°, the string will mark out a measurement of 70° on the protractor.

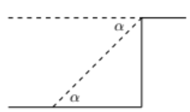


Closely related to the concept of the angle of elevation is the angle of depression. This is the angle that is formed by looking down on something from above.



The angle of elevation is θ.
The angle of depression is α.

In a situation in which the angle of depression is measured, the angle of elevation and the angle of depression are alternate interior angles, which makes them equal.

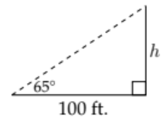


The angle of elevation equals the corrresponding angle of depression.

Example 1

Pacing off 100ft on flat ground from the base of a tree, a forester measures the angle of elevation to the top of the tree as 65°. What is the height of the tree?

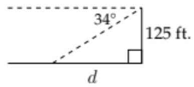
The situation described in the problem creates a diagram like the one below:



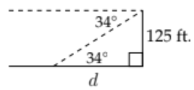
since this is a right triangle, we can use an appropriate trigonometric ratio to find the height of the tree. In this case,

$$\tan 65^{\circ} = \frac{h}{100} \tag{8.4.1}$$

$100 * \tan 65^{\circ} = h$
$214.45 \approx h$
So, the tree is about 214.45 feet tall.
Example 2
From the top of a building 125 feet tall, the angle of depression of an intersection is 34°. How far from the base of the building is the intersection?
As in the previous example, it is often helpful to draw a diagram.



Again, the angle of elevation will be equal to the corresponding angle of depression, so we can use the triangle as seen below to solve the problem:



In this problem

$$\tan 34^{\circ} = \frac{125}{d} \tag{8.4.2}$$

Multiply on both sides by d

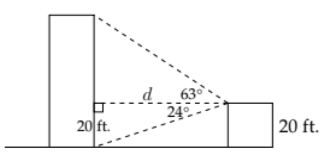
$$d * \tan 34^{\circ} = 125 \tag{8.4.3}$$

Then divide on both sides by $\tan 34^{\circ}$

$$\begin{gathered} \frac{d \tan 34^{\circ}}{\tan 34^{\circ}} = \frac{125}{\tan 34^{\circ}} \\ d \approx 185.32 \end{gathered} \tag{8.4.4}$$

Example 3
Sometimes a problem involves both an angle of elevation and an angle of depression.

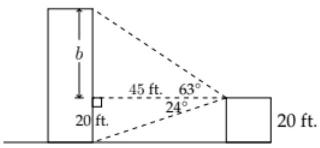
From the roof of a house 20 feet off the ground, the angle of elevation of the top of an apartment building is 63° and the angle of depression to the base of the building is 24°. How far away from the house is the apartment building? How tall is the apartment building?



If we work on the bottom triangle first, then we know that the height of the triangle is 20ft and the angle opposite this side is 24°. So, we can say that:

$$\begin{gathered}\tan 24^\circ = \frac{20}{d} \\ d = \frac{20}{\tan 24^\circ} \\ d \approx 45\text{ft}\end{gathered} \tag{8.4.5}$$

Now that we know that the apartment building is 45 feet away, we can use the upper triangle to determine the height ofthe building.

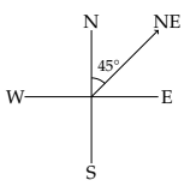


$$\begin{gathered}\tan 63^\circ = \frac{b}{45} \\ 45 * \tan 63^\circ = b \\ 88.3\text{ft.} \approx b \\ 20 + 88.3 = 108.3\text{ft}\end{gathered} \tag{8.4.6}$$

since the variable b only represents the part of the building that is in the second triangle, we need to add 20 feet to b to find the actual height of the building.

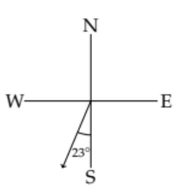
Problems on bearing and direction

Some applications of trigonometry involve ship navigation. One common method used to describe direction in this type of problem is what is known as bearing. Bearing describes a direction by the angle deviation from north or south. For example, the direction we typically describe as northeast is exactly halfway between north and east. The bearing for northeast would be $N45^\circ E$, and is read as "Forty-five degrees east of north."

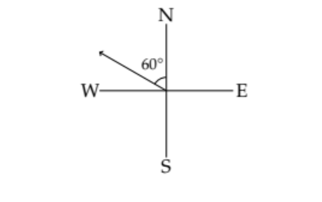


Forty-five degrees East of North: N45°E

Here are a few examples of what a bearing looks like in a N-S-E-W diagram.



Twenty-three degrees West of South: S23°W

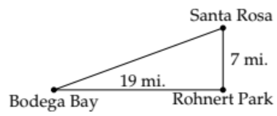


Sixty degrees West of North: N60°W

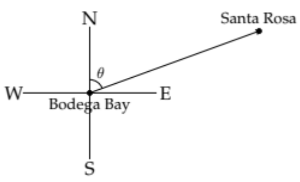
Example 4

Santa Rosa, California is 7 miles due north of Rohnert Park. Bodega Bay is 19 miles due west of Rohnert Park (as the crow flies). What is the bearing of Santa Rosa from Bodega Bay?

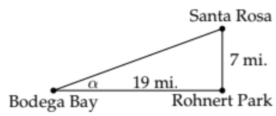
First, it would be helpful to draw a diagram to represent the situation:



To answer the question we'll need another diagram:



If we knew the angle θ, then we could conclude that the bearing of Santa Rosa from Bodega Bay is θ degrees East of North. From the previous diagram:



we can see that we can't find θ directly, but we can find the complement of θ

$$\tan \alpha=\frac{7}{19} \tag{8.4.7}$$

$\alpha \approx 20.2^{\circ}$

$$\text{Therefore, } \theta \approx 90^{\circ}-20.2^{\circ} \approx 69.8^{\circ} \tag{8.4.8}$$

That means that the bearing of Santa Rosa from Bodega Bay is $N 69.8^{\circ} E$, or 69.8° East of North.

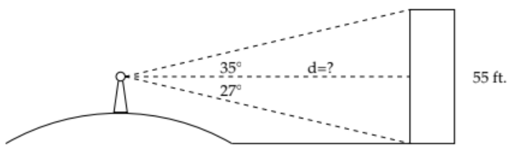
Exercises 1.4

Round answers to the nearest 10^{th}

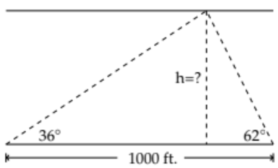
1. From the top of a lighthouse 180 feet above sea level, the angle of depression to a ship in the ocean is 28°. How far is the ship from the base of the lighthouse?
2. A helicopter that is 700 feet in the air measures the angle of depression to a landing pad as 24°. How far is the landing pad from the point directly beneath the helicopter's current position?
3. An 88 foot tree casts a shadow that is 135 feet long. What is the angle of elevation of the sun?
4. A 275 foot guy wire is attached to the top of a communication tower. If the wire makes an angle of 53° with the ground, how

tall is the tower?

5. A woman standing on a hill sees a building that she knows is 55 feet tall. The angle of depression to the bottom of the building is 27° and the angle of elevation to the top of the building is 35°. Find the straight line distance from the woman to the building.



6. To measure the height of the cloud cover at an airport, a spotlight is shined upward at an angle of 62°. An observer 1000 feet away measures the angle of elevation to the spotlight to be 36°. Find the height of the cloud cover.



7. At ground level, a water tower is 430 feet from the base of a building. From one of the upper floors of the building, the angle of elevation to the top of the water tower is 15° and the angle of depression to the bottom of the water tower is 28°. How tall is the water tower? How high off the ground is the observer?

8. A small airplane is flying at the altitude of 7000 feet following a straight road directly beneath it. A car in front of the airplane is sighted with an angle of depression of 72° and a car behind the plane is sighted with an angle of depression of 48°. How far apart are the cars?

9. From a point on the floor, the angle of elevation to the top of a doorway is 43". The angle of elevation to the ceiling directly above the doorway is 56°. If the ceiling is 10 feet above the floor, how high is the doorway? How far in front of the doorway were the angles of elevation measured?

10. A man standing on the roof of a building 70 feet high looks at the building next door. The angle of depression to the roof of the building next door is 36° The angle of depression to the bottom of the building next door is 65°. How tall is the building next door?

11. A boat leaves the harbor and travels 30 miles in the direction of $N38^\circ W$ The boat turns 90° and then travels in the direction $S52^\circ W$ for 12 miles. At that time, how far is the boat from the harbor and what is the bearing of the boat from the harbor entrance?

12. A man walking in the desert travels 1.6 miles in the direction $S57^\circ E$. He then turns 90° and continues walking for 3.2 miles in the direction $N33^\circ E$. At that time, how far is he from his starting point and what is his bearing from the starting point?

13. Madras, Oregon is 26 miles due north of Redmond. Prineville is due east of Redmond and $S34^\circ 42' E$ from Madras. How far is Prineville from Redmond?

14. Raymond, Washington is 22 miles due south of Aberdeen. Montesano is due east of Aberdeen and $N26^\circ 34' E$ from Raymond. How far is Montesano from Raymond?

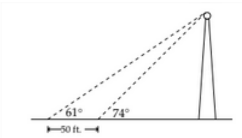
15. A boat travels on a course bearing of $S41^\circ 40' W$ for 84 miles. How far south and how far west is the boat from its starting point?

16. A boat travels on a course bearing of $N17^\circ 10' E$ for 10 miles. How far north and how far east is the boat from its starting point?

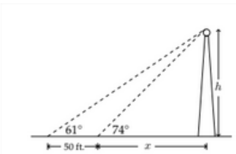
8.5: More Applications

1.5 More Applications

Sometimes solving problems involving right triangles requires the use of a system of equations. A common method for determining the height of an object whose base is inaccessible is that of measuring the angle of elevation from two different places in front of the object. If you measure the angle of elevation to the top of of a radio antenna as 74 , then walk back 50 feet and measure the angle of elevation to the top of the antenna as 61 ', then we would have something like the diagram below:



One of the first things we can do is introduce some labels for the unknown distances:



Then, we can say that:

$\tan 74^{\circ}=\frac{h}{x}$

$\tan 61^{\circ}=\frac{h}{x+50}$

To solve this system of equations, we'll set the first one equal to h :

$$\begin{gathered}\tan 74^{\circ}=\frac{h}{x} \\ x * \tan 74^{\circ}=h\end{gathered} \tag{8.5.1}$$

Then, substitute this into the second equation:

$$\tan 61^{\circ}=\frac{h}{x+30} \tag{8.5.2}$$

$$\tan 61^{\circ}=\frac{x \tan 74^{\circ}}{x+50} \tag{8.5.3}$$

Multiply on both sides by $x+50$:

$(x+50) \tan 61^{\circ}=\frac{x \tan 74^{\circ}}{x+30}(x+50)$

$(x+50) \tan 61^{\circ}=x \tan 74^{\circ}$

There are two options to solve this equation - we can hold on to the tangents
as they are and solve for x in terms $\tan 74^{\circ}$ and $\tan 61^{\circ}$, or we can approximate $\tan 74^{\circ}$ and $\tan 61^{\circ}$ and generate an approximate value for x and h. First we'll approximate:

$(x+50) \tan 61^{\circ}=x \tan 74^{\circ}$

$$(x+50) * 1.804 \approx 3.4874 x \tag{8.5.4}$$

$1.804 x+90.2024 \approx 3.4874 x$

$90.2024 \approx 1.6834 x$

$$53.58 \approx x \tag{8.5.5}$$

$x * \tan 74^{\circ}=h$

$$\begin{gathered}53.58 * \tan 74^{\circ} \approx h \\ 186.87 \text { feet } \approx h\end{gathered} \tag{8.5.6}$$

The other method is a little tricky algebraically:

$$(x+50) \tan 61^{\circ}=x \tan 74^{\circ} \tag{8.5.7}$$

$$
\begin{aligned}
& x \tan 61^{\circ}+50 \tan 61^{\circ}=x \tan 74^{\circ} \\
& 50 \tan 61^{\circ}=x \tan 74^{\circ}-x \tan 61^{\circ} \\
& 50 \tan 61^{\circ}=x\left(\tan 74^{\circ}-\tan 61^{\circ}\right) \\
& \frac{50 \tan 61^{\circ}}{\left(\tan 74^{\circ}-\tan 61^{\circ}\right)}=x
\end{aligned} \tag{8.5.8}
$$

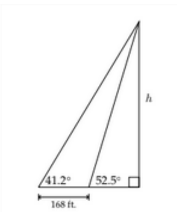
At this point, you can approximate the value of x and solve for h, or express the value of h exactly as

$$
\tan 74^{\circ} * \frac{50 \tan 61^{\circ}}{\left(\tan 74^{\circ}-\tan 61^{\circ}\right)}=h \tag{8.5.9}
$$

Exercises 1.5

1. Find the indicated height h

2. Find the indicated height h



3. A small airplane flying at an altitude of 5300 feet sights two cars in front of the plane traveling on a road directly beneath it. The angle of depression to the nearest car is 62° and the angle of depression to the more distant car is 41°. How far apart are the cars?
4. A hot air balloon is flying above a straight road. In order to estimate their altitude, the people in the balloon measure the angles of depression to two consecutive mile markers on the same side of the balloon. The angle to the doser marker is 17° and the angle to the farther one is 13°. At what altitude is the balloon flying?
5. To estimate the height of a mountain, the angle of elevation from a spot
on level ground to the top of the mountain is measured to be 32°. From a point

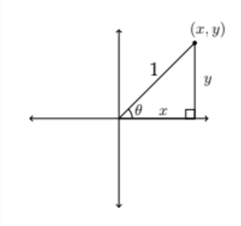
1000 feet closer to the mountain, the angle of elevation is measured to be 35° How high is the mountain above the ground from which the measurements were taken?
6. The angle of elevation from a point on the ground to the top of a pyramid is $35^{\circ}30'$. The angle of elevation from a point 135 feet farther back to the top of the pyramid is $21^{\circ}10$. What is the height of the pyramid?
7. An observer in a lighthouse 70 feet above sea level sights the angle of depression of an approaching ship to be $15^{\circ}50'$. A few minutes later the angle of depression is sighted at $35^{\circ}40'$. Find the distance traveled by the ship during that time.
8. To estimate the height of a tree, one forester stands due west of the tree and another forester stands due north of the tree. The two foresters are the same distance from the base of the tree and they are 45 feet from each other. If the angle of elevation for each forester is 40°, how tall is the tree?
9. A ship is anchored off of a long straight shoreline that runs east to west. From two observation points located 10 miles apart on the shoreline, the bearings of the ship from each observation point are $S35^{\circ}E$ and $S17^{\circ}W$. How far from shore is the ship?
10. From fire lookout Station Alpha the bearing of a forest fire is $N52^{\circ}E$. From lookout Station Beta, sited 6 miles due east of Station Alpha, the bearing is $N38^{\circ}\mathrm{W}$ How far is the fire from Station Alpha?
11. From a point 200 feet from the base of a church, the angle of elevation to the top of the steeple is 28°, while the angle of elevation to the bottom of the steeple is 20°. How high off the ground is the top of the steeple?
12. A television tower 75 feet tall is installed on the top of a building. From a point on the ground in front of the building, the angle of elevation to the top of the tower is 62° and the the angle of elevation to the bottom of the tower is 44°. How tall is the building?

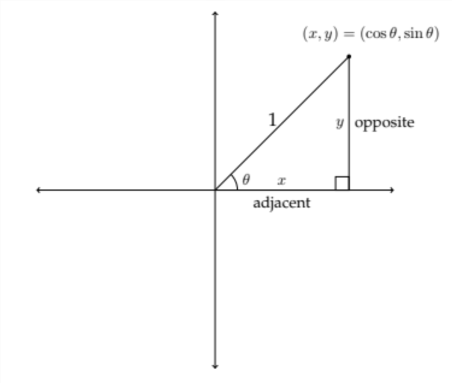
CHAPTER OVERVIEW

9: Graphing the Trigonometric Functions

9.1: Trigonometric Functions of Non-Acute Angles

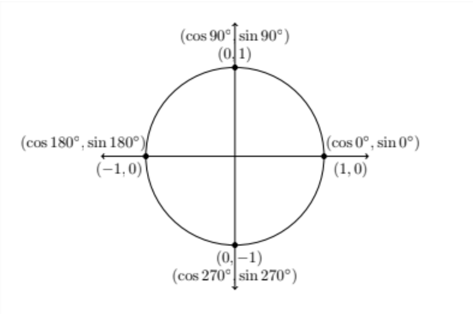
2.1 Trigonometric Functions of Non-Acute Angles
In Chapter 1, we learned about the trigonometric functions of positive acute angles that occur within right triangles. If we wish to extend the definition of the trigonometric functions, then we need to define how to determine the values for the sine and cosine of other angles. To do this, consider a right triangle drawn on the coordinate axes. The positive acute angle θ will be the angle created between the x -axis and the hypotenuse of the triangle. The lengths of the two legs of the triangle will be the x and y coordinates of a point in the first quadrant.





In the picture above we see a triangle in the first quadrant with a hypotenuse of 1. In this situation, the value of $\sin\theta = \frac{opp}{hyp} = \frac{y}{1} = y,$ which is just the y -coordinate of the point at the top of the triangle. Correspondingly, the value of $\cos\theta = \frac{adj}{hyp} = \frac{x}{1} = x$, or the value of the x -coordinate of the same point.

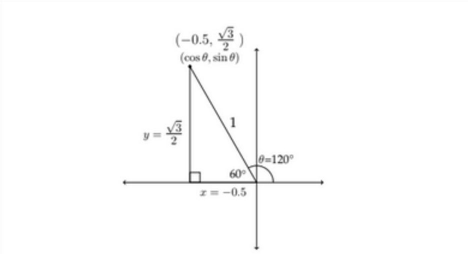
This allows us to find the sine or cosine for what are known as the quadrantal angles - the angles that are multiples of 90°. If we look at the unit circle (the circle with a radius of 1), then we can see the values of the sine and cosine for these angles.



In the previous diagram, we see the values for the sine and cosine of the quadrantal angles:

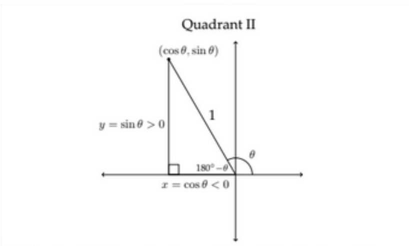
$$\cos 0^{\circ} = 1 \quad \cos 90^{\circ} = 0 \quad \cos 180^{\circ} = -1 \quad \cos 270^{\circ} = 0 \tag{9.1.1}$$

$\sin 0^{\circ} = 0 \quad \sin 90^{\circ} = 1 \quad \sin 180^{\circ} = 0 \quad \sin 270^{\circ} = -1$
If we take a radius of length 1 and rotate it counter-clockwise in the coordinate plane, the x and y coordinates of the point at the tip will correspond to the values of the cosine and sine of the angle that is created in the rotation. Let's look at an example in the second quadrant. If we rotate a line segment of length 1 by 120 ", it will terminate in Quadrant II.

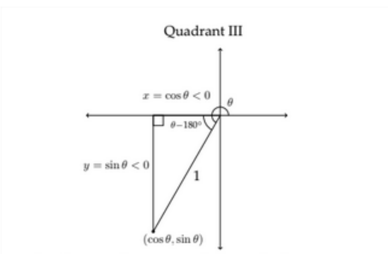


In the diagram above we notice several things. The radius of length 1 has been rotated by 120^{ento} Quadrant II. If we then drop a perpendicular line from the endpoint of the radius to the x -axis, we create a triangle in Quadrant II. Notice that the angle supplementary to 120° appears in the triangle and this allows us to find the lengths of the sides of the triangle and hence the values for the x and y coordinates of the point at the tip of the radius.

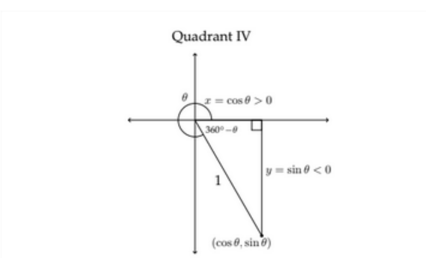
Whenever an angle greater than 90° is created on the coordinate axes, simply drop a perpendicular to the x -axis. The angle created is the reference angle. The values of the trigonometric functions of the angle of rotation and the reference angle will differ only in their sign $(+,-)$. On the next page are examples for Quadrants 1,
III, and IV.



In Quadrant II, the cosine is negative and the sine is positive.

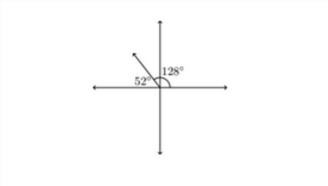


In Quadrant III, the cosine and sine are both negative.

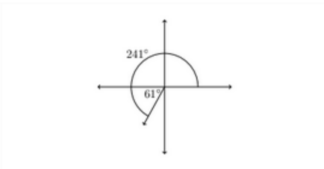


In Quadrant IV, the cosine is positive and the sine is negative.

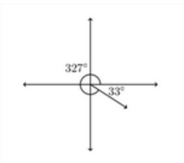
The process for finding reference angles depends on which quadrant the angle terminates in.
Examples
Find the reference angle for the following angles:
1.128°
2.41°
3.327°
1. An angle of 128° terminates in Quadrant II. To find the reference angle, we would subtract the angle from $180^{\circ}: \quad 180^{\circ}-128^{\circ}=52^{\circ}$



2. An angle of 241° terminates in Quadrant III. To find the reference angle, we would subtract 180° from the angle: $241^{\circ}-180^{\circ}=61^{\circ}$

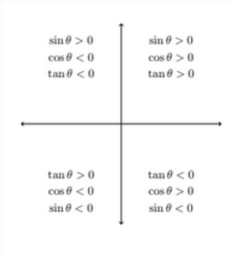


3. An angle of 327° terminates in Quadrant IV. To find the reference angle, we subtract the angle from 360° : $360^{\circ}-327^{\circ}=33^{\circ}$

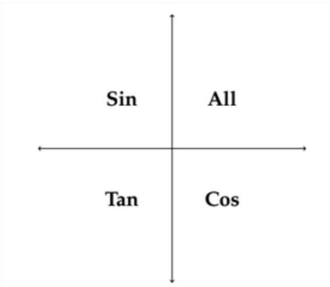


Once we know the reference angle, we can find the trigonometric functions for the original angle itself. In example 1, we had 128°, an angle in Quadrant II with a reference angle of 52°. Therefore, if we want to find the sine, cosine and tangent of 128°, then we should find the sine, cosine and tangent of 52° and apply the appropriate positive or negative sign.

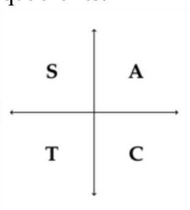
Example 1 Quadrant II
In Quadrant II, x -coordinates are negative and y -coordinates are positive. This means that $\cos\theta<0$ and $\sin\theta>0$. The values for this process are given below:
$\sin 52^{\circ}\approx 0.7880$
$\sin 128^{\circ}\approx 0.7880$
$\cos 52^{\circ}\approx 0.6157 \quad \cos 128^{\circ}\approx -0.6157$
$\tan 52^{\circ}\approx 1.280 \quad \tan 128^{\circ}\approx -1.280^{\circ}$
Example 2 Quadrant III
In Quadrant III, x -coordinates are negative and y -coordinates are also negative. This means that $\cos\theta<0$ and $\sin\theta<0$. The values for this process are given below:
$\sin 61^{\circ}\approx 0.8746 \quad \sin 241^{\circ}\approx -0.8746$
$\cos 61^{\circ}\approx 0.4848$
$\cos 241^{\circ}\approx -0.4848$
$\tan 61^{\circ}\approx 1.8040 \quad \tan 241^{\circ}\approx 1.8040$
Example 3 Quadrant IV
In Quadrant IV, x -coordinates are positive and y -coordinates are negative. This means that $\cos\theta>0$ and $\sin\theta<0$. The values for this process are given below:
$\sin 33^{\circ}\approx 0.5446$
$\sin 327^{\circ}\approx -0.5446$
$\cos 33^{\circ}\approx 0.8387$
$\cos 327^{\circ}\approx 0.8387$
$\tan 33^{\circ}\approx 0.6494 \quad \tan 327^{\circ}\approx -0.6494$



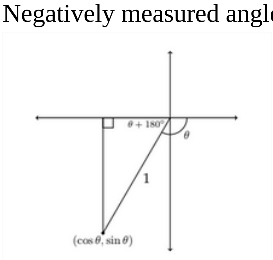
In Quadrant I, ALL the trigonometric functions are positive. In Quadrant II, the SIN function is positive (as well as the CSC).
In Quadrant III, the TAN function is positive (as well as the COT).
In Quadrant IV, the COS function is positive (as well as the SEC).



A common mneumonic device to remember these relationships is the phrase:
"All Students Take Calculus." This can help you remember which trigonometric functions are positive in each of the four quadrants.



Reference Angles for Negative Angles
Negatively measured angles rotate in a clockwise direction.

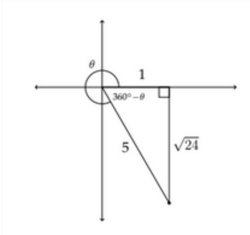


There are a variety of methods for finding the reference angle for a negatively valued angle. You can find a positive angle that is co-terminal with the negative angle and then find the reference angle for the positive angle. You can also drop a perpendicular to the x -axis to find the reference angle for the negative angle directly.

For example, the angle -120° terminates in Quadrant III and is co-terminal with the positive angle 240°. Either way, when you drop a perpendicular to the x -axis, you find that the reference angle is 60°.

