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### My Math GPS: Elementary Algebra Guided Problem Solving (2016 Edition)

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# My Math GPS: Elementary Algebra

Guided Problem Solving

2016 Edition

Jonathan Cornick

G. Michael Guy

Karan Puri



<http://MyMathGPS.com>

GPS 1