If you are given the value of one of the trigonometric functions of an angle θ and know which quadrant θ is located in, you can find the other trigonometric functions for that angle.
Example
Given θ in Quadrant IV with $\cos\theta = \frac{1}{5}$, find $\sin\theta$ and $\tan\theta$
If $\cos\theta = \frac{1}{5}$, then the adjacent side and the hypotenuse must be in a ratio of 15.

We can label these sides as 1 and 5 and then find the length of the third side in the triangle. This will allow us to find $\sin\theta$ and $\tan\theta$



Using the Pythagorean Theorem:
13
$25-10$
$x=1$
we find that the side opposite the reference angle for θ is $\sqrt{24}$ or $2\sqrt{6}$. We can now find $\sin\theta$ and $\tan\theta$:

$$\sin\theta=\frac{\sqrt{24}}{5} \tag{9.1.2}$$

and

$$\tan\theta=\frac{\sqrt{24}}{1}=\sqrt{24} \tag{9.1.3}$$

In the problems in this section, the reciprocal functions secant, cosecant and cotangent are used. Remember that:

$$\sec\theta=\frac{1}{\cos\theta}=\frac{hyp}{ady} \tag{9.1.4}$$

$\csc\theta=\frac{1}{\sin\theta}=\frac{hyp}{\alpha pp}$
$\cot\theta=\frac{1}{\tan\theta}=\frac{adj}{opp}$
Exercises 2.1
Determine the quadrant in which the angle θ lies.
1. $\cos\theta>0,\tan\theta>0$
$\sin\theta<0,\cos\theta>0$
3. $\sec\theta>0,\tan\theta<0$
4. $\cot\theta>0,\cos\theta<0$
5. $\sin\theta>0,\cos\theta<0$
6. $\sin\theta>0,\cot\theta>0$
7. $\sin\theta<0,\cos\theta<0$
8. $\csc\theta>0,\cot\theta<0$
Determine which quadrant the given angle terminates in and find the reference angle for each.
9.195°
10.330°
11.120°
12.210°
13.135°
14.300°
15. -100°
16.225°
17. 315°
18. $\frac{5\pi}{4}$
19. $-\frac{2\pi}{3}$
20. $\frac{7\pi}{3}$
21. $\frac{11\pi}{4}$
22. $\frac{7\pi}{6}$
23. $\frac{11\pi}{6}$

Find $\sin \theta$, $\cos \theta$ and tan θ in each problem.

24. $\sin \theta=-\frac{12}{13}, \theta$ in Quadrant IV
25. $\cos \theta=-\frac{4}{5}, \theta$ in Quadrant II
26. $\cos \theta=\frac{1}{4}, \theta$ in Quadrant I
27. $\tan \theta=\frac{3}{2}, \theta$ in Quadrant III
28. $\tan \theta=-\frac{4}{5}, \theta$ in Quadrant II
29. $\sin \theta=\frac{3}{8}, \theta$ in Quadrant II
30. $\sin \theta=-\frac{1}{3}, \theta$ in Quadrant III
31. $\tan \theta=5, \theta$ in Quadrant I
32. $\sec \theta=-2, \tan \theta<0$
33. $\cot \theta=\sqrt{3}, \cos \theta<0$
34. $\tan \theta=-\frac{1}{3}, \sin \theta<0$
35. $\csc \theta=\sqrt{2}, \cos \theta>0$
36. $\cos \theta=-\frac{2}{5}, \tan \theta>0$
37. $\sec \theta=2, \sin \theta<0$
38. $\sin \theta=\frac{1}{\sqrt{2}}, \cos \theta>0$
39. $\sin \theta=-\frac{2}{3}, \cot \theta>0$

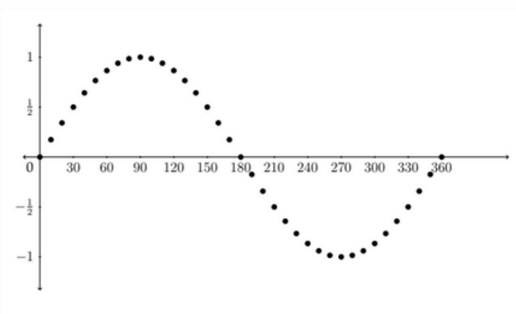
9.2: Graphing Trigonometric Functions

We have seen how to determine the values of trigonometric functions for angles terminating in Quadrants II, III, and IV. This allows us to make a graph of the values of the sine function for any angle. In the chart below, I have listed the values for the sine function for angles between 0° and 360°.

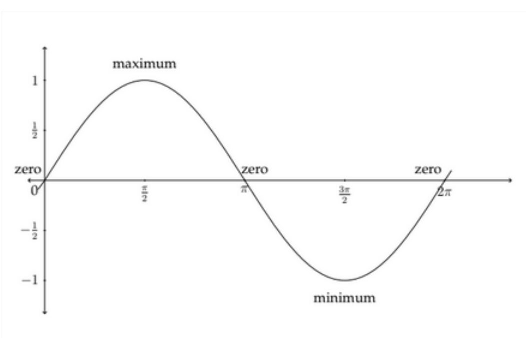
θ	$\sin\theta$		θ	$\sin\theta$
0°	$=0$		100°	≈ 0.9848
10°	≈ 0.1737		110°	≈ 0.9397
20°	≈ 0.3420		120°	≈ 0.8660
30°	$=0.5$		130°	≈ 0.7660
40°	≈ 0.6428		140°	≈ 0.6428
50°	≈ 0.7660		150°	$=0.5$
60°	≈ 0.8660		160°	≈ 0.3420
70°	≈ 0.9397		170°	≈ 0.1737
80°	≈ 0.9848		180°	$=0$
90°	$=1$			

θ	$\sin\theta$		θ	$\sin\theta$
180°	$=0$		280°	≈ -0.9848
190°	≈ -0.1737		290°	≈ -0.9397
200°	≈ -0.3420		300°	≈ -0.8660
210°	$=-0.5$		310°	≈ -0.7660
220°	≈ -0.6428		320°	≈ -0.6428
230°	≈ -0.7660		330°	$=-0.5$
240°	≈ -0.8660		340°	≈ -0.3420
250°	≈ -0.9397		350°	≈ -0.1737
260°	≈ -0.9848		360°	$=0$
270°	$=-1$			

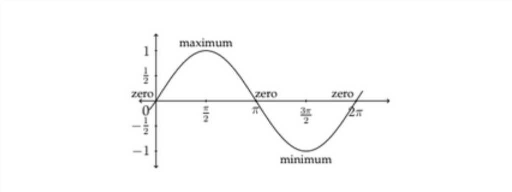
Below we see a graph of these points plotted on the coordinate axes.



In graphing trigonometric functions, we typically use radian measure along the x -axis, so the graph would generally look like this:

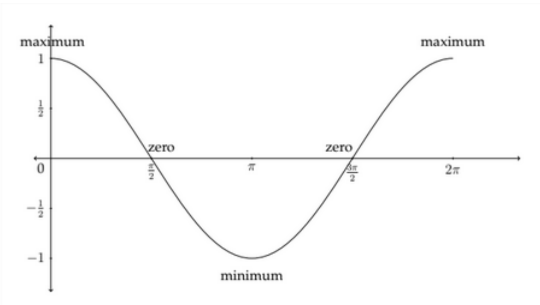


The graph of the standard sine function begins at the zero point, then rises to the maximum value of 1 between 0 and $\frac{7}{3}$ radians. It then decreases back to 0 at
\pi radians before crossing over into the negative values and hitting its minimum value at $\frac{3\pi}{2}$ radians. It then goes back up to 0 at 2π radians before starting all over again.



The standard cosine graph behaves in a similar but slightly different way. We saw earlier that $\cos 0^{\circ} = 1$, so the cosine graph would start at the point $(0, 1)$, then gradually decrease to zero. A picture of the standard cosine graph would look like the figure below:

maximum



The sine and cosine graphs are sometimes referred to as a "sine wave" or "sinusoid" and can be very useful in modeling phenomena that occur in waves. Examples of this are the rise and fall of the tides; sound waves and music; electricity; and the length of day throughout the year. The standard sine and cosine graphs must be modified to fit a particular application so that they will effectively model the situation. The ideas that we examine next will explain how to modify the sine and cosine graphs to fit a variety of different situations.

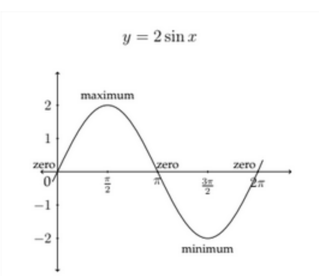
There are four aspects to the sine and cosine functions to take into consideration when making a graph. These are:

1) The Amplitude of the graph
2) The Period of the graph
3) The Vertical Shift of the graph
4) The Phase Shift of the graph

Amplitude

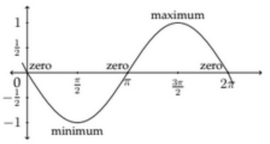
The amplitude of a sine or cosine function refers to the maximum and minimum values of the function. In the standard sine and cosine graphs, the maximum value is 1 and the minimum value is -1. The amplitude is one-half the difference between the maximum and minimum values. In the standard graphs the difference between the maximum and minimum is $1-(-1)=2;$ one-half of this is 1 so the amplitude of the standard sine and cosine functions is 1.

The value of the amplitude is also the absolute value of the coefficient of the sine or cosine expression. In the standard graph, $y=\sin x$, the coefficient of the sine function is 1, so the amplitude is 1. In the function $y=2\sin x$, all the y values will be multiplied by 2 and the amplitude of the function will be 2. The graph for $y=2\sin x$ is shown on the next page.



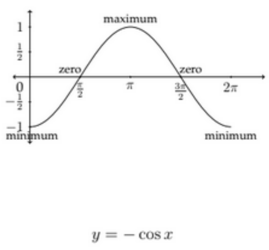
A negative value of the coefficient in front of a trigonometric function will not change the Amplitude of the function, but it will change the shape of the function. For example, the function:

has an amplitude of 1, but the graph will be different from the graph $y=\sin x$ All of the y -values of the function $y=-\sin x$ will have the opposite sign as the y -values of the function $y=\sin x$. The graph for $y=-\sin x$ appears below:



Notice that, because of the negation of the y -values, the graph begins at 0, as does the standard sine function, but the graph of $y=-\sin x$ first goes to a minimum value before crossing through 0 again up to the maximum value.

Likewise, the graph of $y=-\cos x$ begins at the minimum value before crossing through 0 and going to the maximum value, back through 0 and ending at the minimum value again.



Period

The period of the graph refers to how long it takes the graph to complete one full cycle of values. In the standard sine and cosine functions, the period is 2π radians. The function completes a single "wave" and returns to its starting place between 0 and 2π. A coefficient in front of the variable in a sine or cosine function will affect the period of the graph. In the general expression $y=A\sin Bx$, the value of A affects the amplitude of the function and the value of B affects the period of the function.

If we examine the table of values for the standard sine function, we can see how the coefficient of the x -variable will affect the period of the graph. Starting with the table from the standard sine function:

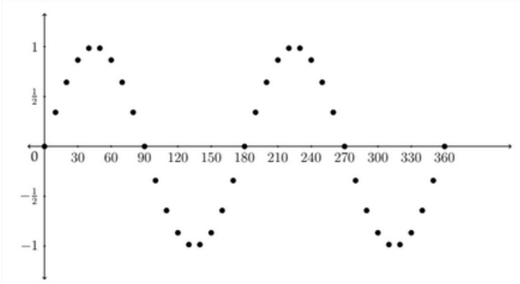
θ	$\sin\theta$		θ	$\sin\theta$
0°	$=0$		100°	≈ 0.9848
10°	≈ 0.1737		110°	≈ 0.9397
20°	≈ 0.3420		120°	≈ 0.8660
30°	≈ 0.5		130°	≈ 0.7660
40°	≈ 0.6428		140°	≈ 0.6428
50°	≈ 0.7660		150°	≈ 0.5
60°	≈ 0.8660		160°	≈ 0.3420
70°	≈ 0.9397		170°	≈ 0.1737
80°	≈ 0.9848		180°	$=0$
90°	$=1$			

θ	$\sin\theta$		θ	$\sin\theta$
180°	$=0$		280°	≈ -0.9848
190°	≈ -0.1737		290°	≈ -0.9397
200°	≈ -0.3420		300°	≈ -0.8660
210°	≈ -0.5		310°	≈ -0.7660
220°	≈ -0.6428		320°	≈ -0.6428
230°	≈ -0.7660		330°	$=-0.5$
240°	≈ -0.8660		340°	≈ -0.3420
250°	≈ -0.9397		350°	≈ -0.1737
260°	≈ -0.9848		360°	$=0$
270°	$=-1$			

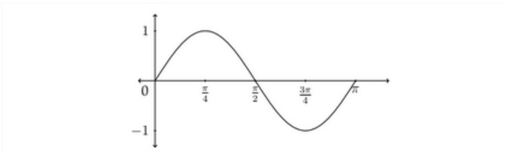
If we create a similar table for the function $y=\sin(2x)$, then we can see how this will affect the graph:

θ	2θ	$\sin(2\theta)$		θ	2θ	$\sin(2\theta)$
0°	0°	$=0$		100°	200°	≈ -0.3420
10°	20°	≈ 0.3420		110°	220°	≈ -0.6428
20°	40°	≈ 0.6428		120°	240°	≈ -0.8660
30°	60°	≈ 0.8660		130°	260°	≈ -0.9848
40°	80°	≈ 0.9848		140°	280°	≈ -0.9848
50°	100°	≈ 0.9848		150°	300°	≈ -0.8660
60°	120°	≈ 0.8660		160°	320°	≈ -0.6428
70°	140°	≈ 0.6428		170°	340°	≈ -0.3420
80°	160°	≈ 0.3420		180°	360°	$=0$
90°	180°	$=0$				

In the previous table we can see that the function $y=\sin(2x)$ completes one full cycle between 0 and π radians instead of the the standard 0 to 2π radians. The graph for these points is shown below. The coordinates for the x -values between π and 2π radians are shown as well.



In this graph, you can see that there are two complete waves between 0 and 2π radians, or one complete wave between 0 and π radians. So, in a sine or cosine function of the form $y=A\sin Bx$, the amplitude will be $|A|$ and the period will be $\frac{2\pi}{b}$. The standard graph for one complete cycle of the function $y=\sin(2x)$ is shown below:



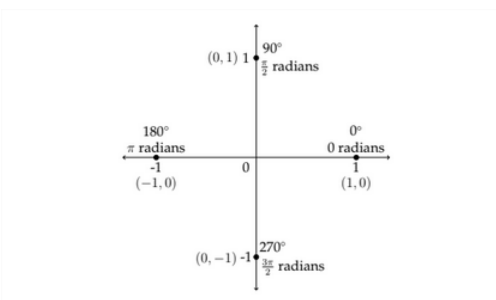
Notice that, because the period has been cut in half, the x -coordinates that correspond to the maximum, minimum, and zero y -coondinates are cut in half as

Example 1

Graph one full period of the function $y=-2\sin 3x$

The amplitude in this case is 2, but since the coefficient is negative, this sine graph will begin by first going to the minimum value. The period of the graph will be $\frac{2\pi}{B}$, or in this case $\frac{2\pi}{3}$ instead of 2π. To determine the x -values for the maximum, minimum and zero y -values, we should examine how these are determined for the standard sine curve.

The maximum, minimum and zero y-values for a standard sine curve occur at the quadrantal angles, that is to say, the angles that separate the four quadrants from each other. The quadrantal angles are 0 ' or 0 radians, 90° or $\frac{\pi}{2}$ radians, 180° or \pi radians, 270° or $\frac{3\pi}{2}$ radians and 360° or 2π radians. These x -values produce the "critical" y-values of the zero, maximum and minimum.



In the standard sine or cosine graph, the distance from each "critical value" of the

graph to the next is always a "jump" of $\frac{\pi}{2}$ along the x -axis. This is one-fourth of the period: $\frac{2\pi}{1}*\frac{1}{4}=\frac{\pi}{2}$. So, to determine the labels for the critical values of the graph along the x -axis, we should take the new period and multiply by $\frac{1}{4}$

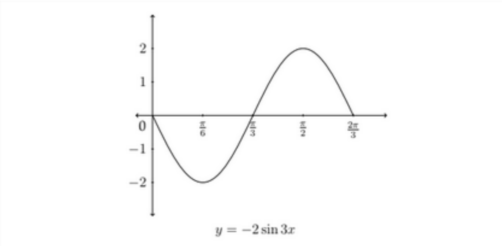
The function we are working with is $y=-2\sin 3x$, so to find the new period we calculated $\frac{2\pi}{B}$, which was $\frac{2\pi}{3}$. Then, in order to label the x -axis properly we should next take $\frac{2\pi}{3}$ and multiply by $\frac{1}{4}$

$$\frac{2\pi}{3}*\frac{1}{4}=\frac{2\pi}{12}=\frac{\pi}{6} \tag{9.2.1}$$

So, the critical values along the x -axis will be:

$$\frac{1\pi}{6},\frac{2\pi}{6},\frac{3\pi}{6}, \text{ and } \frac{4\pi}{6} \tag{9.2.2}$$

We want to express these in lowest terms, so we would label them as $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$, and $\frac{2\pi}{3}$. The graph will start at zero, then (because the value of the coefficient A is negative) it will go down to a minimum value at $\frac{\pi}{6}$, back to zero at $\frac{\pi}{3}$, then up to the maximum at $\frac{\pi}{2}$ and back down to zero at $\frac{2\pi}{3}$ to complete one full period of the graph. The graph for this function is pictured below. Notice that the minimum y-value is -2 and the maximum y -value is 2 because $A=2$



Let's look at an example using the cosine graph.

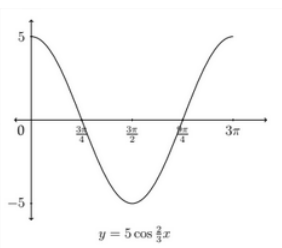
Example 2

Graph one full period of the function $y=5\cos\frac{2}{3}x$

The amplitude of the function is 5 because $A=5$, so the maximum y-value will be 5 and the minimum y -value will be -5. The period of the graph will be $\frac{2\pi}{B}$ which in this case is $\frac{2\pi}{7}=2\pi * \frac{3}{2}=3\pi$. So the period is 3 π. The critical values along the x -axis will start at 0 and be separated by "jumps" of $3\pi * \frac{1}{4}=\frac{3\pi}{4}$. So the critical values along the x -axis will be:

$$0, \frac{3\pi}{4}, \frac{6\pi}{4}, \frac{9\pi}{4}, \text{ and } \frac{12\pi}{4} \tag{9.2.3}$$

We want to express these in lowest terms so we would label them as $\frac{3\pi}{4}, \frac{3\pi}{2}, \frac{9\pi}{4}$ and 3π. The graph will start at the maximum y -value of 5 at $x=0$, then it will go to zero at $x=\frac{3\pi}{4}$, down to the minimum y -value of -5 at $x=\frac{3\pi}{2}$, back through 0 at $x=\frac{9\pi}{4}$, and then up to the maximum y -value of 5 at $x=3\pi$ to complete one full period of the graph. The graph of $y=5\cos\frac{2}{3}x$. is shown below.

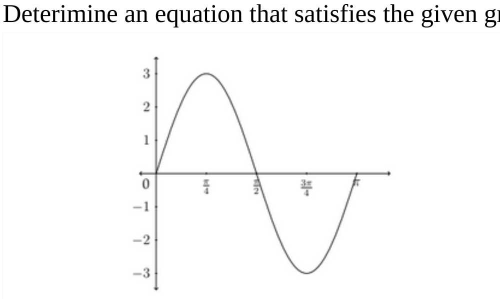


Determining an equation from a graph

Sometimes, you will be given a graph and asked to determine an equation which satisfies the conditions visible in the graph. So far, we have only discussed two of the possible transformations of a trigonometric function - the amplitude and period. Remember that in an equation of the form $y=A\sin Bx$ or $y=A\cos Bx$, the amplitude is $|A|$ and the period is $\frac{2\pi}{B}$. So, to write an equation for a trigonometric function, we need to determine the values of A and B.

Example 3

Deterimine an equation that satisfies the given graph.



First note that the maximum y -value for the graph is 3 and the minimum is -3 This means that the amplitude is 3. Next we see that there is one complete period of the function between 0 and π, this means that the period is π. From this information, we know that $A=3$ and that the period for the graph is π. Since the period $P=\frac{2\pi}{B}$, then we know that $B=\frac{2\pi}{P}$. So, $B=\frac{2\pi}{\pi}=2$

Lastly, this graph starts with a y -value of 5, which is the maximum y -value. It then goes to 0 and down to the minimum, back through 0 and then back to the maximum to form one complete wave. since this is the signature of the cosine function, the answer to this problem would be:

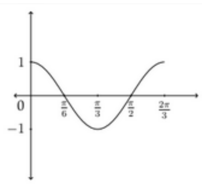
$$y=5\cos\frac{8}{9}x \tag{9.2.4}$$

Exercises 2.2

Determine the amplitude and the period for each problem and graph one period of the function. Identify important points on the x and y axes.

1. $y=\cos 4x$
2. $y=-\sin 2x$
3. $y=3\sin 3x$
4. $y=-2\cos 5x$
5. $y=4\cos\frac{1}{2}x$
6. $y=2\sin\frac{1}{3}x$
7. $y=-\frac{1}{2}\sin\frac{2}{3}x$
8. $y=-3\cos\frac{1}{5}x$
9. $y=-4\sin 6x$
10. $y=3\sin 4x$
11. $y=2\cos\frac{1}{2}x$
12. $y=3\cos\frac{1}{3}x$

Determine an equation that satifies the given graph.

13.


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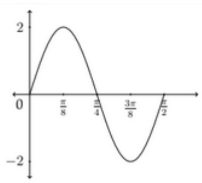

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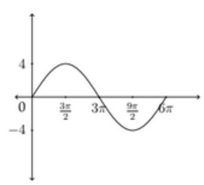

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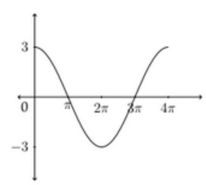
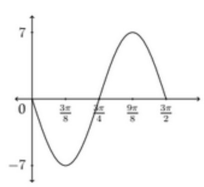
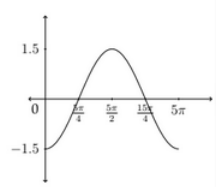


17.


18.


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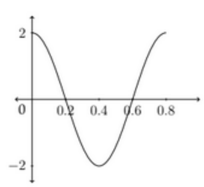
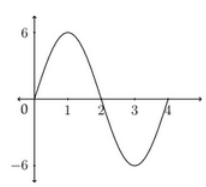
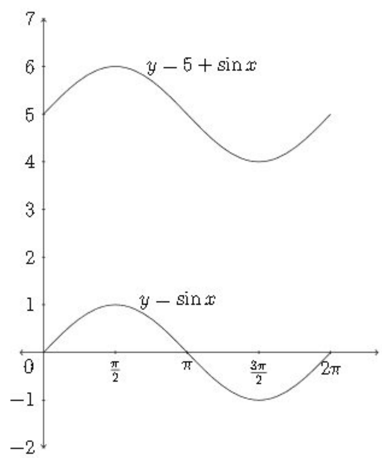


20.


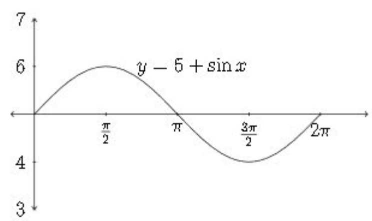
9.3: The Vertical Shift of a Trigonometric Function

If a constant is added or subtracted to a trigonometric function, this will affect the y -values of the function If we consider the function $y-5+\sin x,$ then each of the standard y -values would have 5 added to if, which would shift the graph up 5 units.
The chart below considers just the quadrantal values for the sine function:

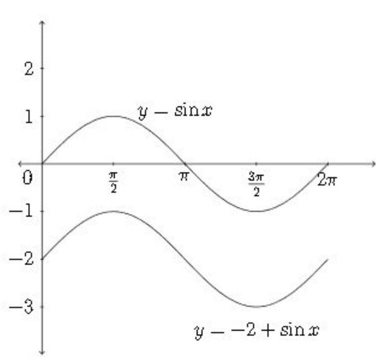
```
\begin{tabular}{|c|c|c|}
\hline\(\theta\) & \(\sin \theta\) & \(5+\sin \theta\) \\
\hline 0 & 0 & 5 \\
\hline\(\pi / 2\) & 1 & 6 \\
\hline\(\pi\) & 0 & 5 \\
\hline \(3 \pi / 2\) & -1 & 4 \\
\hline \(2 \pi\) & 0 & 5 \\
\hline
\end{tabular}
```



Sometimes the x -axis is drawn through the line that is the new "zero" or "midline" for the function - in this case it would be $y-5$



Likewise, a negative constant would move the graph down, as each y-value would be less than the corresponding y -value in the standard sine function



In the previous examples the constant has been written in front of the sine function for clarity. Often the constant is written after the function:

$$\begin{gathered} y-\sin x+5 \\ \text{or} \\ y-\sin x-2 \end{gathered} \tag{9.3.1}$$

We have now examined three of the four transformations of trigonometric functions that are discussed in this chapter - amplitude, period and vertical shift. A general equation for a sinusoid that involves these three transformations would be:

$$y=A\sin(Bx)+D \tag{9.3.2}$$

Or

$$y=A\cos(Bx)+D \tag{9.3.3}$$

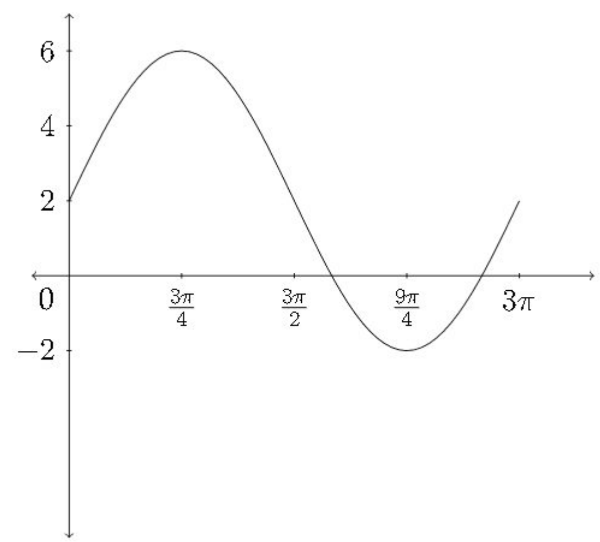
In determining an equation from a graph that involves a vertical shift, the value of A will be half the distance between the maximum and minimum values:

$$A=\frac{max-min}{2} \tag{9.3.4}$$

and the value of D will be the average of the maximum and minimum values:

$$D=\frac{max+min}{2} \tag{9.3.5}$$

Example
Determine an equation that satisfies the given graph.



In this graph, the maximum y -value is 6 and the minimum y -value is -2. The average of these two:

$$\frac{max+min}{2}=\frac{6+(-2)}{2}=\frac{4}{2}=2=D \tag{9.3.6}$$

is the value of D, the vertical shift.
The distance between 6 and -2 is $6-(-2)=8$. Half the distance between the $\max$ and $\min$ is 4, which is the value of A

$$\frac{max-min}{2}=\frac{6-(-2)}{2}=\frac{8}{2}=4=A \tag{9.3.7}$$

The graph completes one full cycle between 0 and 3π, so the period would be 3π and the value of B would be $B=\frac{2\pi}{P}=\frac{2\pi}{3\pi}=\frac{2}{3}=B$. So a correct equation for the graph would be:

$$y = 4\sin\frac{2}{3}x + 2 \tag{9.3.8}$$

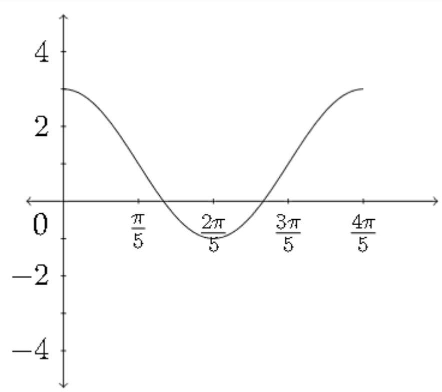
Exercises 2.3

Determine the Amplitude, Period and Vertical Shift for each function below and graph one period of the function. Identify the important points on the x and y axes.

1. $y = \sin x + 1$
2. $y = \cos x - 1$
3. $y = 2\cos x - \frac{1}{2}$
4. $y = 5\sin x + 4$
5. $y = -\sin\left(\frac{1}{4}x\right) + 1$
6. $y = -\cos(2x) + 7$
7. $y = \frac{1}{3}\sin(\pi x) - 4$
8. $y = -\frac{1}{2}\cos(2\pi x) + 2$
9. $y = 5\cos\left(\frac{1}{2}x\right) + 1$
10. $y = 4\sin\left(\frac{1}{3}x\right) - 1$
11. $y = 3\cos x + 2$
12. $y = 2\sin x + 3$
13. $y = 2 - 4\cos(3x)$
14. $y = 5 - 3\sin(2x)$

Determine an equation that satisfies the given graph.

15.



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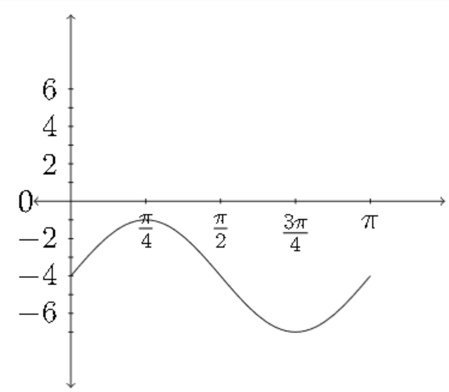
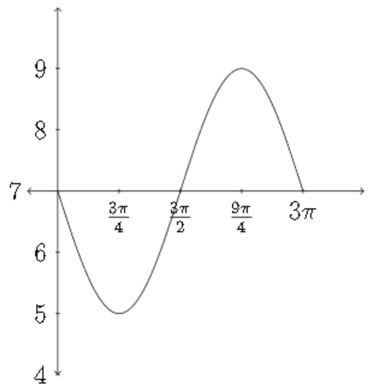
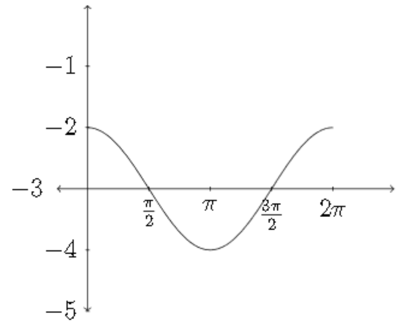


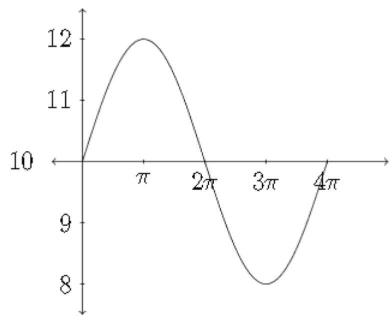
17.


9
8
7
6
5
4
$\frac{3\pi}{4}$
$\frac{3\pi}{2}$
$\frac{9\pi}{4}$
3π

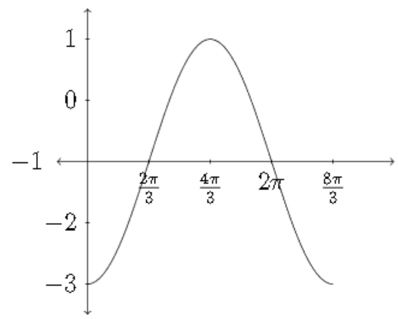
18.


-1
-2
-3
-4
-5
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

19.


12
11
10
9
8
π
2π
3π
4π

20.


1
0
-1
-2
-3
$\frac{2\pi}{3}$
$\frac{4\pi}{3}$
2π
$\frac{8\pi}{3}$

9.4: Phase Shift

The last form of transformation we will discuss in the graphing of trigonometric functions is the phase shift, or horizontal displacement. So far, we have considered the amplitude, period and vertical shift transformations of trigonometric functions. In the standard equation $y = A\sin(Bx) + D$, these corrrespond to the coefficients A, B and D. Notice that the amplitude and vertical shift coefficients $(A \text{ and } D)$, which affect the y -axis occur outside of the trigonometric function, whereas the coefficient that affects the period of the graph along the x -axis occurs within the sine function. This is true of the phase shift as well.
If we consider a general equation of:

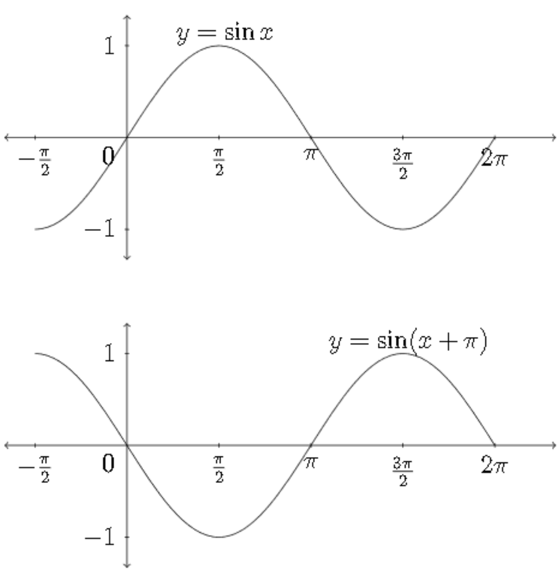
$$y = A\sin(Bx + C) + D \tag{9.4.1}$$

the constant C will affect the phase shift, or horizontal displacement of the function. Let's look at a simple example.

Graph at least one period of the given function: $y = \sin(x + \pi)$ Be sure to indicate important points along the x and y axes.
Let's examine this function by looking at a table of values.

x	$x+\pi$	$\sin(x+\pi)$
0	π	0
$\pi/2$	$3\pi/2$	-1
π	2π	0
$3\pi/2$	$5\pi/2$	1
2π	3π	0

Now let's look at a graph of $y = \sin(x + \pi)$ as compared to the standard graph of $y = \sin x$



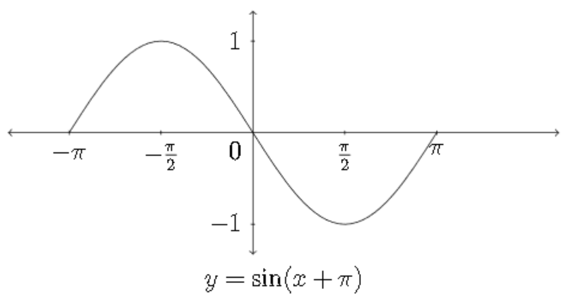
Notice that if we take the standard graph of $y = \sin x$ and drag it backwards along the x -axis a distance of π, we would have the graph of $y = \sin(x + \pi)$ That's because each x value is having π added to it, so to arrive at the x value that produces a particular y -value, we would need to subtract π. Here's an example:

$x+\pi$	$y=\sin(x+\pi)$
0	0
$\pi/2$	1
π	0
$3\pi/2$	-1
2π	0

In the table above we see the standard x and y values for the graph of the sine function. In the table below, we add a column that shows the value that x would need to be for $x+\pi$ to be the standard values:

x	$x+\pi$	$y=\sin(x+\pi)$
$-\pi$	0	0
$-\pi/2$	$\pi/2$	1
0	π	0
$\pi/2$	$3\pi/2$	-1
π	2π	0

Here's a graph of these values:



This is the same graph of $y=\sin(x+\pi)$ that we saw on the previous page, but anchored to different points on the x -axis. Either graph would be a correct response to a question asking for at least one period of the graph of $y=\sin(x+\pi)$

Let's look at another example:

Example 2

Graph at least one period of the given function: $y=\sin\left(x+\frac{\pi}{3}\right)$ Be sure to indicate important points along the x and y axes.

In this simplified example, we really have only one transformation to worry about - the phase shift. Notice that the amplitude, period and vertical shift have all been left out. When considering a sine or cosine graph that has a phase shift, a good way to start the graph of the function is to determine the new starting point of the graph. In the previous example, we saw how the function $y=\sin(x+\pi)$

shifted the graph a distance of π to the left and made the new starting point of the sine curve $-\pi$

In graphing the standard sine curve we're generally interested in the quadrantal angles that produce the maximum, minimum and zero points of the graph. In graphing the function $y=\sin\left(x+\frac{\pi}{3}\right)$, we want to know which values of x will produce the quadrantal angles when we add $\frac{\pi}{3}$ to them.

So, to determine the new starting point we want to know the solution to the equation $x+\frac{\pi}{3}=0$

$$\begin{aligned} x+\frac{\pi}{3}&=0 \\ -\frac{\pi}{3}&-\frac{\pi}{3} \\ x&=-\frac{\pi}{3} \end{aligned} \tag{9.4.2}$$

This is the new starting point for the graph $y = \sin\left(x + \frac{\pi}{3}\right)$. Because this graph has a standard period, the "jump" between each of the quadrantal angles will be
$\frac{\pi}{2}$. To graph one period of a typical trigonometric function we'll need at least five quadrantal angle values. So, if our new starting point is $-\frac{\pi}{3}$, then the next critical value along the x -axis will be:

$$-\frac{\pi}{3} + \frac{\pi}{2} = -\frac{2\pi}{6} + \frac{3\pi}{6} = \frac{\pi}{6} \tag{9.4.3}$$

Then the subsequent critical values would be:

$$\begin{gathered} \frac{\pi}{6} + \frac{\pi}{2} = \frac{\pi}{6} + \frac{3\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3} \\ \frac{4\pi}{6} + \frac{3\pi}{6} = \frac{7\pi}{6} \\ \frac{7\pi}{6} + \frac{3\pi}{6} = \frac{10\pi}{6} = \frac{5\pi}{3} \end{gathered} \tag{9.4.4}$$

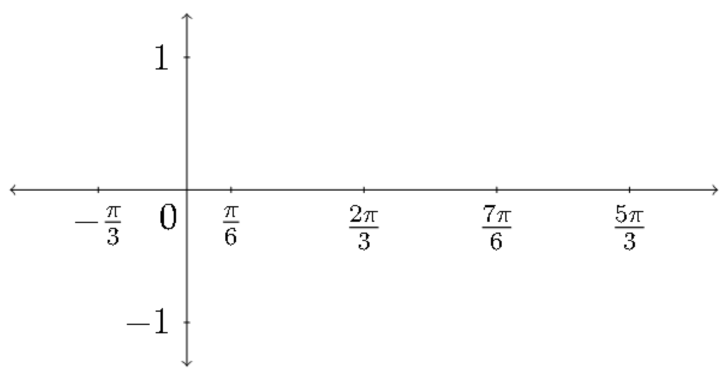
So the five critical values along the x -axis are:

$$-\frac{2\pi}{6}, \frac{\pi}{6}, \frac{4\pi}{6}, \frac{7\pi}{6} \text{ and } \frac{10\pi}{6} \tag{9.4.5}$$

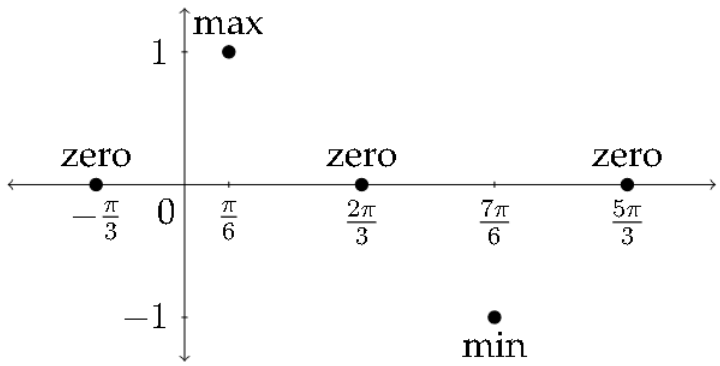
or, in reduced form:

$$-\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6} \text{ and } \frac{5\pi}{3} \tag{9.4.6}$$

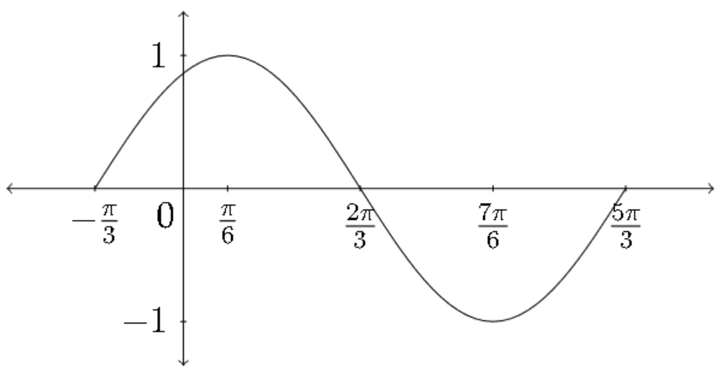
In order to graph the function, we would put these values along the x -axis and plot the standard quadrantal y -values to match up with them:



The y -values for the sine function start at zero, go up to the maximum, back down through zero to the minimum and then back to zero:



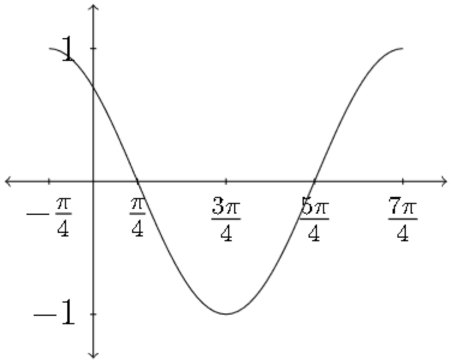
Connecting these points to make a sine curve produces the following graph:



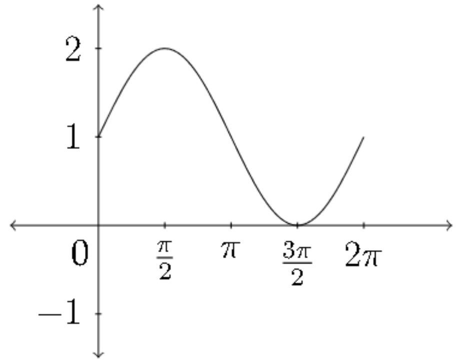
Exercises 2.4

Match each function with the appropriate graph.

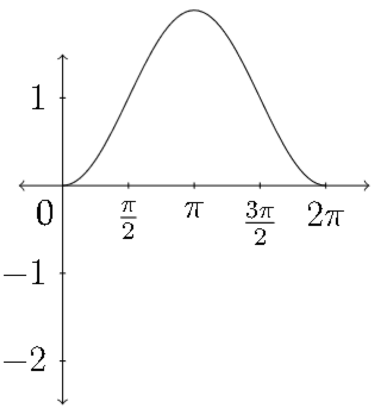
1. $y = \cos\left(x - \frac{\pi}{4}\right)$
2. $y = \sin\left(x + \frac{\pi}{4}\right)$
3. $y = \cos x - 1$
4. $y = \sin x + 1$
5. $y = \sin\left(x - \frac{\pi}{4}\right)$
6. $y = 1 - \cos x$
7. $y = \sin x - 1$
8. $y = \cos\left(x + \frac{\pi}{4}\right)$


1
−1
−π/4
π/4
3π/4
5π/4
7π/4

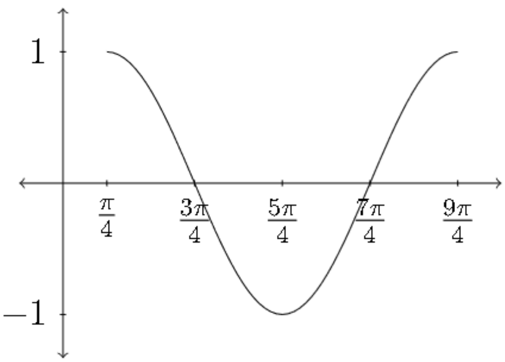
A.


2
1
0
−1
π/2
π
3π/2
2π

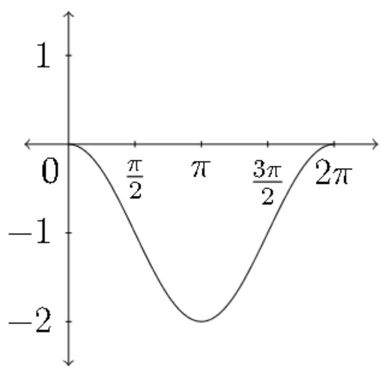
B.


1
0
−1
−2
π/2
π
3π/2
2π

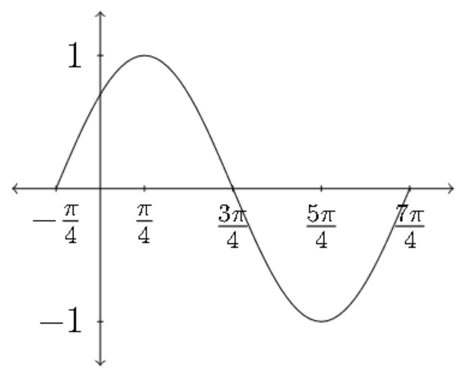
C.


1
−1
π/4
3π/4
5π/4
7π/4
9π/4

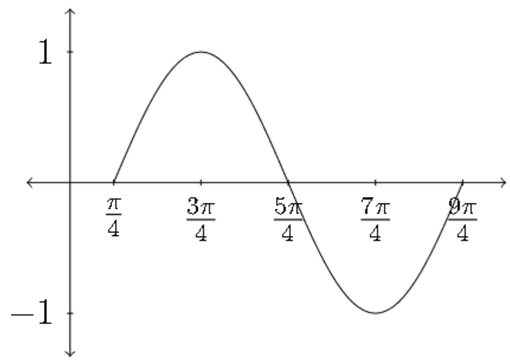
D.


1
0
−1
−2
π/2
π
3π/2
2π

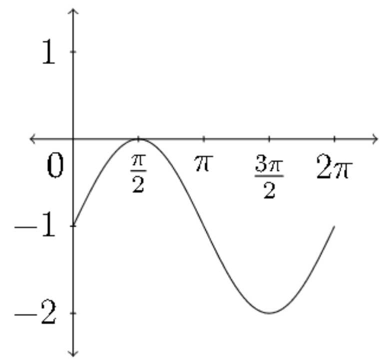
E.


1
−1
−π/4
π/4
3π/4
5π/4
7π/4

F.



G.



H.

Sketch at least one period for each function. Be sure to include the important values along the x and y axes.

9. $y=\sin\left(x+\frac{\pi}{6}\right)$
10. $y=\cos\left(x-\frac{\pi}{6}\right)$
11. $y=\cos\left(x-\frac{\pi}{3}\right)$
12. $y=\sin\left(x+\frac{\pi}{3}\right)$
13. $y=\sin\left(x-\frac{3\pi}{4}\right)$
14. $y=\cos\left(x+\frac{3\pi}{4}\right)$
15. $y=\cos\left(x+\frac{2\pi}{3}\right)$
16. $y=\sin\left(x-\frac{2\pi}{3}\right)$

9.5: Combining the Transformations

In the previous sections, we have seen how the various transformations act on the trigonometric functions and we have worked with the first three (amplitude, period and vertical shift) in combination with each other. Combining the phase shift with the other transformations is tricky because of the way that the period and the phase shift interact with each other.
Now we have two standard equations for the sinusoid:

$$y = A\sin(Bx + C) + D \tag{9.5.1}$$

and

$$y = A\cos(Bx + C) + D \tag{9.5.2}$$

A and D, the amplitude and the vertical shift affect the y -axis, while B and C affect the x -axis.

y-axis	x-axis
Amplitude$= \lvert A\rvert$	Period$=\frac{2\pi}{B}$
Vertical Shift$= D$	Phase Shift$=-\frac{C}{B}$

Let's look at an example in which we need to combine a change in the period of the graph with a phase shift.

Example 1
Graph at least one period of the given function. Indicate the important values along the x and y axes.
$y = \cos(4x + \pi)$
The transformations in this example only affect the x -axis. The period of the function is $\frac{2\pi}{B} = \frac{2\pi}{4} = \frac{\pi}{2}$. So, the function will complete one full cycle over a distance of $\frac{\pi}{2}$ along the x -axis.

However, because of the phase shift, this graph will not start at 0 and end at $\frac{\pi}{2}$. We need to find the new starting point that is caused by the phase shift. So, we take what is called the "argument," or what it is we're finding the cosine of: $(4x + \pi)$ and set that equal to zero.

$$\begin{aligned} 4x + \pi &= 0 \\ 4x &= -\pi \\ x &= -\tfrac{\pi}{4} \end{aligned} \tag{9.5.3}$$

This is our new starting point. To identify the critical values along the x -axis, we'll need to determine how far each "jump$''$ would be given a period of $\frac{\pi}{2}$

$$\frac{\pi}{2} * \frac{1}{4} = \frac{\pi}{8} \tag{9.5.4}$$

So, each subsequent critical value along the x -axis will be a distance of $\frac{\pi}{8}$ from the previous one. If we start at our new starting point for this function $-\frac{\pi}{4}$, then if we add $\frac{\pi}{8}$ a total of 4 times, we will arrive at each of the five critical values for this function.

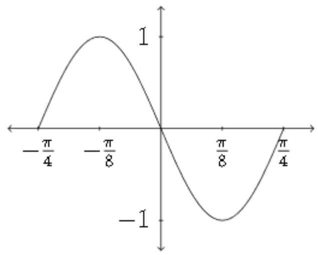
So, each subsequent critical value along the x -axis will be a distance of $\frac{\pi}{8}$ from the previous one. If we start at our new starting point for this function $-\frac{\pi}{4}$, then if we add $\frac{\pi}{8}$ a total of 4 times, we will arrive at each of the five critical values for this function

$$\begin{gathered} -\tfrac{\pi}{4} + \tfrac{\pi}{8} = -\tfrac{2\pi}{8} + \tfrac{\pi}{8} = -\tfrac{\pi}{8} \\ -\tfrac{\pi}{8} + \tfrac{\pi}{8} = 0 \\ 0 + \tfrac{\pi}{8} = \tfrac{\pi}{8} \\ \tfrac{\pi}{8} + \tfrac{\pi}{8} = \tfrac{2\pi}{8} = \tfrac{\pi}{4} \end{gathered} \tag{9.5.5}$$

So the critical values along the x -axis would be:

$$-\frac{\pi}{4}, -\frac{\pi}{8}, 0, \frac{\pi}{8}, \text{ and } \frac{\pi}{4} \tag{9.5.6}$$

Notice that the distance between the starting point $-\frac{\pi}{4}$ and the ending point $\frac{\pi}{4}$ is equal to the period we found at the beginning of the problem, which was $\frac{\pi}{2}$. Now let's graph the function:



since there were no changes to the y -axis, the amplitude for the function is 1 and the vertical shift is 0. Along the x -axis, we see a positive sine function that starts at $\left(-\frac{\pi}{4}, 0\right)$ then goes up to $\left(-\frac{\pi}{8}, 1\right)$, back down through (0,0) to $\left(\frac{\pi}{8}, -1\right)$ and back up to $\left(\frac{\pi}{4}, 0\right)$ to complete one full cycle of the graph.

Let's look at an example in which there are some changes to the y -axis as well as the x -axis.

Example 2

Graph at least one period of the given function. Be sure to identify critical values along the x and y axes.

$$y=-\frac{5}{2}+\cos(3x-\pi) \tag{9.5.7}$$

Remember which coefficients affect which axis in graphing:

y-axis	x-axis
Amplitude$= \lvert A \rvert$	Period$=\frac{2\pi}{B}$
Vertical Shift$= D$	Phase Shift$=-\frac{C}{B}$

In this example, the amplitude is 1, since there is no coefficient in front of the cosine function. The vertical shift is $-\frac{5}{2}$, which will shift the function down a distance of 2.5 on the y -axis. So, the mid-line or "zero" points of the graph will be at -2.5, the maximum y value will be -1.5 and the minimum y value will be -3.5

Along the x -axis, the period for the graph will be $\frac{2\pi}{B}=\frac{2\pi}{3}$, since the coefficient B in this problem is 3. To find the new starting point, we'll take the argument of the cosine function and set it equal to zero.

$$\begin{gathered} 3x-\pi=0 \\ 3x=\pi \\ x=\pi * \frac{1}{3}=\frac{\pi}{3} \end{gathered} \tag{9.5.8}$$

So, our new starting point will be at $\frac{\pi}{3}$. To determine the other critical values along the x -axis, we can find out how far each "jump" between the critical values would be. To do this, we take the period $\left(\frac{2\pi}{3}\right)$ and divide it by 4 (or multiply by $\left.\frac{1}{4}\right)$

$$\frac{2\pi}{3} * \frac{1}{4}=\frac{2\pi}{12}=\frac{\pi}{6} \tag{9.5.9}$$

Now we can add this value to our new starting point four times to determine the other critical values along the x -axis.

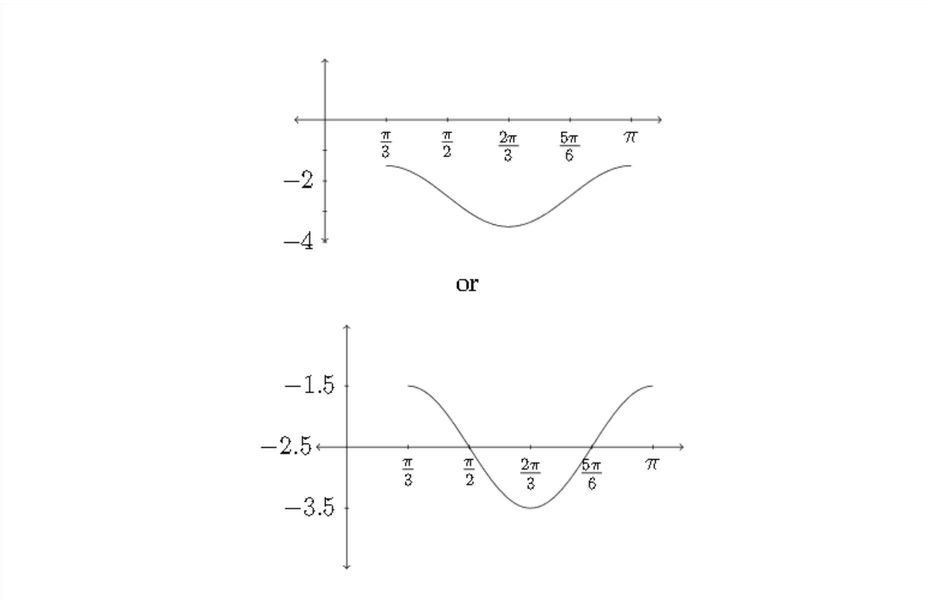
$$\begin{gathered} \frac{\pi}{3}+\frac{\pi}{6}=\frac{2\pi}{6}+\frac{\pi}{6}=\frac{3\pi}{6}=\frac{\pi}{2} \\ \frac{3\pi}{6}+\frac{\pi}{6}=\frac{4\pi}{6}=\frac{2\pi}{3} \\ \frac{4\pi}{6}+\frac{\pi}{6}=\frac{5\pi}{6} \\ \frac{5\pi}{6}+\frac{\pi}{6}=\pi \end{gathered} \tag{9.5.10}$$

So the critical values along the x -axis would be:

$$\frac{2\pi}{6}, \frac{3\pi}{6}, \frac{4\pi}{6}, \frac{5\pi}{6}, \text{ and } \frac{6\pi}{6} \tag{9.5.11}$$

$$\text{or}$$
$$\tfrac{\pi}{3}, \tfrac{\pi}{2}, \tfrac{2\pi}{3}, \tfrac{5\pi}{6}, \text{ and } \pi \tag{9.5.12}$$

Again, notice that the distance along the x -axis from the starting point to the ending point is the period: $\frac{2\pi}{3}$. Now let's graph the function:



Let's look at one more example.
Example 3
Sometimes the coefficient B appears factored out of the argument as it does in the problem below.

Graph at least one period of the given function. Be sure to include the critical values along the x and y axes.
$y = 4\sin 2\left(x + \frac{\pi}{3}\right) - 1$
First let's see how the x and y axes are affected by the transformations in this problem.

y-axis	x-axis
Amplitude$= \lvert A \rvert$	Period$=\frac{2\pi}{B}$
Vertical Shift$= D$	Phase Shift$=-\frac{C}{B}$

$$y = 4\sin 2\left(x + \frac{\pi}{3}\right) - 1 \tag{9.5.13}$$

The amplitude in this problem is 4 and the vertical shift is -1
The period for this graph is $\frac{2\pi}{B} = \frac{2\pi}{2} = \pi$. Notice that the value of B is 2 in this example, even though it's been factored out from the rest of the argument.

The new starting point for the graph is actually easier to find in problems of this type. If we take the argument as it is and set it equal to zero:

$$2\left(x + \frac{\pi}{3}\right) = 0 \tag{9.5.14}$$

we can divide through on both sides by 2 to cancel out the factor of B :

$$\begin{gathered}\frac{2\left(x+\frac{\pi}{3}\right)}{2}=\frac{0}{2} \\ x+\frac{\pi}{3}=0 \\ x=-\frac{\pi}{3}\end{gathered} \tag{9.5.15}$$

So, the new starting point for the function is $-\frac{\pi}{3}$

Now let's find the rest of the critical values along the x -axis. The period for this graph is π, so the "jump " between the critical values along the x -axis will be:

$$\pi * \frac{1}{4}=\frac{\pi}{4} \tag{9.5.16}$$

To find the rest of the critical values we'll need to add $\frac{\pi}{4}$ to the starting point of the graph $\left(-\frac{\pi}{3}\right)$ four times:

$$\begin{gathered}-\frac{\pi}{3}+\frac{\pi}{4}=-\frac{4\pi}{12}+\frac{3\pi}{12}=-\frac{\pi}{12} \\ -\frac{\pi}{12}+\frac{3\pi}{12}=\frac{2\pi}{12}=\frac{\pi}{6} \\ \frac{2\pi}{12}+\frac{3\pi}{12}=\frac{5\pi}{12} \\ \frac{5\pi}{12}+\frac{3\pi}{12}=\frac{8\pi}{12}=\frac{2\pi}{3}\end{gathered} \tag{9.5.17}$$

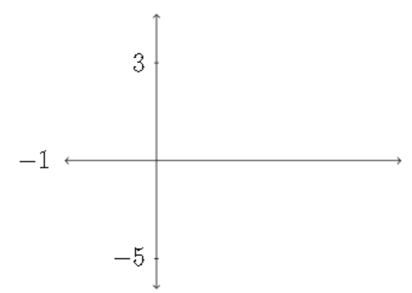
So the critical values along the x -axis would be:

$$-\frac{4\pi}{12},-\frac{1\pi}{12},\frac{2\pi}{12},\frac{5\pi}{12}, \text{ and } \frac{8\pi}{12} \tag{9.5.18}$$

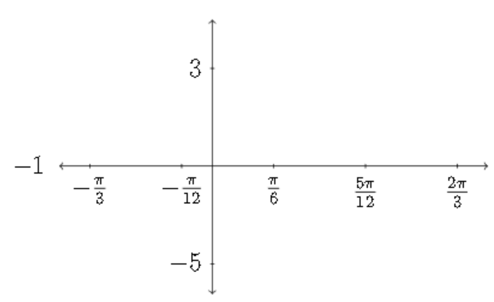
Or

$$-\frac{\pi}{3},-\frac{\pi}{12},\frac{\pi}{6},\frac{5\pi}{12}, \text{ and } \frac{2\pi}{3} \tag{9.5.19}$$

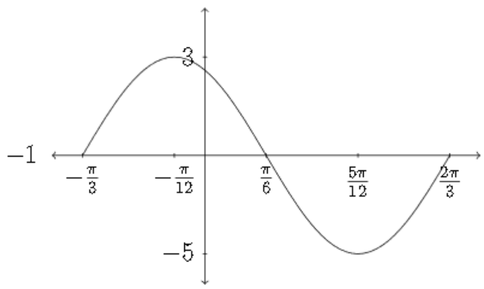
Now that we've addressed each of the four transformations let's use this information to draw the graph. First the y -axis - the amplitude is 4 and the vertical shift is -1:



Now, let's fill in the information for the x -axis. The critical values along the x -axis are $-\frac{\pi}{3},-\frac{\pi}{12},\frac{\pi}{6},\frac{5\pi}{12}$, and $\frac{2\pi}{3}$



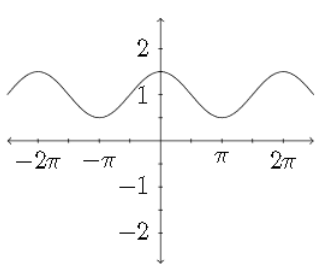
The function we're graphing is a positive sine function, so it will start at the "midline" or zero value (which in this case is -1), go up to the maximum, back through the mid-line to the minimum and back to the mid-line:



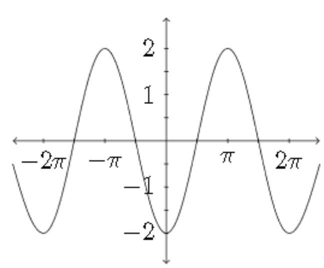
Determine the Amplitude, Period, Vertical Shift and Phase Shift for each function and graph at least one complete period. Be sure to identify the critical values along the x and y axes.

1. $y=\sin\left(x+\frac{\pi}{2}\right)$
2. $y=\sin(x-\pi)$
3. $y=3\cos\left(x-\frac{\pi}{2}\right)$
4. $y=\frac{1}{2}\cos(x+\pi)$
5. $y=3+\cos\left(x-\frac{\pi}{4}\right)$
6. $y=-2+\sin\left(x+\frac{\pi}{6}\right)$
7. $y=\sin(2x-\pi)$
8. $y=\sin\left(4x+\frac{\pi}{4}\right)$
9. $y=2\cos\left(\frac{x}{2}+\pi\right)$
10. $y=-3\sin(6x-\pi)$
11. $y=-\frac{1}{3}\sin\left(2x+\frac{\pi}{4}\right)$
12. $y=\frac{1}{2}\cos\left(\frac{x}{2}-\pi\right)$
13. $y=2\sin\left(2x-\frac{\pi}{3}\right)-1$
14. $y=1+2\cos\left(3x+\frac{\pi}{2}\right)$
15. $y=3\cos 2\left(x+\frac{\pi}{6}\right)$
16. $y=-4\sin 2\left(x+\frac{\pi}{2}\right)$
17. $y=\sin\frac{1}{2}\left(x+\frac{\pi}{4}\right)$
18. $y=3+2\sin 3\left(x+\frac{\pi}{2}\right)$

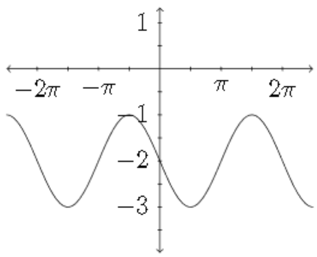
In problems $19-22$, determine an equation for the function that is shown.



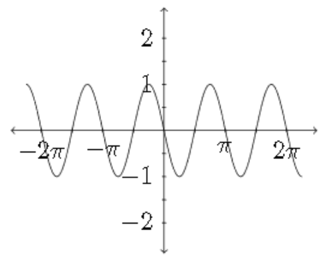
19.



20.



21.



22.

Match the function to the appropriate graph

23. $y = -\cos 2x$

24. $y = \frac{1}{2}\sin x - 2$

25. $y = 2\cos\left(x + \frac{\pi}{2}\right)$

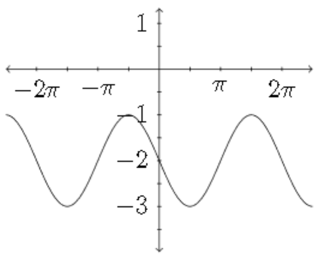
26. $y = -3\sin\frac{1}{2}x - 1$

27. $y = \sin(x - \pi) - 2$

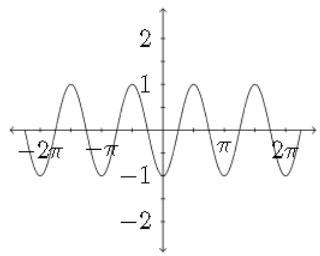
28. $y = -\frac{1}{2}\cos\left(x - \frac{\pi}{4}\right)$

29. $y = \frac{1}{3}\sin 3x$

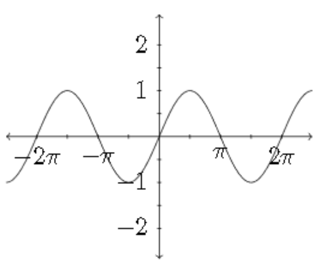
30. $y = \cos\left(x - \frac{\pi}{2}\right)$



A.



B.



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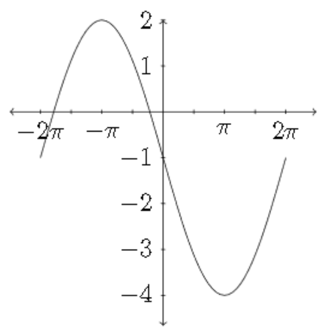

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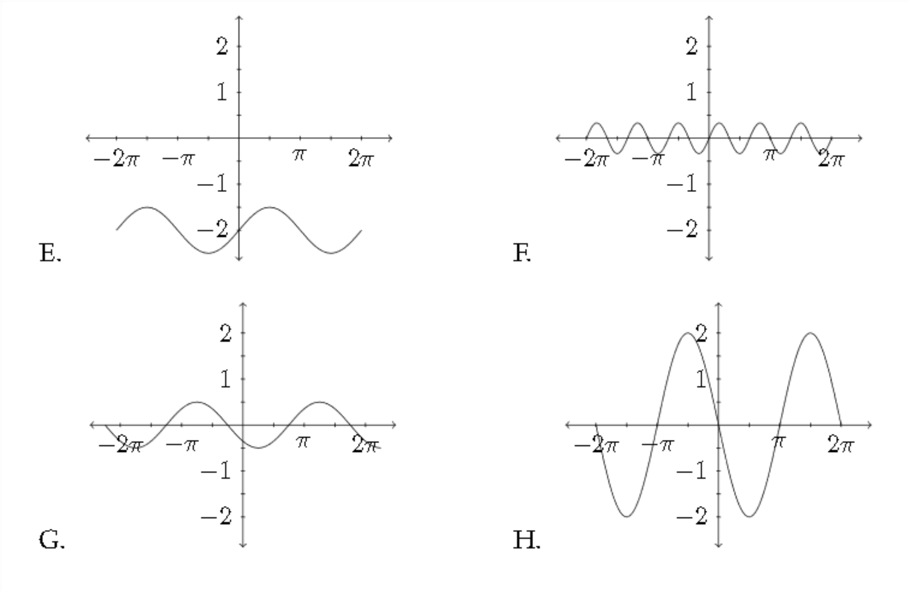


Determine the Amplitude, Period, Vertical Shift and Phase Shift for each function and graph at least one complete period. Be sure to identify the critical values along the x and y axes.

31. $y=2\cos\left(2x+\frac{\pi}{2}\right)-1$
32. $y=-4\cos(3x-2\pi)$
33. $y=\sin\left(2x-\frac{\pi}{4}\right)$
34. $y=-\sin(3x+\pi)$
35. $y=3\cos\left(x+\frac{\pi}{3}\right)+1$
36. $y=-2\sin\left(3x-\frac{\pi}{2}\right)+4$
37. $y=-\frac{1}{2}\sin\left(x-\frac{\pi}{2}\right)-2$
38. $y=2-\cos\left(2x-\frac{\pi}{3}\right)$

CHAPTER OVERVIEW

10: Trigonometric Identities and Equations

Due to the nature of the trigonometric ratios, they have some interesting properties that make them useful in a number of mathematical problem-solving situations. One of the hallmarks of mathematical problem-solving is to change the appearance of the problem without changing its value. Trigonometric identities can be very helpful in changing the appearance of a problem.

The process of demonstrating the validity of a trigonometric identity involves changing one trigonometric expression into another, using a series of clearly defined steps. We'll look at a few examples briefly, but first, let's examine some of the fundamental trigonometric identities.

10.1: Reciprocal and Pythagorean Identities

The two most basic types of trigonometric identities are the reciprocal identities and the Pythagorean identities. The reciprocal identities are simply definitions of the reciprocals of the three standard trigonometric ratios:

$$\sec\theta = \frac{1}{\cos\theta} \quad \csc\theta = \frac{1}{\sin\theta} \quad \cot\theta = \frac{1}{\tan\theta} \tag{10.1.1}$$

Also, recall the definitions of the three standard trigonometric ratios (sine, cosine and tangent):

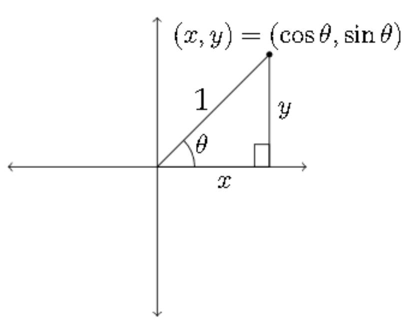
$$\begin{aligned} \sin\theta &= \tfrac{opp}{hyp} \\ \cos\theta &= \tfrac{adj}{hyp} \\ \tan\theta &= \tfrac{opp}{ady} \end{aligned} \tag{10.1.2}$$

If we look more closely at the relationships between the sine, cosine and tangent, we'll notice that $\frac{\sin\theta}{\cos\theta} = \tan\theta$

$$\frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{opp}{hyp}\right)}{\left(\frac{adj}{hyp}\right)} = \frac{opp}{hyp} * \frac{hyp}{adj} = \frac{opp}{adj} = \tan\theta \tag{10.1.3}$$

Pythagorean Identities

The Pythagorean Identities are, of course, based on the Pythagorean Theorem. If we recall a diagram that was introduced in Chapter 2, we can build these identities from the relationships in the diagram:



Using the Pythagorean Theorem in this diagram, we see that $x^2 + y^2 = 1^2$, so $x^2 + y^2 = 1$. But, also remember that, in the unit circle, $x = \cos\theta$ and $y = \sin\theta$

Substituting this equality gives us the first Pythagorean Identity:

$$x^2 + y^2 = 1 \tag{10.1.4}$$

or

$$\cos^2\theta + \sin^2\theta = 1 \tag{10.1.5}$$

This identity is usually stated in the form:

$$\sin^2\theta + \cos^2\theta = 1 \tag{10.1.6}$$

If we take this identity and divide through on both sides by $\cos^2\theta$, this will result in the first of two additional Pythagorean Identities:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \tag{10.1.7}$$

or

$$\tan^2\theta + 1 = \sec^2\theta \tag{10.1.8}$$

Dividing through by $\sin^2\theta$ gives us the second:

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \tag{10.1.9}$$

or

$$1 + \cot^2\theta = \csc^2\theta \tag{10.1.10}$$

So, the three Pythagorean Identities we will be using are:

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \tan^2\theta + 1 &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned} \tag{10.1.11}$$

These Pythagorean Identities are often stated in other terms, such as:

$$\begin{aligned} \sin^2\theta &= 1 - \cos^2\theta \\ \cos^2\theta &= 1 - \sin^2\theta \\ \tan^2\theta &= \sec^2\theta - 1 \\ \cot^2\theta &= \csc^2\theta - 1 \end{aligned} \tag{10.1.12}$$

At the beginning of this chapter, we discussed verifying trigonometric identities. Now that we have some basic identities to work with, let's use them to verify the equality of some more complicated statements. The process of verifying trigonometric identities involves changing one side of the given expression into the other side. since these are not really equations, we will not treat them the way we treat equations. That is to say, we won't add or subtract anything to both sides of the statement (or multiply or divide by anything on both sides either).

Another reason for not treating a trigonometric identity as an equation is that, in practice, this process typically involves just one side of the statement. In problem solving, mathematicians typically use trigonometric identities to change the appearance of a problem without changing its value. In this process, a trigonometric expression is changed into another trigonometric expression rather than showing that two trigonometric expressions are the same, which is what we will be doing.

Example 1

Verify the identity $(\sin\theta)(\cot\theta) = \cos\theta$

This is a very straightforward identity and it can solved by using one of the fundamental approaches to working with trigonometric identities. This is the approach of writing everything in terms of sines and cosines.

Beginning with the original statement:

$$(\sin\theta)(\cot\theta) = \cos\theta \tag{10.1.13}$$

Replace $\cot\theta$ with $\frac{\cos\theta}{\sin\theta}$

$$(\sin\theta)\frac{\cos\theta}{\sin\theta} = \cos\theta \tag{10.1.14}$$

Then canceling out the $\sin\theta$:

$$\cos\theta = \cos\theta \tag{10.1.15}$$

There are four fundamental approaches to verifying trigonometric identities:

1. write everything in terms of sines and cosines
2. make a common denominator and add fractions
3. split a fraction
4. factor and cancel

Not all of these can be used in every problem and some problems will use combinations of these strategies. Here is another example.

Example 2

Verify the identity $\tan\theta+\cot\theta=\sec\theta\csc\theta$

First we'll write everything in terms of sines and cosines:

$$\begin{gathered}\tan\theta+\cot\theta=\sec\theta\csc\theta\\ \frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}=\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}\end{gathered}\tag{10.1.16}$$

Next, on the left hand side, we can add the two fractions together by making a common denominator of $\cos\theta\sin\theta$

$$\begin{aligned}\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}&=\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}\\ \frac{\sin\theta}{\sin\theta}\cdot\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\cdot\frac{\cos\theta}{\cos\theta}&=\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}\\ \frac{\sin^2\theta}{\sin\theta\cos\theta}+\frac{\cos^2\theta}{\sin\theta\cos\theta}&=\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}\\ \frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}&=\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}\\ \frac{1}{\sin\theta\cos\theta}&=\frac{1}{\sin\theta\cos\theta}\end{aligned}$$

In this example, you can see that we have first written everything in terms of sines and cosines, then created common denominators and added the fractions on the left hand side together. After this is done, we can replace the expression $\sin^2\theta+\cos^2\theta$ with 1, since this is the fundamental Pythagorean Identity.

Example 3

Verify the identity $\frac{\tan\theta-\cot\theta}{\sin\theta\cos\theta}=\sec^2\theta-\csc^2\theta$

We'll begin this problem by splitting the fraction over the denominator. This can be helpful in problems in which there is no addition or subtraction in the denominator. The idea here is that since $\frac{a}{x}+\frac{b}{x}=\frac{a+b}{x}$, then we can reverse this process and say that $\frac{a+b}{x}=\frac{a}{x}+\frac{b}{x}$

In the problem above we'll say that:

$$\frac{\tan\theta-\cot\theta}{\sin\theta\cos\theta}=\sec^2\theta-\csc^2\theta\tag{10.1.17}$$

$$\frac{\tan\theta}{\sin\theta\cos\theta}-\frac{\cot\theta}{\sin\theta\cos\theta}=\sec^2\theta-\csc^2\theta\tag{10.1.18}$$

$$\frac{\frac{\sin\theta}{\cos\theta}}{\sin\theta\cos\theta}-\frac{\frac{\cos\theta}{\sin\theta}}{\sin\theta\cos\theta}=\sec^2\theta-\csc^2\theta\tag{10.1.19}$$

$$\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta\cos\theta}-\frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta\cos\theta}=\sec^2\theta-\csc^2\theta\tag{10.1.20}$$

$$\frac{\sin\cancel{\theta}}{\cos\theta}\cdot\frac{1}{\sin\cancel{\theta}\cos\theta}-\frac{\cos\cancel{\theta}}{\sin\theta}\cdot\frac{1}{\sin\theta\cos\cancel{\theta}}=\sec^2\theta-\csc^2\theta\tag{10.1.21}$$

$$\frac{1}{\cos^2\theta}-\frac{1}{\sin^2\theta}=\sec^2\theta-\csc^2\theta\tag{10.1.22}$$

$$\sec^2\theta-\csc^2\theta=\sec^2\theta-\csc^2\theta\tag{10.1.23}$$

Example 4

Verify the identity $\frac{\tan^2\theta-\cos^2\theta}{1-\cos^2\theta}=\sec^2\theta-\cot^2\theta$

On the left-hand side, notice the expression $1-\cos^2\theta$ in the denominator. We can replace this with $\sin^2\theta$, which is a simpler expression. It is often helpful to have a simpler expression in the denominator rather than a more complicated expression.

$$\begin{gathered}\frac{\tan^2\theta-\cos^2\theta}{1-\cos^2\theta}=\sec^2\theta-\cot^2\theta\\ \frac{\tan^2\theta-\cos^2\theta}{\sin^2\theta}=\sec^2\theta-\cot^2\theta\end{gathered}\tag{10.1.24}$$

Next, we can split the fraction over the denominator of $\sin^2\theta$

$$\frac{\tan^2\theta-\cos^2\theta}{\sin^2\theta}=\sec^2\theta-\cot^2\theta \tag{10.1.25}$$

$$\frac{\tan^2\theta}{\sin^2\theta}-\frac{\cos^2\theta}{\sin^2\theta}=\sec^2\theta-\cot^2\theta \tag{10.1.26}$$

We can see on the left-hand side that the expression $\frac{\cos^2\theta}{\sin^2\theta}$ is equivalent to $\cot^2\theta$ but the first piece on the left-hand side needs to be simplified a little more. We'll rewrite $\tan^2\theta$ as $\frac{\sin^2\theta}{\cos^2\theta}$ and then simplify the complex fraction.

$$\begin{gathered}\frac{\tan^2\theta}{\sin^2\theta}-\frac{\cos^2\theta}{\sin^2\theta}=\sec^2\theta-\cot^2\theta\\ \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\sin^2\theta}-\cot^2\theta=\sec^2\theta-\cot^2\theta\\ \frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{1}{\sin^2\theta}-\cot^2\theta=\sec^2\theta-\cot^2\theta\\ \frac{\cancel{\sin^2\theta}}{\cos^2\theta}\cdot\frac{1}{\cancel{\sin^2\theta}}-\cot^2\theta=\sec^2\theta-\cot^2\theta\end{gathered} \tag{10.1.27}$$

After we cancel out the $\sin^2\theta$, we're almost done:

$$\begin{aligned}\frac{\cancel{\sin^2\theta}}{\cos^2\theta}\cdot\frac{1}{\cancel{\sin^2\theta}}-\cot^2\theta&=\sec^2\theta-\cot^2\theta\\ \frac{1}{\cos^2\theta}-\cot^2\theta&=\sec^2\theta-\cot^2\theta\\ \sec^2\theta-\cot^2\theta&=\sec^2\theta-\cot^2\theta\end{aligned}$$

The trigonometric identities we have discussed in this section are summarized below:

Pythagorean Identities	Reciprocal Identities
$\sin^2\theta+\cos^2\theta=1$	$\tan\theta=\frac{\sin\theta}{\cos\theta}$
$\tan^2\theta+1=\sec^2\theta$	$\cot\theta=\frac{\cos\theta}{\sin\theta}$
$1+\cot^2\theta=\csc^2\theta$	$\sec\theta=\frac{1}{\cos\theta}$
	$\csc\theta=\frac{1}{\sin\theta}$

In the examples above and in the exercises, the form $\sin\theta$ or $\cos\theta$ is typically used, however any letter may be used to represent the angle in question so long as it is the SAME letter in all expressions. For example, we can say that:

$$\sin^2\theta+\cos^2\theta=1 \tag{10.1.28}$$

or we can say that

$$\sin^2 x+\cos^2 x=1 \tag{10.1.29}$$

however:

$$\sin^2\theta + \cos^2 x \neq 1 \tag{10.1.30}$$

because θ and x could be different angles!

Exercises 3.1

In each problem verify the given trigonometric identify.

1. $\cos\theta(\sec\theta - \cos\theta) = \sin^2\theta$

2. $\tan\theta(\cot\theta + \tan\theta) = \sec^2\theta$

3. $\tan\theta(\csc\theta + \cot\theta) - \sec\theta = 1$

4. $\cot\theta(\sec\theta + \tan\theta) - \csc\theta = 1$

5. $\tan^2\theta\csc^2\theta - \tan^2\theta = 1$

6. $\sin^2\theta\cot^2\theta + \sin^2\theta = 1$

7. $\dfrac{\sin\theta\tan\theta + \sin\theta}{\tan\theta + \tan^2\theta} = \cos\theta$

8. $\dfrac{\cos\theta\cot\theta + \cos\theta}{\cot\theta + \cot^2\theta} = \sin\theta$

9. $\dfrac{(\sin\theta + \cos\theta)^2}{\cos\theta} - \sec\theta = 2\sin\theta$

10. $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$

11. $\cos\theta(\tan\theta + \cot\theta) = \csc\theta$

12. $\sin\theta(\cot\theta + \tan\theta) = \sec\theta$

13. $\dfrac{\cos\theta}{\tan\theta} - \csc\theta = -\sin\theta$

14. $\dfrac{\sin\theta}{\cot\theta} - \sec\theta = -\cos\theta$

15. $\dfrac{\csc\theta}{\cos\theta} - \dfrac{\cos\theta}{\csc\theta} = \dfrac{\cot^2\theta + \sin^2\theta}{\cot\theta}$

16. $\dfrac{\sec\theta + \csc\theta}{\tan\theta + \cot\theta} = \sin\theta + \cos\theta$

17. $\dfrac{\sin\theta}{1 + \sin\theta} - \dfrac{\sin\theta}{1 - \sin\theta} = -2\tan^2\theta$

18. $\dfrac{\cos\theta}{1 + \cos\theta} - \dfrac{\cos\theta}{1 - \cos\theta} = -2\cot^2\theta$

19. $\dfrac{\cot\theta}{1 + \csc\theta} - \dfrac{\cot\theta}{1 - \csc\theta} = 2\sec\theta$

20. $\dfrac{\tan\theta}{1 + \sec\theta} - \dfrac{\tan\theta}{1 - \sec\theta} = 2\csc\theta$

21. $\dfrac{\sec^2\theta}{1+\cot^2\theta} = \tan^2\theta$

22. $\dfrac{\csc^2\theta}{1+\tan^2\theta} = \cot^2\theta$

23. $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$

24. $\csc^4\theta - \csc^2\theta = \cot^4\theta + \cot^2\theta$

25. $1 - \dfrac{\cos^2\theta}{1+\sin\theta} = \sin\theta$

26. $1 - \dfrac{\sin^2\theta}{1+\cos\theta} = \cos\theta$

27. $\dfrac{\sec\theta}{\csc\theta} + \dfrac{\sin\theta}{\cos\theta} = 2\tan\theta$

28. $\dfrac{1-\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1-\sin\theta} = 2\sec\theta$

29. $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{1+\sin\theta}{\cos\theta} = 2\sec\theta$

30. $\dfrac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta} = \sin^2\theta - \cos^2\theta$

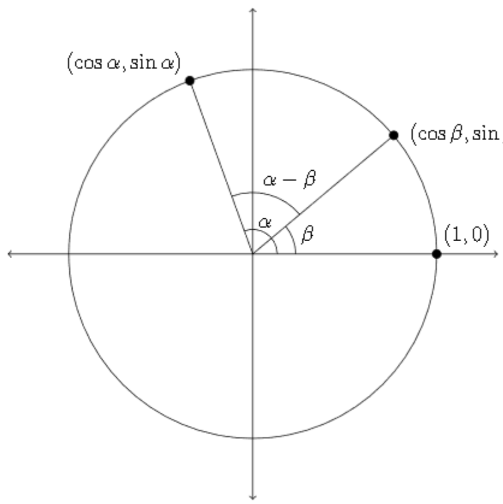
31. $\dfrac{\sec\theta-\cos\theta}{\sec\theta+\cos\theta} = \dfrac{\sin^2\theta}{1+\cos^2\theta}$

32. $\dfrac{\sec\theta+\tan\theta}{\cot\theta+\cos\theta} = \tan\theta\sec\theta$

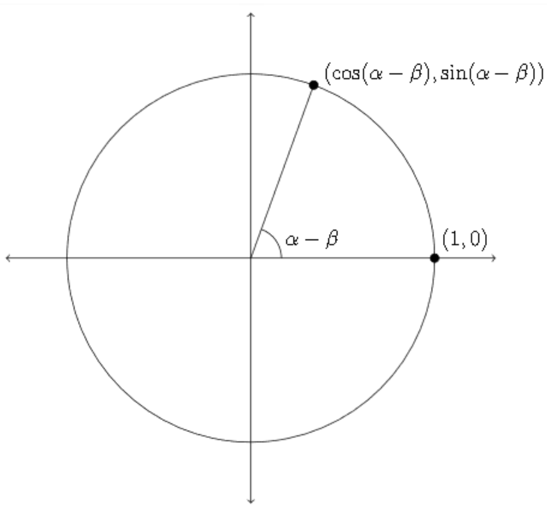
10.2: Double-Angle Identities

In this section we will include several new identities to the collection we established in the previous section. These new identities are called "Double-Angle Identities$''$ because they typically deal with relationships between trigonometric functions of a particular angle and functions of "two times" or double the original angle.

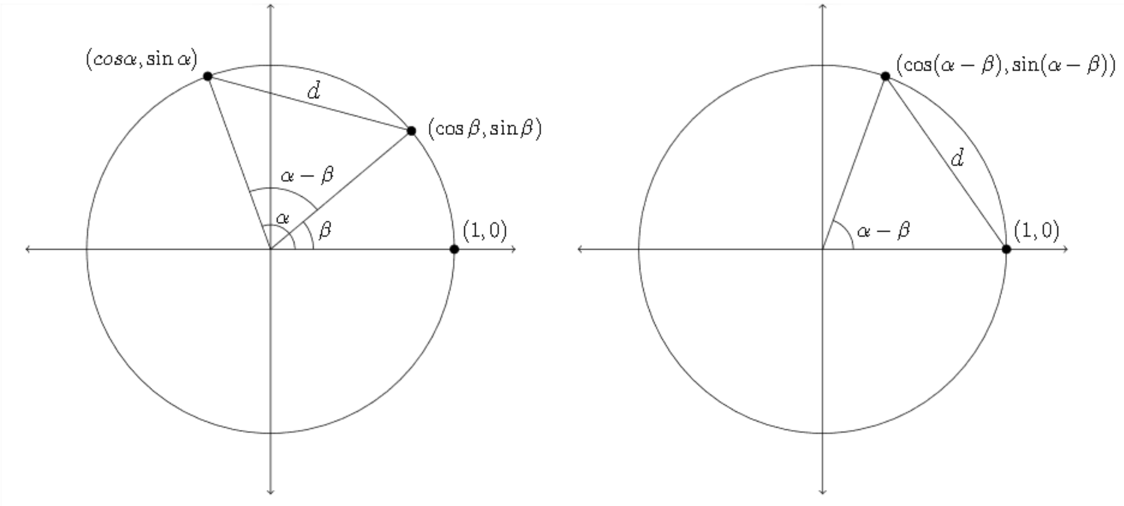
To establish the validity of these identities we need to use what are known as the Sum and Difference Identities. These are identities that deal with expressions such as $\sin(\alpha+\beta)$. First we will establish an expression that is equivalent to $\cos(\alpha-\beta)$
Let's start with the unit circle:



If we rotate everything in this picture clockwise so that the point labeled $(\cos\beta, \sin\beta)$ slides down to the point labeled $(1,0)$, then the angle of rotation in the diagram will be $\alpha-\beta$ and the corresponding point on the edge of the circle will be:
$(\cos(\alpha-\beta), \sin(\alpha-\beta))$
The diagram that represents this rotation is on the next page.



since the the second diagram is created by rotating the lines and points from the first diagram, the distance between the points $(\cos\alpha, \sin\alpha)$ and $(\cos\beta, \sin\beta)$ in the first diagram is the same as the distance between $(\cos(\alpha-\beta), \sin(\alpha-\beta))$ and the point (1,0) in the second diagram.



In the diagram above the length of d in each picture is the same.

We can represent this distance d with the distance formula used to calculate the distance between two points in the coordinate plane:
The distance between the points (x_1, y_1) and (x_2, y_2) is

$$d=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2} \tag{10.2.1}$$

So, in the first diagram the distance d will be:

$$d=\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2} \tag{10.2.2}$$

In the second diagram the distance d will be:

$$d=\sqrt{(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2} \tag{10.2.3}$$

since these distances are the same, we can set them equal to each other:

$$\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2}=\sqrt{(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2} \tag{10.2.4}$$

We'll square both sides to clear the radicals:

$$(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2=(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2 \tag{10.2.5}$$

Next, we'll rewrite $(\sin(\alpha-\beta)-0)^2$ as $\sin^2(\alpha-\beta)$

$$(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2=(\cos(\alpha-\beta)-1)^2+\sin^2(\alpha-\beta) \tag{10.2.6}$$

Now we'll work to simplify the expressions on the left-hand side of this equation.

$$(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2=(\cos(\alpha-\beta)-1)^2+\sin^2(\alpha-\beta) \tag{10.2.7}$$

First, each one needs to be squared:

$$\begin{aligned}(\cos\alpha-\cos\beta)^2&=\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta\\(\sin\alpha-\sin\beta)^2&=\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta\end{aligned} \tag{10.2.8}$$

So, the left-hand side will now be:

$$\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta \tag{10.2.9}$$

If we rearrange this a little, it will simplify nicely:

$$\begin{gathered}\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta\\ \sin^2\alpha+\cos^2\alpha+\sin^2\beta+\cos^2\beta-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta\end{gathered}\tag{10.2.10}$$

Notice the Pythagorean Identities at the front of this expression - these are each equal to 1:

$$\begin{gathered}\sin^2\alpha+\cos^2\alpha+\sin^2\beta+\cos^2\beta-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta\\ 1+1-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta\\ 2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta\\ 2(1-\cos\alpha\cos\beta-\sin\alpha\sin\beta)\end{gathered}\tag{10.2.11}$$

Now that we've simplified the left-hand side, we'll simplify the right-hand side First we'll square the expression $(\cos(\alpha-\beta)-1)^2$

$$(\cos(\alpha-\beta)-1)^2=\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1\tag{10.2.12}$$

So, the right-hand side is now:

$$\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta)\tag{10.2.13}$$

If we rearrange this expression, we'll again have a nice Pythagorean Identity:

$$\begin{gathered}\sin^2(\alpha-\beta)+\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1\\ 1-2\cos(\alpha-\beta)+1\\ 2-2\cos(\alpha-\beta)\\ 2(1-\cos(\alpha-\beta))\end{gathered}\tag{10.2.14}$$

So the left-hand side was equal to:

$$2(1-\cos\alpha\cos\beta-\sin\alpha\sin\beta)\tag{10.2.15}$$

And the right-hand side was equal to:

$$2(1-\cos(\alpha-\beta))\tag{10.2.16}$$

So, our original statement in simplified form is:

$$2(1-\cos\alpha\cos\beta-\sin\alpha\sin\beta)=2(1-\cos(\alpha-\beta))\tag{10.2.17}$$

If we divide by 2 on both sides, we'll have:

$$1-\cos\alpha\cos\beta-\sin\alpha\sin\beta=1-\cos(\alpha-\beta)\tag{10.2.18}$$

then subtract 1

$$-\cos\alpha\cos\beta-\sin\alpha\sin\beta=-\cos(\alpha-\beta)\tag{10.2.19}$$

and multiply through by -1

$$\begin{gathered}\cos\alpha\cos\beta+\sin\alpha\sin\beta=\cos(\alpha-\beta)\\ \text{So}, \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta\end{gathered}\tag{10.2.20}$$

This will help us to generate the double-angle formulas, but to do this, we don't want $\cos(\alpha-\beta)$, we want $\cos(\alpha+\beta)$ (you'll see why in a minute).

So, to change this around, we'll use identities for negative angles. Recall that in the fourth quadrant the sine function is negative and the cosine function is positive. For this reason, $\sin(-\theta)=-\sin(\theta)$ and $\cos(-\theta)=\cos(\theta)$
Now we can say that $\cos(2\theta)=\cos(\theta+\theta)=\cos(\theta-(-\theta))$. Going back to our identity for $\cos(\alpha-\beta)$, we can say that:

$$\begin{aligned}\cos(\theta-(-\theta)) &= \cos\theta\cos(-\theta)+\sin\theta\sin(-\theta)\\ \cos(\theta-(-\theta)) &= \cos\theta\cos\theta+\sin\theta(-\sin\theta)\\ \cos(\theta-(-\theta)) &= \cos\theta\cos\theta-\sin\theta\sin\theta\\ \cos(\theta-(-\theta)) &= \cos^2\theta-\sin^2\theta\\ \cos(\theta+\theta) &= \cos^2\theta-\sin^2\theta\\ \cos(2\theta) &= \cos^2\theta-\sin^2\theta\end{aligned}$$

This is the double-angle identity for the cosine: $\cos(2\theta)=\cos^2\theta-\sin^2\theta$

This identity actually appears in any one of three forms because the Pythagorean Identities can be applied to this to change its appearance:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta-\sin^2\theta\\ \cos(2\theta) &= 1-\sin^2\theta-\sin^2\theta\\ \cos(2\theta) &= 1-2\sin^2\theta\end{aligned} \tag{10.2.21}$$

If we substitute for the $\sin^2\theta$ term:

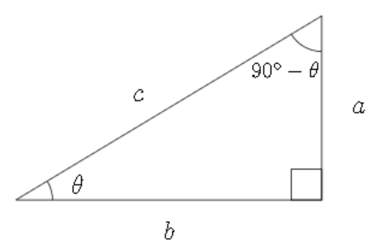
$$\begin{aligned}\cos(2\theta) &= \cos^2\theta-\sin^2\theta\\ \cos(2\theta) &= \cos^2\theta-\left(1-\cos^2\theta\right)\\ \cos(2\theta) &= \cos^2\theta-1+\cos^2\theta\\ \cos(2\theta) &= 2\cos^2\theta-1\end{aligned} \tag{10.2.22}$$

So, the three forms of the cosine double angle identity are:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta-\sin^2\theta\\ &= 2\cos^2\theta-1\\ &= 1-2\sin^2\theta\end{aligned}$$

The double-angle identity for the sine function uses what is known as the cofunction identity. Remember that, in a right triangle, the sine of one angle is the same as the cosine of its complement (which is the other acute angle). This is because the adjacent side for one angle is the opposite side for the other angle. The denominator in both cases is the hypotenuse, so the cofunctions of complementary angles are equal.

In the diagram below, we can see this more clearly:



In the diagram above note that:

$$\sin\theta=\frac{a}{c}=\cos(90^\circ-\theta) \tag{10.2.23}$$

So, if we want an identity for $\sin(\theta+\theta)$, we'll start with $\sin(\alpha+\beta)$ which is equivalent to $\cos(90^\circ-(\alpha+\beta))$. We'll use a trick here and restate this as:

$$\begin{aligned}\sin(\alpha+\beta) &= \cos(90^\circ-(\alpha+\beta))\\ &= \cos(90^\circ-\alpha-\beta)\\ &= \cos((90^\circ-\alpha)-\beta)\\ &= \cos(90^\circ-\alpha)\cos\beta+\sin(90^\circ-\alpha)\sin\beta\\ &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\end{aligned}$$

Now, we can use this to find an expression for $\sin 2\theta = \sin(\theta+\theta)$

$$\begin{aligned} \sin 2\theta &= \sin(\theta+\theta) \\ &= \sin\theta\cos\theta + \cos\theta\sin\theta \\ &= 2\sin\theta\cos\theta \end{aligned}$$

Exercises 3.2

Here is a summary of all the identities we've worked with in this chapter:

Pythagorean Identities
$\sin^2\theta + \cos^2\theta = 1$
$\tan^2\theta + 1 = \sec^2\theta$
$1 + \cot^2\theta = \csc^2\theta$

Reciprocal Identities
$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$\cot\theta = \frac{\cos\theta}{\sin\theta}$
$\sec\theta = \frac{1}{\cos\theta}$
$\csc\theta = \frac{1}{\sin\theta}$

Double-Angle Identities
$\cos(2\theta) = \cos^2\theta - \sin^2\theta$
$\cos(2\theta) = 2\cos^2\theta - 1$
$\cos(2\theta) = 1 - 2\sin^2\theta$
$\sin(2\theta) = 2\sin\theta\cos\theta$

Working problems involving double-angle identities is very similar to the other identities we've worked with previously- you just have more identities to choose from!

Example
Verify the given identity: $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$
We have three possible identities to choose from for the left-hand side, so we'll wait on that for a moment while we simplify the right-hand side.

$$\begin{aligned} \frac{1-\tan^2 x}{1+\tan^2 x} &= \frac{1-\tan^2 x}{\sec^2 x} \\ &= \frac{1}{\sec^2 x} - \frac{\tan^2 x}{\sec^2 x} \\ &= \cos^2 x - \frac{\left(\frac{\sin^2 x}{\cos^2 x}\right)}{\left(\frac{1}{\cos^2 x}\right)} \\ &= \cos^2 x - \frac{\sin^2 x}{\cos^2 x}\cdot\frac{\cos^2 x}{1} \\ &= \cos^2 x - \frac{\sin^2 x}{\cancel{\cos^2 x}}\cdot\frac{\cancel{\cos^2 x}}{1} \\ &= \cos^2 x - \sin^2 x \end{aligned}$$

This is one of the identities for $\cos(2\theta)$ so we can stop and simply state $\cos(2x) = \cos^2 x - \sin^2 x$

In each problem verify the given trigonometric identity.
1. $\frac{2\sin x\cos x}{\cos^2 x - \sin^2 x} = \tan(2x)$
2. $\sec(2x) = \frac{\sec^2 x}{2-\sec^2 x}$
3. $\sin(2x)\csc x = 2\cos x$
4. $\frac{2\cos x}{\sin(2x)} = \csc x$
5. $\frac{\cos(2x)}{\sin x} + \sin x = \frac{\cot x}{\sec x}$

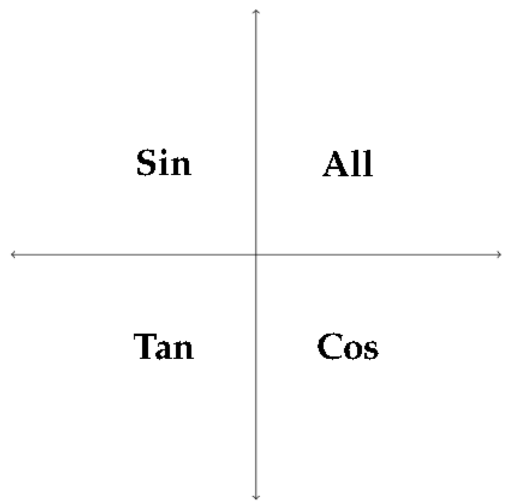
6. $\frac{\sin x+\sin(2x)}{\sec x+2}=\sin x\cos x$
7. $(\sin x+\cos x)^2=1+\sin(2x)$
8. $\left(\sin^2 x-1\right)^2=\sin^4 x+\cos(2x)$
9. $2\cos x-\frac{\cos(2x)}{\cos x}=\sec x$
10. $\frac{1+\cos(2x)}{1-\cos(2x)}=\cot^2 x$
11. $\frac{\cos(2x)}{\sin^2 x}=\cot^2 x-1$
12. $\frac{\cos(2x)}{\sin^2 x}=\csc^2 x-2$
13. $\frac{\cot x-\tan x}{\cot x+\tan x}=\cos 2x$
14. $\sin 2x=\frac{2(\tan x-\cot x)}{\tan^2 x-\cot^2 x}$
15. $\frac{2\cos 2x}{\sin 2x}=\cot x-\tan x$
16. $\tan 2x=\frac{2}{\cot x-\tan x}$
17. $\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x}=2\csc x$
18. $\tan x+\cot x=2\csc(2x)$
19. $\cos(2x)=\frac{\cot^2 x-1}{\cot^2 x+1}$
20. $\sin(2x)=\frac{2\tan x}{1+\tan^2 x}$
21. $\frac{2\sin^2 x}{\sin(2x)}+\cot x=\sec x\csc x$
22. $\sec^2 x\cos(2x)=\sec^2 x-2\tan^2 x$
23. $\frac{\cos(2x)}{\sin x}+\sin x=\csc x-\sin x$
24. $\frac{2\tan x-\sin(2x)}{2\sin^2 x}=\tan x$

10.3: Trigonometric Equations

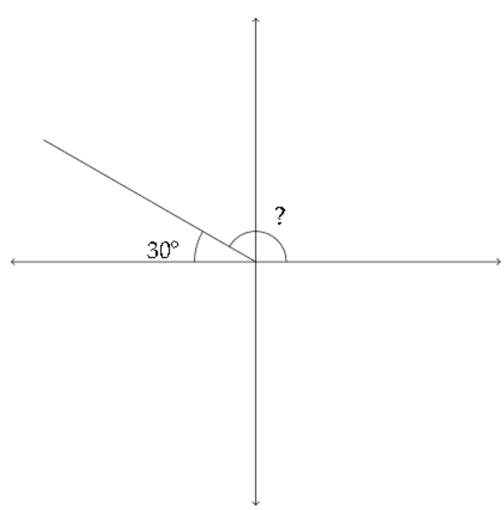
In the previous section on trigonometric identities we worked with equations that would be true for all values of a particular angle θ. These are sort of like the algebraic equations whose solution set is "all real numbers," like $2x+10=2(x+1)+8$. In this section, we will solve trigonometric equations whose solution set involves only certain values for the angle in question. Because of the cyclical nature of the angles we're working with, there will often be an infinite number of solutions although not "all real numbers."

Example 1

Here's an example. Suppose that we consider the equation $\sin x=0.5$. Whether we use technology, a table or reasoning to solve this equation, it's clear that one solution is 30°. However, remember from the beginning of Chapter 2 that the sine function is positive in Quadrant Π. That means that a second quadrant angle with a reference angle of 30° also has a sine equal to 0.5. Recall the ASTC diagram from Chapter 2 :



So, the sine function is positive in Quadrants I and Π. This means that in addition to a solution of 30°, there is another solution in Quadrant Π. As mentioned above, this second quadrant solution has a reference angle of 30°



To find this angle, we simply subtract $180^{\circ}-30^{\circ}=150^{\circ}$
In Quadrant Π, we subtract the reference angle from 180°

In Quadrant Π, we add the reference angle to 180°

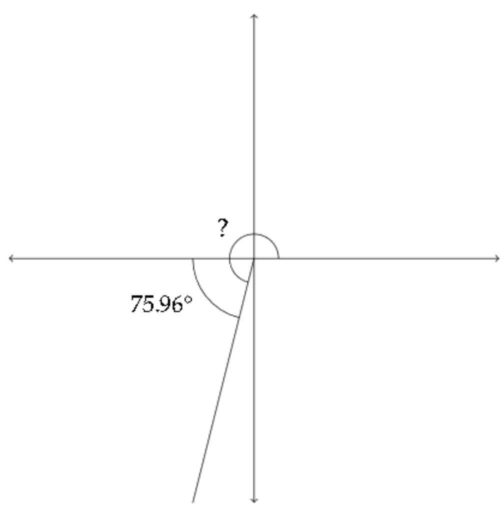
In Quadrant ΠV, we subtract the reference angle from 360°
So, the solutions to the equation $\sin x = 0.5$ between 0° and 360° are $x = 30^{\circ}, 150^{\circ}$ In this chapter we will consider mainly solutions with this restriction:

$$0^{\circ} \leq x < 360^{\circ} \tag{10.3.1}$$

The infinite solutions to this equation can be expressed as:

$$30^{\circ} + n \cdot 360^{\circ} \text{ and } 150^{\circ} + n \cdot 360^{\circ} \tag{10.3.2}$$

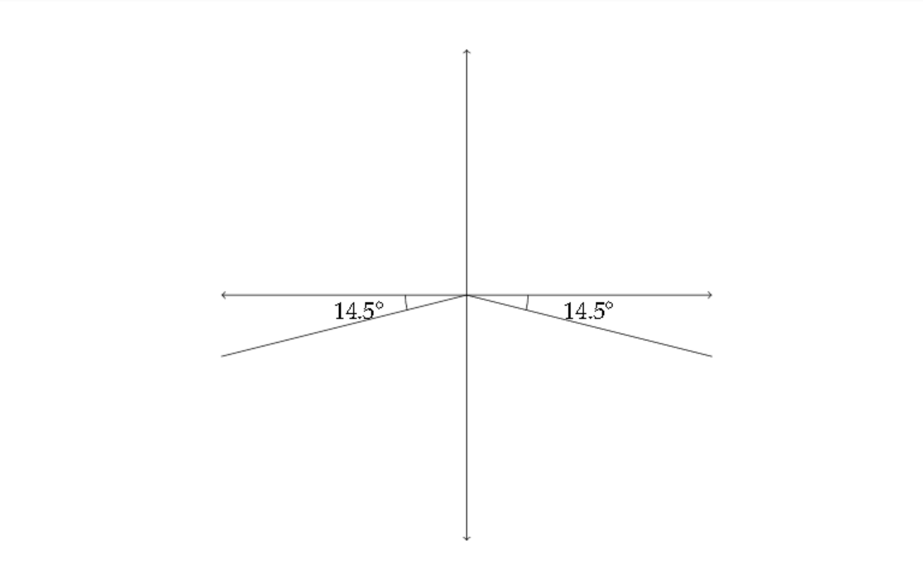
Let's look at another example:
Example 2
Find all solutions of the given equation for $0^{\circ} \leq x < 360^{\circ}$
$\tan x = 4$
Using a calculator to find $\tan^{-1}(4)$, we find that it returns an answer of $x \approx 75.96^{\circ}$ So this is the solution to the equation that lies in QuadrantI. The tangent function is also positive in Quadrant Π, so we should also consider the third quadrant angle with a reference angle of 75.96°



In Quadrant Π, we add the reference angle to 180°
$180^{\circ} + 75.96^{\circ} = 255.96^{\circ}$, so our solutions for this equation are $x \approx 75.96^{\circ}, 255.96^{\circ}$
Often, calculators are programmed to return an angle value that is not between $0^{\circ} \leq x < 360^{\circ}$

Example 3
Find all solutions of the given equation for $0^{\circ} \leq x < 360^{\circ}$
$\sin x = -0.25$
Solving this on a TI calculator would generally return a value of -14.5°. However, -14.5° is clearly not between 0° and 360°, so we need to use this information to find the solutions that are between 0° and 360°

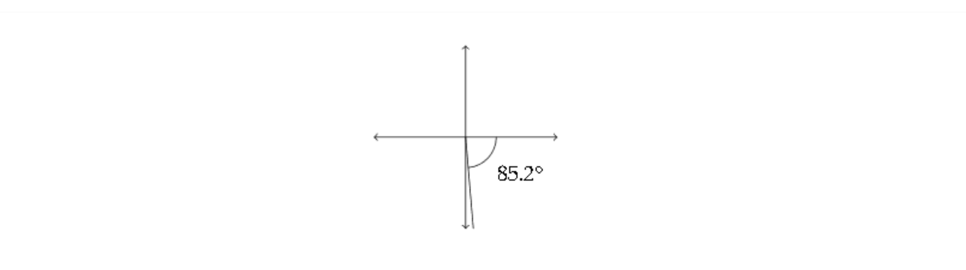
With the calculator returning a vlue of -14.5°, we know that the reference angle for all answers will be 14.5°. Knowing this, we can say that the sine is negative in Quadrants III and IV, so we'll need angles in those quadrants with reference angles of 14.5°



In Quadrant Π we'll add 180° to the reference angle: $180^{\circ}+14.5^{\circ}=194.5^{\circ}$
In Quadrant Π we'll subtract the reference angle from 360° : $360^{\circ}-14.5^{\circ}=345.5^{\circ}$
So, $x \approx 194.5^{\circ}, 345.5^{\circ}$

Some trigonometric equations have no real number solutions. The equation $\sin x=2$ has no real number solutions. Recall that the sine ratio was originally defined as the ratio of the side opposite an angle to the hypotenuse. The hypotenuse is always the longest side in a right triangle so there is no way the sine function could be greater than 1 if we're working with real-valued angles. However, in the same way that complex numbers are used to solve equations like $x^{2}=-7$, complex-valued angles can be used to solve equations such as $\sin x=2$ We won't go into this here, however, there is a relatively straightforward way to solve these equations.
If you encounter an equation like $\cos x=3$ and are solving for values of x $0^{\circ} \leq x<360^{\circ}$, then the proper response is "no solution" or "no real solution." However, remember that the tangent function can take any value between $-\infty$ and ∞

Example 4
Solving an equation that includes a reciprocal trigonometric function simply involves the extra step of finding the reciprocal:
Find all solutions of the given equation for $0^{\circ} \leq x<360^{\circ}$
$\sec x=12$
The trick here is to restate the equation so that we can use the preprogrammed values from a calculator to find the solution.
If $\sec x=12$ then $\cos x=\frac{1}{12}$. Finding $\cos ^{-1}\left(\frac{1}{12}\right)$ gives a solution of $x \approx 85.2^{\circ}$
The cosine and the secant are both positive in Quadrant ΠV, so we'll also want a fourth quadrant angle whose reference angle is 85.2°



In Quadrant Π, we'll subtract the reference angle from 360°

$$\begin{gathered}360^{\circ}-85.2^{\circ} \approx 274.8^{\circ} \\ x \approx 85.2^{\circ}, 274.8^{\circ}\end{gathered} \tag{10.3.3}$$

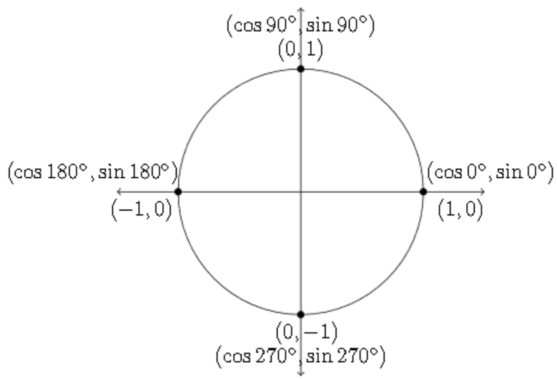
Example 5
Solving a quadratic trigonometric equation often involves the use of the quadratic formula:

Find all solutions of the given equation for $0^{\circ} \leq x < 360^{\circ}$
$2 \sin^2 x - \sin x - 2 = 0$
Using the quadratic formula we arrive at approximate values for $\sin x$ of $\sin x \approx -0.7808, 1.2808$
The solution $\sin x \approx 1.2808$ yields no real solutions, so we will focus on solving $\sin x \approx -0.7808$
Finding $\sin^{-1}(-0.7808)$ gives us an answer of $\approx -51.3^{\circ}$. This means our answers will lie in Quadrants III and IV with reference angles of 51.3°. In Quadrant III we'll say $180^{\circ} + 51.3^{\circ} \approx 231.3^{\circ}$. In Quadrant IV, we'll subtract the reference angle from $360^{\circ} : 360^{\circ} - 51.3^{\circ} \approx 308.7^{\circ}$
So, $x \approx 231.3^{\circ}, 308.7^{\circ}$

Exercises 3.3
Find all solutions for $0^{\circ} \leq x < 360^{\circ}$ Round all angle measures to the nearest 10^{th} of a degree.
1. $\cos x - 0.75 = 0$
2. $\sin x + 0.432 = 0$
3. $3 \sin x - 5 = 0$
4. $\sin x - 4 = 0$
5. $3 \sec x + 8 = 0$
6. $4 \csc x + 9 = 0$
7. $3 - 5 \sin x = 4 \sin x + 1$
8. $4 \cos x - 5 = \cos x - 3$
9. $3 \tan^2 x + 2 \tan x = 0$
10. $4 \cos^2 x - \cos x = 0$
11. $3 \cos^2 x + 5 \cos x - 2 = 0$
12. $2 \cot^2 x - 7 \cot x + 3 = 0$
13. $2 \tan^2 x - \tan x - 10 = 0$
14. $2 \sin^2 x + 5 \sin x + 3 = 0$
15. $2 \cos^2 x - 5 \cos x - 5 = 0$
16. $3 \sin^2 x - \sin x - 1 = 0$

10.4: More Trigonometric Equations

When the solution to a trigonometric equation is one of the quadrantal angles $(0^{\circ}, 90^{\circ}, 180^{\circ}, 270^{\circ} \text{ and so on })$, then determining all the solutions between 0° and 360° can work a little differently. The calculator will return some of these values, but in some cases it may not. If we go back to the unit circle, we can see this more clearly:



In the diagram above we can see the sine and cosine for $0^{\circ}, 90^{\circ}, 180^{\circ}$, and 270° since $\tan \theta = \frac{\sin \theta}{\cos \theta}$, then we can see that $\tan 0^{\circ} = 0, \tan 90^{\circ}$ is undefined, $\tan 180^{\circ} =$
0 and $\tan 270^{\circ}$ is also undefined.

The real issue with the quadrantal angles is finding $\sin^{-1}(0), \cos^{-1}(0)$ or $\tan^{-1}(0)$ The calculator returns values of:

$$\begin{aligned} \sin^{-1}(0) &= 0^{\circ} \\ \cos^{-1}(0) &= 90^{\circ} \\ \tan^{-1}(0) &= 0^{\circ} \end{aligned} \tag{10.4.1}$$

In each case, there is another possibility than differs from the given angle by 180° so:

$$\begin{aligned} \sin^{-1}(0) &= 0^{\circ}, 180^{\circ} \\ \cos^{-1}(0) &= 90^{\circ}, 270^{\circ} \\ \tan^{-1}(0) &= 0^{\circ}, 180^{\circ} \end{aligned} \tag{10.4.2}$$

Let's look at how this is used in solving an equation:

Example 1
Solve the given equation for $0^{\circ} \leq x < 360^{\circ}$

$$\tan^{2} x - \tan x = 0 \tag{10.4.3}$$

We could use the quadratic formula to solve this, but we can also solve by factoring:

$$\begin{aligned} \tan^{2} x - \tan x &= 0 \\ \tan x(\tan x - 1) &= 0 \\ \tan x = 0 \text{ or } \tan x &= 1 \end{aligned} \tag{10.4.4}$$

Using a calculator to find $\tan^{-1}(0)$ and $\tan^{-1}(1)$ returns values of $\tan^{-1}(0) = 0^{\circ}$ and $\tan^{-1}(1) = 45^{\circ}$. Once we know the reference angle for $\tan^{-1}(1)$, then we know that since the tangent is also positive in Quadrant ΠI, the solutions here are 45° and 225°. The calculator returns an answer of 0° for $\tan^{-1}(0)$, but we just saw that $\tan 180^{\circ} = 0$ as well.
The answers for this equation are $x = 45^{\circ}, 225^{\circ}, 0^{\circ}, 180^{\circ}$

Another approach to solving trigonometric equations involves using Pythagorean Identities to make a substitution that so that the equation can be simply solved by the quadratic formula. Here's an example:

Example 2
Solve the given equation for $0^{\circ} \leq x < 360^{\circ}$

$$\sin^2\theta - 6\cos\theta = 4 \tag{10.4.5}$$

Notice that, unlike the problems we saw in the previous section, this equation involves both the sine and the cosine. To remedy this, we can replace the $\sin^2\theta$ term with the expression $1-\cos^2\theta$

$$\begin{aligned} \sin^2\theta - 6\cos\theta &= 4 \\ 1-\cos^2\theta - 6\cos\theta &= 4 \\ 0 &= \cos^2\theta + 6\cos\theta + 3 \end{aligned} \tag{10.4.6}$$

using the quadratic formula:

$$\cos\theta \approx -5.449, -0.5505 \tag{10.4.7}$$

since $\cos^{-1}(-5.449)$ is not a real-valued angle, we can focus on the other answer:
$\cos^{-1}(-0.5505) \approx 123.4^{\circ}$. since the cosine function is also negative in the third quadrant, we need to find the reference angle that will help us identify the third quadrant angle that is a solution for this equation:

$$180^{\circ} - 123.4^{\circ} = 56.6^{\circ} \tag{10.4.8}$$

So the reference angle is 56.6°

$$180^{\circ} + 56.6^{\circ} = 236.6^{\circ} \tag{10.4.9}$$

The solutions are $\theta \approx 123.4^{\circ}, 236.6^{\circ}$

Example 3
Solve the given equation for $0^{\circ} \leq x < 360^{\circ}$

$$2\cos^2\theta - \sin\theta = \sin^2\theta + 1 \tag{10.4.10}$$

First, we'll substitute $1-\sin^2\theta$ for the $\cos^2\theta$

$$\begin{aligned} 2\cos^2\theta - \sin\theta &= \sin^2\theta + 1 \\ 2\left(1-\sin^2\theta\right) - \sin\theta &= \sin^2\theta + 1 \\ 2 - 2\sin^2\theta - \sin\theta &= \sin^2\theta + 1 \\ 0 &= 3\sin^2\theta + \sin\theta - 1 \end{aligned} \tag{10.4.11}$$

Solving this with the quadratic formula gives us solutions of $\sin\theta \approx -0.7676, 0.43426$

$$\sin^{-1}(-0.7676) \approx -50.1^{\circ} \tag{10.4.12}$$

$$\sin^{-1}(0.43426) \approx 25.7^{\circ} \tag{10.4.13}$$

We'll work with the positive solution first. since the sine is also positive in Quadrant II, the other angle will be $180^{\circ} - 25.7^{\circ} = 154.3^{\circ}$

For the negative solution, we know that the sine is negative in Quadrants III and IV, so with a reference angle of 50.1°, in the third quadrant $180^{\circ} + 50.1^{\circ} = 230.1^{\circ}$ and in the fourth quadrant $360^{\circ} - 50.1^{\circ} = 309.9^{\circ}$
The solution set is $\theta \approx 25.7^{\circ}, 154.3^{\circ}, 230.1^{\circ}, 309.9^{\circ}$

Exercises 3.4
Solve the given equations for $0^{\circ} \leq x < 360^{\circ}$

1. $9\sin^2\theta - 6\sin\theta = 1$
2. $4\cos^2\theta + 4\cos\theta = 1$
3. $\sec^2\alpha - 2\sec\alpha - 3 = 0$
4. $\csc^2\beta + 4\csc\beta - 10 = 0$
5. $\csc^2 x + 4\csc x - 7 = 0$
6. $3\cot^2 x - 3\cot x - 1 = 0$
7. $2\sin^2 x = 1 - \cos x$

8. $\cos^2\alpha+4=2\sin\alpha-3$
9. $\cos^2\beta-3\sin\beta+2\sin^2\beta=0$
10. $\sin^2\theta=2\cos\theta+3\cos^2\theta$
11. $\sec^2 x=2\tan x+4$
12. $3\tan^2 x=\sec x+2$
13. $\cos\alpha+1=2\cos 2\alpha$
14. $\cos 2x-3\sin x-2=0$
15. $\csc^2\theta=\cot\theta+5$
16. $\csc\theta+5=2\cot^2\theta+2$

CHAPTER OVERVIEW

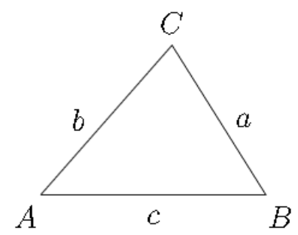
11: The Law of Sines and The Law of Cosines

Previously, we used the fundamental trigonometric relationships in right triangles to find unknown distances and angles. Unfortunately, in many problem solving situations, it is inconvenient to use right triangle relationships. Therefore, from the right triangle relationships, we can derive relationships that can be used in any triangle.

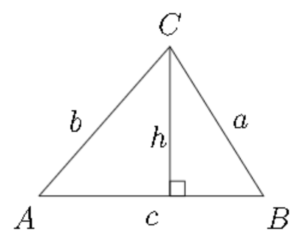
Thumbnail: Law of cosines with acute angles. (CC BY SA 3.0 Unported; Scaler via Wikipedia)

11.1: The Law of Sines

The Law of sines is based on right triangle relationships that can be created with the height of a triangle. Often, in this type of a problem, the angles are labeled with capital letters and their corresponding sides are labeled with lower case letters.



If we drop a perpendicular to the base of the triangle from the vertex point at $\angle C$ this creates two right triangles with which we can make use of the right triangle trigonometry covered in Chapter 1. This perpendicular would be the height of the triangle.



The Law of sines is derived from this configuration and allows us to calculate the value of sides and angles in a triangle without a right angle, based on information about known sides and angles. Given the right triangles in the diagram above, we can see that:

$$\sin B=\frac{h}{a} \tag{11.1.1}$$

and

$$\sin A=\frac{h}{b} \tag{11.1.2}$$

Clearing the denominator in each fraction, we can see that:

$$a \sin B=h \tag{11.1.3}$$

and

$$\begin{gathered} b \sin A=h \\ \text{so} \\ a \sin B=b \sin A \end{gathered} \tag{11.1.4}$$

To put this in the form in which the Law of sines is normally stated, we can divide on both sides of the previous expression by ab :

$$\begin{gathered} a \sin B=b \sin A \\ \frac{a \sin B}{ab}=\frac{b \sin A}{ab} \\ \frac{\sin B}{b}=\frac{\sin A}{a} \end{gathered} \tag{11.1.5}$$

A similar process will show that $\frac{\sin C}{c}$ is equivalent to $\frac{\sin B}{b}$ and $\frac{\sin A}{a}$. The diagram we derived this from used an acute triangle in which all the angles were less than 90°. The process to show that this is true for an obtuse triangle (which has one angle larger than 90°) is relatively simple and is

left to the reader to discover or look up in another resource.
The Law of sines

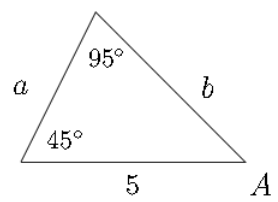
$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c} \tag{11.1.6}$$

Sometimes it is handy to set up a problem with the side lengths in the numerator:
The Law of sines

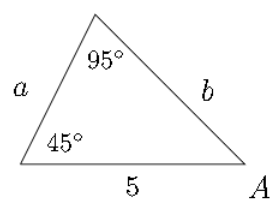
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \tag{11.1.7}$$

\

Solve the triangle. Round side lengths to the nearest 100^{th} .



In this problem we're given two angles and one side. It's important that the side we're given corresponds to one of the known angles, otherwise we wouldn't be able to use the Law of sines.



since we know two of the angles, then the third will just be $180^{\circ}-(45^{\circ}+95^{\circ})=\ 180^{\circ}-140^{\circ}=40^{\circ}=\angle A.$ To find the lengths of the unknown sides, we'll use the Law of sines. We should start by choosing a side-angle pair for which we know both the side and the angle. In this case, we know that $\angle C=95^{\circ}$ and side $c=5$

$$\begin{gathered}\frac{c}{\sin C}=\frac{b}{\sin B}\\ \frac{5}{\sin 95^{\circ}}=\frac{b}{\sin 45^{\circ}}\end{gathered} \tag{11.1.8}$$

If we multiply on both sides by $\sin 45^{\circ}$, then

$$\sin 45^{\circ}*\frac{5}{\sin 95^{\circ}}=b \tag{11.1.9}$$

To arrive at an approximate value for $\sin 45^{\circ}*\frac{5}{\sin 95^{\circ}}$, we can say:

$$\begin{gathered}0.7071*\frac{5}{0.9962}\approx b\\ 3.55\approx b\end{gathered} \tag{11.1.10}$$

To find the length of side a, I would recommend that we use the exact side-angle pair that was given in the problem, rather than using the approximate value of side b that we just solved for.

This will make our value for side a more accurate:

$$\begin{aligned}\frac{c}{\sin C} &= \frac{a}{\sin A}\\ \frac{5}{\sin 95^\circ} &= \frac{a}{\sin 40^\circ}\end{aligned} \tag{11.1.11}$$

Multiplying on both sides by $\sin 40^\circ$, then

$$\sin 40^\circ * \frac{5}{\sin 95^\circ} = a \tag{11.1.12}$$

To arrive at an approximate value for $\sin 40^\circ * \frac{5}{\sin 95^\circ}$, we can say:

$$\begin{aligned}0.6428 * \frac{5}{0.9962} &\approx a\\ 3.23 &\approx a\end{aligned} \tag{11.1.13}$$

$$\begin{array}{ll}\angle A = 40^\circ & a \approx 3.23\\ \angle B = 45^\circ & b \approx 3.55\\ \angle C = 95^\circ & c = 5\end{array} \tag{11.1.14}$$

Example 2
Some problems don't come with diagrams:
Solve the triangle if: $\angle A = 40^\circ, \quad \angle B = 20^\circ, \quad a = 2$
Round side lengths to the nearest 100^{th}.
Just as in the previous example, we can begin by finding the measure of the third angle $\angle C$. This would be $180^\circ - (40^\circ + 20^\circ) = 180^\circ - 60^\circ = 120^\circ = \angle C$
To find the missing sides, we should use the complete side-angle pair that is given in the problem: $\angle A = 40^\circ$ and $a = 2$

We can find side b first or side c first, it doesn't matter which:

$$\begin{aligned}\frac{a}{\sin A} &= \frac{b}{\sin B}\\ \frac{2}{\sin 40^\circ} &= \frac{b}{\sin 20^\circ}\\ \sin 20^\circ * \frac{2}{\sin 40^\circ} &= b\end{aligned} \tag{11.1.15}$$

Then,

$$0.3420 * \frac{2}{0.6428} \approx b \tag{11.1.16}$$

$$1.06 \approx b \tag{11.1.17}$$

For side c:

$$\begin{aligned}\frac{a}{\sin A} &= \frac{c}{\sin C}\\ \frac{2}{\sin 40^\circ} &= \frac{c}{\sin 120^\circ}\\ \sin 120^\circ * \frac{2}{\sin 40^\circ} &= c\end{aligned} \tag{11.1.18}$$

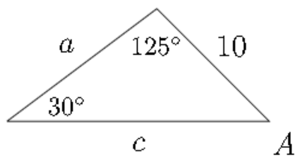
Then,

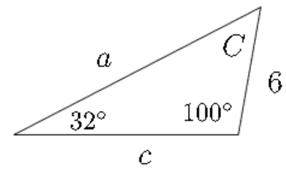
$$0.8660 * \frac{2}{0.6428} \approx c \tag{11.1.19}$$

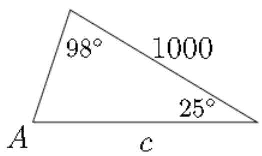
$$2.69 \approx c \tag{11.1.20}$$

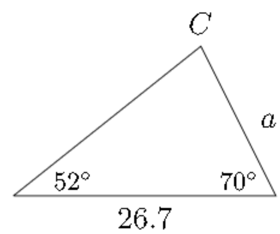
$$\begin{array}{ll}\angle A = 40^\circ & a = 2\\ \angle B = 20^\circ & b \approx 1.06\\ \angle C = 120^\circ & c \approx 2.69\end{array} \tag{11.1.21}$$

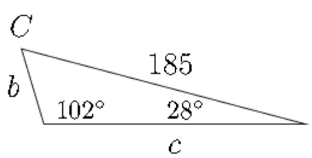
Exercises 4.1
In each problem below, solve the triangle. Round side lengths to the nearest 100^{th}

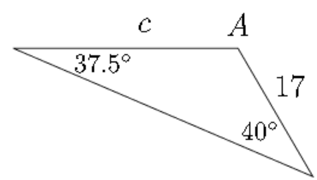
1. 

2. 

3. 

4. 

5. 

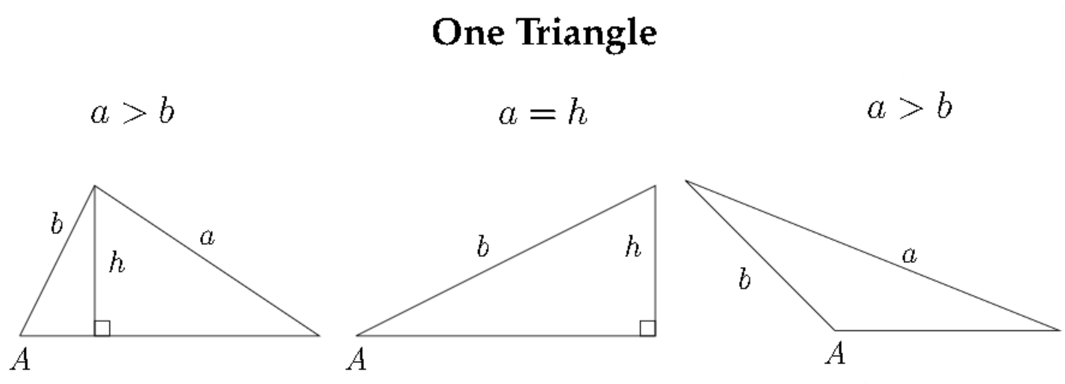
6. 

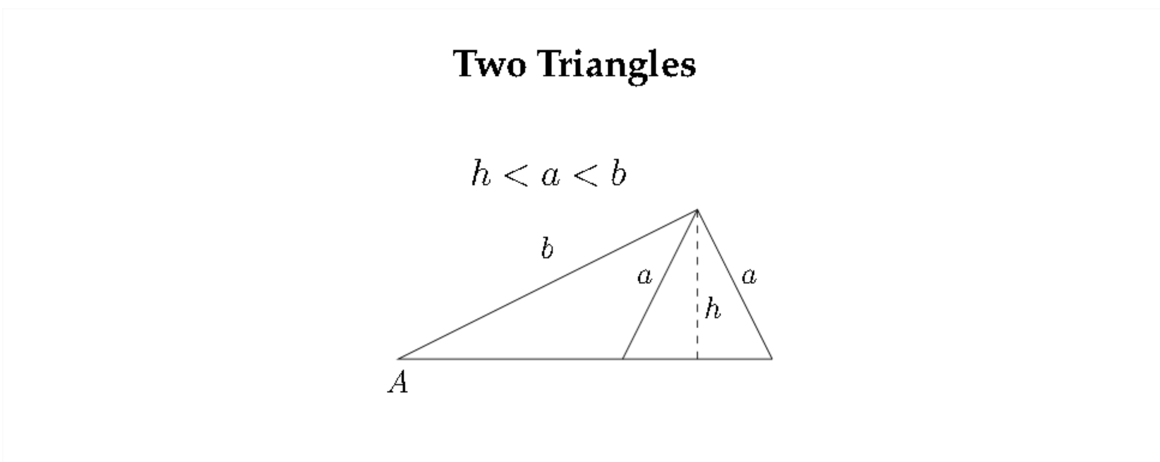
7. $\angle A=50^{\circ}, \quad \angle C=27^{\circ}, \quad a=3$
8. $\angle B=70^{\circ}, \quad \angle C=10^{\circ}, \quad b=5$
9. $\angle A=110^{\circ}, \quad \angle C=30^{\circ}, \quad c=3$
10. $\angle A=50^{\circ}, \quad \angle B=68^{\circ}, \quad a=230$
11. $\angle A=23^{\circ}, \quad \angle B=110^{\circ}, \quad c=50$
12. $\angle A=22^{\circ}, \quad \angle B=95^{\circ}, \quad a=420$
13. $\angle B=10^{\circ}, \quad \angle C=100^{\circ}, \quad c=11$
14. $\angle A=30^{\circ}, \quad \angle C=65^{\circ}, \quad b=10$
15. $\angle A=82^{\circ}, \quad \angle B=65.4^{\circ}, \quad b=36.5$
16. $\angle B=28^{\circ}, \quad \angle C=78^{\circ}, \quad c=44$
17. $\angle A=42^{\circ}, \quad \angle B=61^{\circ}, \quad a=12$
18. $\angle A=42.5^{\circ}, \quad \angle B=71.4^{\circ}, \quad a=215$

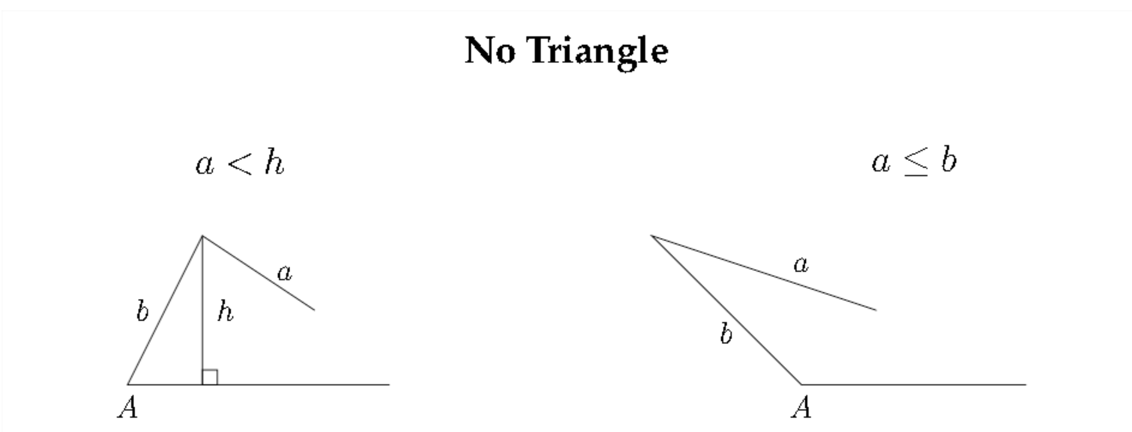
11.2: The Law of Sines - the Ambiguous Case

In all of the examples and problems in Section 4.1, notice that we were always given two angles and one side, although we could use the Law of sines if we were given one angle and two sides (as long as one of the sides corresponded to the given angle). This is because when we use the Law of sines to find an angle, an ambiguity can arise due to the sine function being positive in Quadrant I and Quadrant II.

We saw in Chapter 3 that multiple answers arise when we use the inverse trigonometric functions. For problems in which we use the Law of sines given one angle and two sides, there may be one possible triangle, two possible triangles or no possible triangles. There are six different scenarios related to the ambiguous case of the Law of sines: three result in one triangle, one results in two triangles and two result in no triangle.







We'll look at three examples: one for one triangle, one for two triangles and one for no triangles.

Example 11.2.1

Solve the triangle if: $\angle A = 112^{\circ}, \quad a = 45, \quad b = 24$
Round the angles and side lengths to the nearest 10^{th}

Solution

Using the Law of sines, we can say that:

$$\begin{gathered}\frac{\sin 112^{\circ}}{45} = \frac{\sin B}{24} \\ \frac{0.9272}{45} \approx \frac{\sin B}{24} \\ 24 * \frac{0.9272}{45} \approx \sin B \\ 0.4945 \approx \sin B\end{gathered} \tag{11.2.1}$$

Then, we find $\sin^{-1}(0.4945) \approx 29.6^{\circ}$. Remember from Chapter 3 that there is a Quadrant II angle that has $\sin\theta \approx 0.4945$, with a reference angle of 29.6°. So, $\angle B$ could also be $\approx 150.4^{\circ}$. However, with $\angle A = 112^{\circ}$, there is no way that another angle of 150.4° would fit inside the same triangle. For this reason, we know then that $\angle B$ must be 29.6°

$$29.6^{\circ} \approx B \tag{11.2.2}$$

So now

$$\begin{gathered}\angle A = 112^{\circ} \\ \angle B \approx 29.6^{\circ} \\ \text{and} \angle C = 180^{\circ} - (112^{\circ} + 29.6^{\circ}) = 180^{\circ} - 141.6^{\circ} \approx 38.4^{\circ} \\ \angle C \approx 38.4^{\circ}\end{gathered} \tag{11.2.3}$$

We already know that $a = 45$ and $b = 24$. To find side c, I would recommend using the most exact values possible in the Law of sines calculation. This will provide the most accurate result in finding the length of side c

$$\begin{gathered}\frac{45}{\sin 112^{\circ}} = \frac{c}{\sin 38.4^{\circ}} \\ \frac{45}{0.9272} \approx \frac{c}{0.6211} \\ 0.6211 * \frac{45}{0.9272} \approx c \\ 30.1 \approx c \\ \angle A = 112^{\circ} \qquad a = 45 \\ \angle B \approx 29.6^{\circ} \qquad b = 24 \\ \angle C \approx 38.4^{\circ} \qquad c \approx 30.1\end{gathered} \tag{11.2.4}$$

Example 11.2.2

Solve the triangle if: $\angle A = 38^{\circ}, \quad a = 40, \quad b = 52$
Round the angles and side lengths to the nearest 10^{th}.

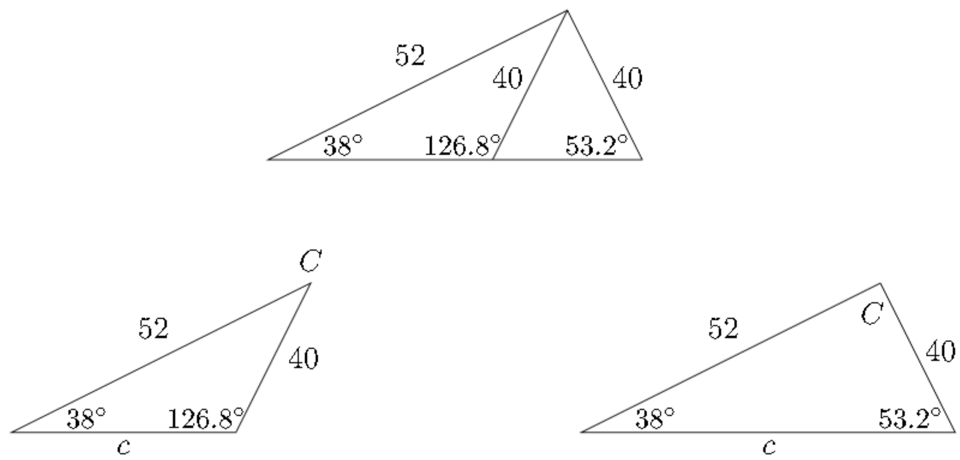
Solution

Using the Law of sines, we can say that:

$$\begin{gathered}\frac{\sin 38^{\circ}}{40} = \frac{\sin B}{52} \\ \frac{0.6157}{40} \approx \frac{\sin B}{52} \\ 52 * \frac{0.6157}{40} \approx \sin B \\ 0.8004 \approx \sin B\end{gathered} \tag{11.2.5}$$

Just as in the previous example, we can find $\sin^{-1}(0.8004) \approx 53.2^{\circ}$. But again, there is a Quadrant II angle whose sine has the same value ≈ 0.8004. The angle 126.8° has a sine ≈ 0.8004 and a reference angle of 53.2°. With $\angle A = 38^{\circ}$, both of these angles (53.2° and 126.8°) could potentially fit in the triangle with angle A

If we go back to the diagrams we looked at earlier in this section, we can see how this would happen:



In the first possibility $\angle C$ would be $\approx 15.2^{\circ}$
In the second possibility $\angle C$ would be $\approx 88.8^{\circ}$

To find the two possible lengths for side c, we'll need to solve two Law of sines calculations, one with $\angle C \approx 15.2^{\circ}$ and one with the $\angle C \approx 88.8^{\circ}$

$$\begin{gathered} \frac{40}{\sin 38^{\circ}} = \frac{c}{\sin 15.2^{\circ}} \\ \frac{40}{0.6157} \approx \frac{c}{0.2622} \\ 0.2622 * \frac{40}{0.6157} \approx c \\ 17.0 \approx c \end{gathered} \tag{11.2.6}$$

With $\angle C \approx 88.8^{\circ}$:

$$\begin{gathered} \frac{40}{\sin 38^{\circ}} = \frac{c}{\sin 88.8^{\circ}} \\ \frac{40}{0.6157} \approx \frac{c}{0.9998} \\ 0.9998 * \frac{40}{0.6157} \approx c \\ 65.0 \approx c \end{gathered} \tag{11.2.7}$$

So, our two possible solutions would be:

$$\begin{array}{ll} \angle A = 38^{\circ} & a = 40 \\ \angle B \approx 126.8^{\circ} & b = 52 \\ \angle C \approx 15.2^{\circ} & c \approx 17.0 \end{array} \tag{11.2.8}$$

OR

$$\angle A = 38^{\circ} \quad a = 40 \tag{11.2.9}$$

$$\angle B \approx 53.2^{\circ} \quad b = 52$$
$$\angle C \approx 88.8^{\circ} \quad c \approx 65.0 \tag{11.2.10}$$

Example 11.2.3

Solve the triangle if:

$$\angle B = 73^{\circ}, \quad b = 51, \quad a = 92. \tag{11.2.11}$$

Round the angles and side lengths to the nearest 10^{th}.

Solution

Using the Law of sines, we can say that:

$$\frac{\sin 73^{\circ}}{51} = \frac{\sin A}{92} \tag{11.2.12}$$

$$\frac{0.9563}{51} \approx \frac{\sin A}{92}$$
$$92 * \frac{0.9563}{51} \approx \sin A \tag{11.2.13}$$
$$1.7251 \approx \sin A$$

As we saw previously, no real-valued angle has a sine greater than 1. Therefore, no triangle is possible.

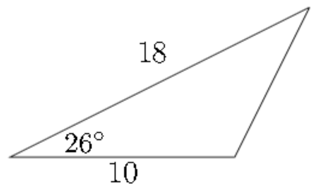
Exercises

In each problem, solve the triangle. Round side lengths to the nearest 100^{th} and angle measures to the nearest 10^{th}.

1. $\angle A = 50^{\circ}, \quad b = 20, \quad a = 32$
2. $\angle B = 40^{\circ}, \quad b = 4, \quad c = 3$
3. $\angle A = 43^{\circ}, \quad a = 23, \quad b = 29$
4. $\angle C = 20^{\circ}, \quad c = 43, \quad a = 55$
5. $\angle B = 62^{\circ}, \quad b = 4, \quad a = 5$
6. $\angle A = 75^{\circ}, \quad b = 8, \quad a = 3$
7. $\angle B = 24^{\circ}, \quad a = 17, \quad b = 8$
8. $\angle A = 40^{\circ}, \quad a = 4, \quad c = 5$
9. $\angle A = 108^{\circ}, \quad a = 12, \quad b = 7$
10. $\angle B = 117^{\circ}, \quad b = 19.6, \quad c = 10.5$
11. $\angle A = 42^{\circ}, \quad a = 18, \quad c = 11$
12. $\angle C = 27^{\circ}, \quad a = 42, \quad c = 37$
13. $\angle C = 125^{\circ}, \quad c = 2.7, \quad b = 5.2$
14. $\angle B = 115^{\circ}, \quad b = 68, \quad a = 92$
15. $\angle A = 43^{\circ}, \quad a = 31, \quad b = 37$
16. $\angle A = 28^{\circ}, \quad b = 3.5, \quad a = 4.3$
17. $\angle C = 132^{\circ}, \quad c = 22, \quad b = 16$
18. $\angle B = 114.2^{\circ}, \quad b = 87.2, \quad a = 12.1$
19. $\angle B = 52^{\circ}, \quad c = 82.7, \quad b = 70$
20. $\angle C = 65^{\circ}, \quad b = 7.6, \quad c = 7.1$

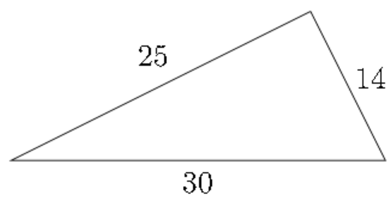
11.3: The Law of Cosines

There are situations in which the Law of sines cannot be used to solve a triangle. In the diagram below, we have information about two sides and the included angle:



The problem above lacks a complete angle-side pair which is necessary to set up the Law of sines calculation.

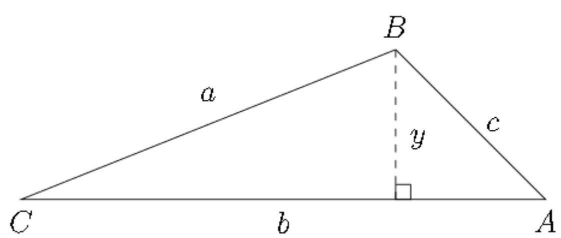
Another common situation involves a triangle in which all three sides are known but no angles are known:



Again, the lack of an angle-side pair would prevent us from setting up a Law of sines calculation.

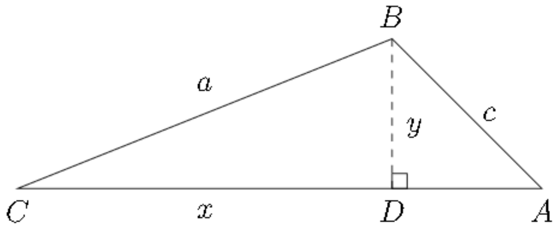
The Law of Cosines is one way to get around this difficulty. Using the Law of cosines is more complicated than using the Law of sines, however, as we have just seen, the Law of sines will not always be enough to solve a triangle.

To derive The Law of cosines, we begin with an arbitrary triangle, like the one seen on the next page:



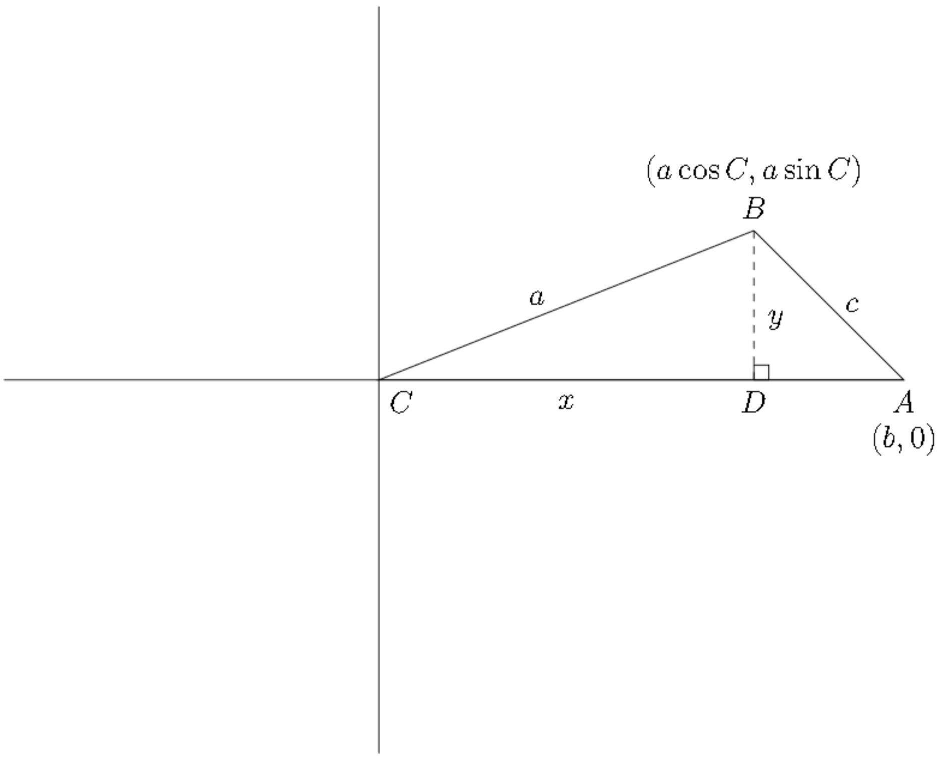
In this diagram we have taken the arbitrary triangle and created a perpendicular with length y. From this, we can say that $\sin C = \frac{y}{a}$ and that $a \sin C = y$

We can split side AC into two pieces AD and CD, as seen below, and label the distance CD as x



Then, we can say that $\cos C = \frac{x}{a}$ and that $a \cos C = x$

If we then put this triangle onto the coordinate axes with $\angle C$ at the origin (0,0) we can derive the Law of cosines. The coordinate of the vertex at $\angle B$ will be $(a \cos C, a \sin C)$, and the coordinates of the vertex at $\angle A$ will be $(b, 0)$



Using the distance formula, we can say that:

$$c = \sqrt{(a \cos C - b)^2 + (a \sin C - 0)^2} \tag{11.3.1}$$

Squaring both sides:

$$c^2 = (a \cos C - b)^2 + (a \sin C - 0)^2 \tag{11.3.2}$$

and

$$c^2 = a^2 \cos^2 C - 2ab \cos C + b^2 + a^2 \sin^2 C \tag{11.3.3}$$

or

$$c^2 = a^2 \sin^2 C + a^2 \cos^2 C + b^2 - 2ab\cos C \tag{11.3.4}$$

Factoring out the a^2 and replacing $\sin^2 C + \cos^2 C$ with 1 , we come out with one of the most common forms of the Law of cosines:

$$\begin{gathered} c^2 = a^2 \sin^2 C + a^2 \cos^2 C + b^2 - 2ab\cos C \\ c^2 = a^2\left(\sin^2 C + \cos^2 C\right) + b^2 - 2ab\cos C \\ c^2 = a^2(1) + b^2 - 2ab\cos C \\ c^2 = a^2 + b^2 - 2ab\cos C \end{gathered} \tag{11.3.5}$$

Any letter may be used to represent each of the sides, but the order that the letters are used in is very important. The side of the triangle isolated on the left hand side must correspond to the angle used on the right hand side.

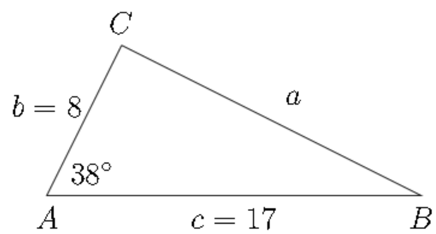
The Law of cosines

$$\begin{gathered} a^2 = b^2 + c^2 - 2bc\cos A \\ b^2 = a^2 + c^2 - 2ac\cos B \\ c^2 = a^2 + b^2 - 2ab\cos C \end{gathered} \tag{11.3.6}$$

We'll look at three examples- two in which two sides and the included angle are given and one in which the three sides of the triangle are given.

Example 1

Solve the triangle: $\angle A = 38^\circ, c = 17, b = 8$ Round angle measures and side lengths to the nearest 10^{th} .



It's usually a good idea to see if you can use the Law of sines first, since it is easier to calculate. In this case we can't because we don't have a complete angle-side pair. So, using the Law of cosines to find side a :

$$\begin{gathered} a^2 = b^2 + c^2 - 2bc\cos A \\ a^2 = 8^2 + 17^2 - 2 * 8 * 17 * \cos 38^\circ \\ a^2 \approx 64 + 289 - 272 * 0.7880 \\ a^2 \approx 353 - 214.336 \\ a^2 \approx 138.664 \\ a \approx 11.8 \end{gathered} \tag{11.3.7}$$

Once we know that $a \approx 11.8$ we can use this to find the other angles using the Law of sines. Because of the issue of the ambiguous case in using the Law of sines, it's often a good idea to find the angles that correspond to the two shortest sides in the triangle, because if there is an obtuse angle in the triangle it will have to correspond to the longest side. If we find the two smaller angles, we can be assured that they will both be acute and we can subtract from 180° to find the largest angle.

$$\begin{gathered} \frac{\sin 38^\circ}{11.8} = \frac{\sin B}{8} \\ 8 \times \frac{0.61566}{11.8} = \sin B \\ 0.4174 \approx \sin B \\ 24.7^\circ \approx B \end{gathered} \tag{11.3.8}$$

So with $\angle A=38^{\circ}$ and $\angle B\approx 24.7^{\circ}$, then:

$$\angle C\approx 180^{\circ}-(38^{\circ}+24.7^{\circ})\approx 180^{\circ}-62.7^{\circ}\approx 117.3^{\circ} \tag{11.3.9}$$

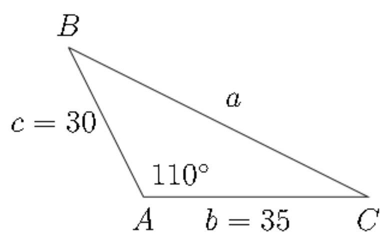
So, the angles and sides of the triangle would be:

$$\begin{array}{ll}\angle A=38^{\circ} & a\approx 11.8\\ \angle B\approx 24.7^{\circ} & b=8\\ \angle C\approx 117.3^{\circ} & c=17\end{array} \tag{11.3.10}$$

In example 2, we'll look at a problem in which an obtuse angle is given.

Example 2

Solve the triangle: $\angle A=110^{\circ}, c=30, b=35$ Round angle measures and side lengths to the nearest 10^{th}.



The calculation for this problem is slightly different from the last one because the cosine of 110° will be negative:

$$\begin{gathered}a^2=b^2+c^2-2bc\cos A\\ a^2=35^2+30^2-2*35*30*\cos 110^{\circ}\\ a^2\approx 1225+900-2100*(-0.3420)\\ a^2\approx 2125+718.2\\ a^2\approx 2843.2\\ a\approx 53.3\end{gathered} \tag{11.3.11}$$

In this problem, since we were given an obtuse angle, then the other two angles must be acute and we don't have to worry about the ambiguous case in using the Law of sines.

$$\frac{\sin 110^{\circ}}{53.3}=\frac{\sin B}{35} \quad \frac{\sin 110^{\circ}}{53.3}=\frac{\sin C}{30} \tag{11.3.12}$$

$$35*\frac{0.9397}{53.3}=\sin B \quad 30*\frac{0.9397}{53.3}=\sin C \tag{11.3.13}$$

$$0.61706\approx\sin B \quad 0.5289\approx\sin C \tag{11.3.14}$$

$$38.1^{\circ}\approx B \quad 31.9^{\circ}\approx C \tag{11.3.15}$$

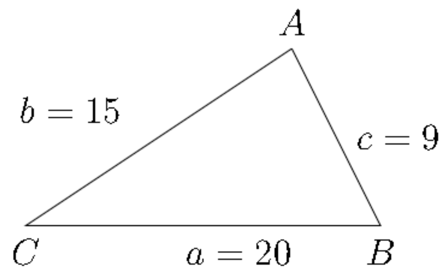
So the angles and sides of the triangle would be:

$$\begin{array}{ll}\angle A=110^{\circ} & a\approx 53.3\\ \angle B\approx 38.1^{\circ} & b=35\\ \angle C\approx 31.9^{\circ} & c=30\end{array} \tag{11.3.16}$$

In example 3, we'll look at a problem in which three side lengths are given and we find an angle using the Law of cosines.

Example 3

Solve the triangle: $a=20, \quad c=9, \quad b=15$ Round angle measures to the nearest 10^{th}



It doesn't matter which angle we choose to solve, but whichever angle we choose must correspond to the side isolated on the left-hand side of the formula. If we want to solve for $\angle B$, we would say:

$$\begin{gathered} b^2 = a^2 + c^2 - 2ac\cos B \\ 15^2 = 20^2 + 9^2 - 2*20*9*\cos B \\ 225 = 400 + 81 - 360*\cos B \\ 225 = 481 - 360\cos B \\ -256 = -360\cos B \\ \frac{-256}{-360} = \frac{-360\cos B}{-360} \\ 0.7\overline{1} = \cos B \\ 44.7^\circ \approx B \end{gathered} \tag{11.3.17}$$

Once we know the measure of $\angle B$, we'll use this to find the measure of $\angle C$, which corresponds to side c, the smallest side. Then we'll subtract to find the biggest angle.

$$\begin{gathered} \frac{\sin 44.7^\circ}{15} = \frac{\sin C}{9} \\ 9*\frac{0.7034}{15} = \sin C \\ 0.42204 \approx \sin C \\ 25.0^\circ \approx C \end{gathered}$$

So, with $\angle B \approx 44.7^\circ$ and $\angle C \approx 25.0^\circ$, then:

$$\angle A \approx 180^\circ - (44.7^\circ + 25.0^\circ) \approx 180^\circ - 69.7^\circ \approx 110.3^\circ \tag{11.3.18}$$

So the angles and sides of the triangle would be:

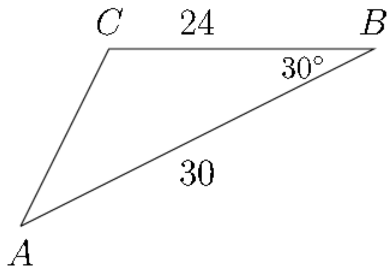
$$\begin{array}{ll} \angle A = 110.3^\circ & a \approx 53.3 \\ \angle B \approx 44.7^\circ & b = 35 \\ \angle C \approx 25.0^\circ & c = 30 \end{array} \tag{11.3.19}$$

If we had used the Law of sines to find $\angle A$, the calculator would have returned the value of the reference angle for $\angle A$, rather than the angle that is actually in the triangle described in the problem!

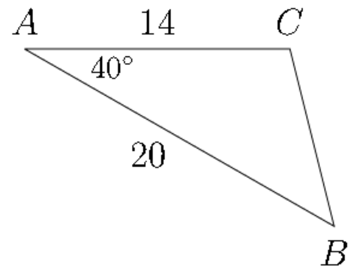
Exercises 4.3

In each problem, solve the triangle. Round side lengths to the nearest 100^{th} and angle measures to the nearest 10^{th}.

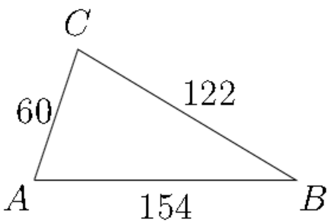
1.



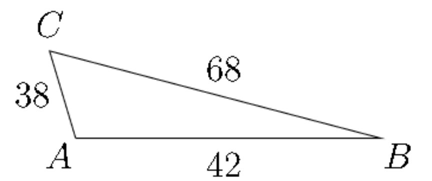
2.



3.



4.



5. $\angle A=52^{\circ}, \quad c=27, \quad b=36$
6. $\angle B=75^{\circ}, \quad a=32, \quad c=59$
7. $\angle B=135^{\circ}, \quad a=12, \quad c=18$
8. $\angle C=120^{\circ}, \quad b=22, \quad a=30$
9. $a=21, \quad b=26, \quad c=23$
10. $a=11, \quad b=13, \quad c=17$
11. $a=25, \quad b=32, \quad c=40$
12. $a=60, \quad b=88, \quad c=120$
13. $\angle A=77.4^{\circ}, \quad b=444, \quad c=390$
14. $\angle B=10^{\circ}, \quad a=18, \quad c=30$
15. $a=112.7, \quad b=96.5, \quad c=130.2$
16. $a=4.7, \quad b=3.2, \quad c=5.9$

11.4: Applications

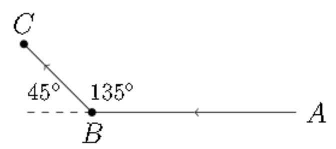
In the previous sections on applications, we saw situations in which right triangle trigonometry was used to find distances and angles. In this section, we will use the Law of sines and the Law of cosines to find distances and angles.

Example 11.4.1

A car travels along a straight road, heading west for 1 hour, then traveling on another straight road northwest for a half hour. If the speed of the car was a constant 50 mph how far is the car from its starting point?

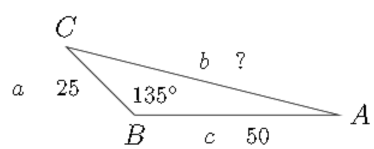
Solution

First, let's draw a diagram:



In the picture above, we know the angles 45° and 135° because of the direction the car was traveling. The direction northwest cuts exactly halfway between north and west creating a 45° angle. On the other side of this 45° angle is a 135° angle which is in the triangle we'll use to answer the question (triangle ABC).

The length of $\overline{AB}$ is 50 miles and the length of $\overline{BC}$ is 25 miles. This comes from the information about the speed and traveling time given in the problem. So the triangle we need to answer the question is pictured below:



We can use the Law of cosines to solve this problem:

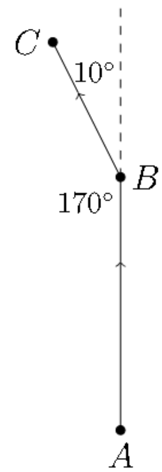
$$\begin{gathered} b^2=a^2+c^2-2ac\cos B \\ b^2=25^2+50^2-2*25*50*\cos 135^\circ \\ b^2\approx 625+2500-2500*(-0.7071) \\ b^2\approx 3125+1767.75 \\ b^2\approx 4892.75 \\ b\approx 69.9\text{miles} \end{gathered} \tag{11.4.1}$$

Example 11.4.2

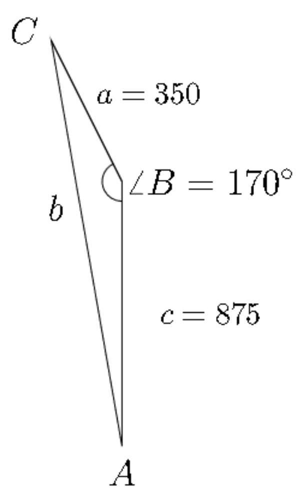
A pilot flies an airplane in a straight path for 2.5 hours and then makes a course correction, heading 10° to the left of the original course. The pilot then flies in this direction for 1 hour. If the speed of the plane is a constant 350mph, how far is the plane from its starting position?

Solution

Again, we'll start by making a diagram:



In this problem, we'll be working with triangle ABC, shown below. We can calculate the lengths of $\overline{AB}$ and $\overline{BC}$ from the information given in the problem and use this to calculate the length of $\overline{AC}$:

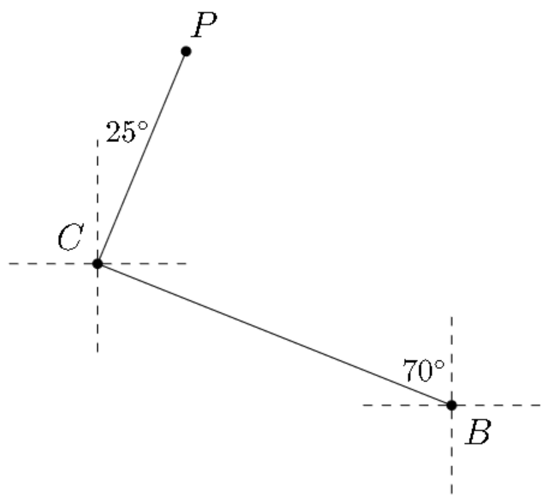


Using the Law of cosines:

$$\begin{gathered} b^2 = a^2 + c^2 - 2ac\cos B \\ b^2 = 350^2 + 875^2 - 2 * 350 * 1050 * \cos 170^\circ \\ b^2 \approx 122,500 + 765,625 - 735,000 * (-0.9848) \\ b^2 \approx 888,125 + 723,828 \\ b^2 \approx 1,611,953 \\ b \approx 1270\text{miles} \end{gathered} \tag{11.4.2}$$

Example 11.4.3

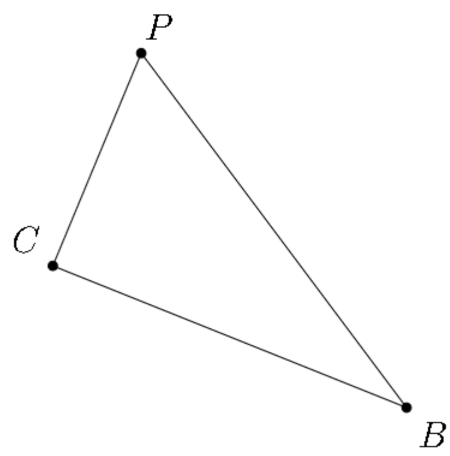
A pilot leaves the airport in Bend, headed towards Corvallis with the bearing $N70^\circ W$. He travels the 103 miles and makes a delivery before taking off and flying at a bearing of $N25^\circ E$ for 72 miles to arrive in Portland.



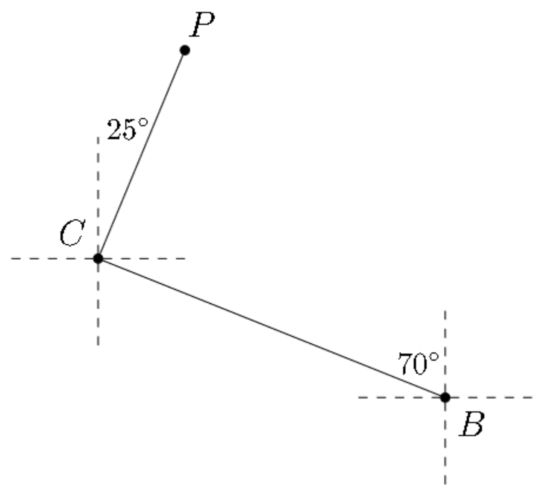
a) Based on this information, find the air distance between Portland and Bend.
b) Find the bearing from Portland to Bend.

Solution

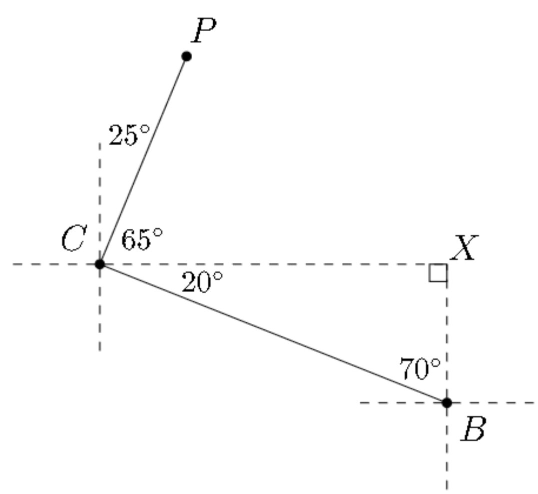
In this problem, a diagram has been given. We'll amend this to make it into a triangle:



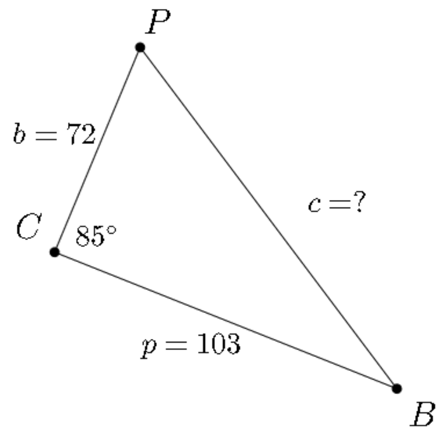
Filling the measures of the angles is tricky in this problem, so let's look at the original diagram again:



If we extend the dashed line east from Corvallis so that it meets the dashed line running north from Bend, we can create a triangle that shows us that the angle $\angle BCX = 20^{\circ}$. Also, notice that $\angle PCX = (90^{\circ} - 25^{\circ}) = 65^{\circ}$



This means that $\angle BCP = 85^{\circ}$. We know from the problem that $\overline{BC} = 103$ and $\overline{CP} = 72$. We'll need to find the length of $\overline{BP}$ and the measure of $\angle CPB$ to answer the questions.



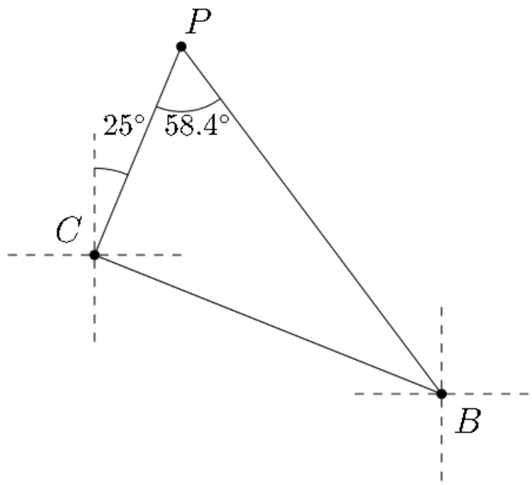
Now we're working with a triangle like the one pictured above, so we can use the Law of cosines to find the air distance from Portland to Bend:

$$\begin{gathered} c^2 = b^2 + p^2 - 2bp\cos C \\ c^2 = 72^2 + 103^2 - 2*72*103*\cos 85^\circ \\ c^2 \approx 5184 + 10,609 - 14,832*(0.087156) \\ c^2 \approx 15,793 - 1292.7 \\ c^2 \approx 14,500.3 \\ c \approx 120.4 \text{ miles} \end{gathered} \tag{11.4.3}$$

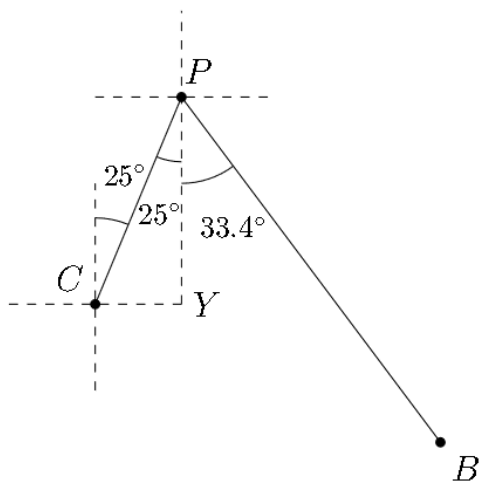
To find $\angle P$, we'll use the Law of sines:

$$\begin{gathered} \frac{\sin 85^\circ}{120.4} = \frac{\sin P}{103} \\ 103*\frac{\sin 85^\circ}{120.4} = \sin P \\ 103*\frac{0.9962}{120.4} \approx \sin P \\ 0.85223 \approx \sin P \\ 58.4^\circ \approx P \end{gathered} \tag{11.4.4}$$

Now that we know the measure of $\angle P$, we can determine the bearing of Bend from Portland.

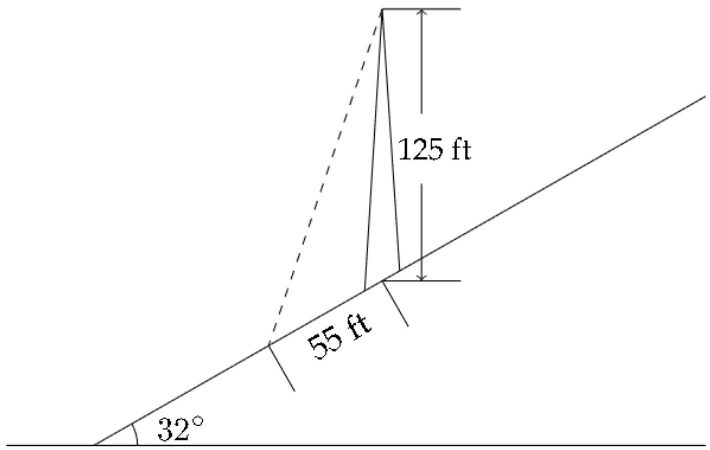


In the picture below notice that $\angle YPC = 25^{\circ}$. This means that the bearing from Portland to Bend will be east of south by the difference between $\angle P = 58.4^{\circ}$ and $\angle YPC = 25^{\circ}$. This makes the bearing of Bend from Portland equal to $S33.4^{\circ}E$



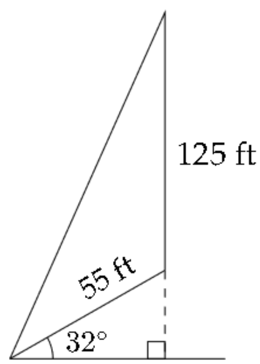
Example 11.4.4

A 125 foot tower is located on the side of a mountain that is inclined 32° to the horizontal. A guy wire is to be attached to the top of the tower and anchored at a point 55 feet downhill from the base of the tower. Find the shortest length of wire needed.



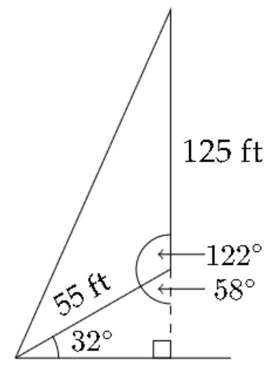
Solution

An important aspect in solving this problem is to identify a triangle in the problem which involves the unknown quantity we're being asked to find. If we're looking for the length of the guy wire, we can use a triangle that involves the wire, the distance from the wire to the center of the tower and the height of the tower:



The angle between the horizontal and the hill will stay 32° at any point on the hill. If we drop a perpendicular to the horizontal, we'll be able to find the angle included between the two given sides.

In the little right triangle, we know the 32 'angle. That means the other acute angle must be 58°, and the supplementary angle (which is in the triangle we're interested in $)$ will be 122°



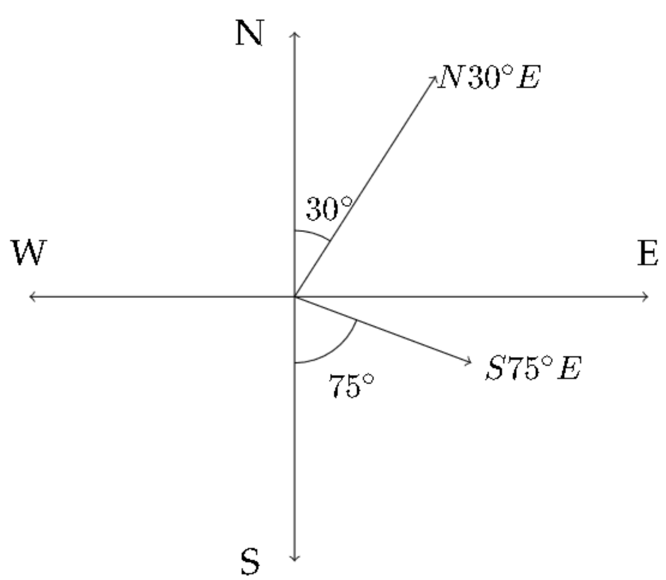
Now we can use the Law of cosines to find the length of the guy wire:

$$\begin{aligned} x^2 &= 125^2 + 55^2 - 2 * 125 * 55 * \cos 122^\circ \\ x^2 &= 15,625 + 3025 - 13,750 * \cos 122^\circ \\ x^2 &\approx 18,650 + 7286.39 \\ x^2 &\approx 25,936.39 \\ x &\approx 161 \text{ feet} \end{aligned} \tag{11.4.5}$$

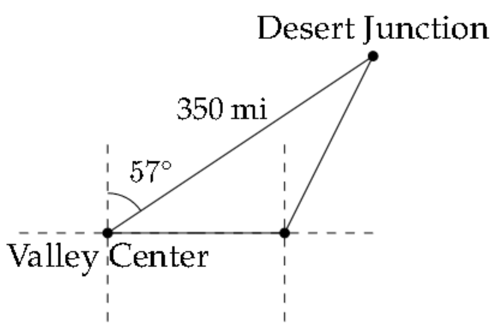
Exercises

1. Two straight roads diverge at an angle of 50°. Two cars leave the intersection at 1pm, one traveling 60mph and the other traveling 45mph. How far apart are the cars (as the crow flies) at $1:30\text{pm}$?

2. Two boats leave the same port at the same time. One travels at a speed of 40mph in the direction $N30^\circ E$ and the other travels at a speed of 28 mph in the direction $S75^\circ E$. How far apart are the two boats after one hour?



3. The airport in Desert Junction is 350 miles from the airport in Valley Center at a bearing of $N57^{\circ}E$. A pilot who wants to fly from Valley Center to Desert Junction mistakenly flies due east at 225 mph for 30 minutes before correcting the error. How far is the plane from its destination when the pilot notices the error? What bearing should the plane use in order to arrive at Desert Junction?



4. An airplane leaves airport A and travels 520 miles to airport B at a bearing of $N35^{\circ}W$. The plane leaves airport B and travels to airport C 310 miles away at a bearing of $S65^{\circ}W$ from airport B. Find the distance from airport A to airport C.

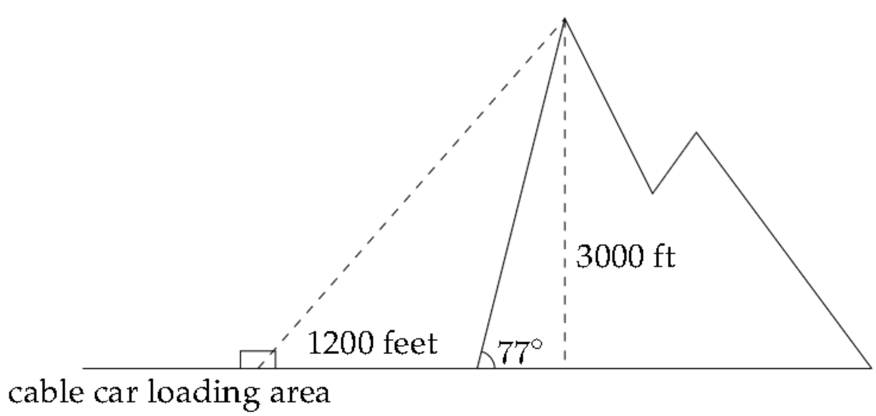
5. Two planes take off at the same time from an airport. The first plane flies at 300 mph at a bearing of $S45^{\circ}E$. The second plane is flying at a bearing of $S5^{\circ}W$ with a speed of 330 mph. How far apart are they after 3 hours?

6. Two planes leave an airport at the same time. Their speeds are 180mph and 110mph, and the angle between their flight paths is 43°. How far apart are they after 2.5 hours?

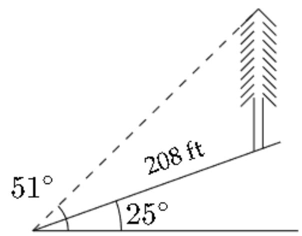
7. Two ships leave a harbor entrance at the same time. The first travels at a speed of 23 mph and the second travels at 17 mph. If the angle between the courses of the ships is 110°, how far apart are they after one hour?

8. A ship leaves the entrance to a harbor and travels 15 miles with a bearing $S10^{\circ}W$, then turns and travels 45 miles with a bearing of $N43^{\circ}W$. How far from the harbor entrance is the ship and what is the bearing of the ship from the harbor?

9. A steep mountain is inclined 77° to the horizontal and rises 3000 feet above the surrounding plain. A cable car is to be installed that will connect the plain to the top of the mountain. The distance from the foot of the mountain to the cable car entry loading area is 1200 feet (see diagram below). Find the shortest necessary length of the cable.



10. A tree on a hillside casts a shadow 208ft down the hill. If the angle of inclination of the hillside is 25° to the horizontal and the angle of elevation of the sun is 51°, find the height of the tree.


51°
208 ft
25°

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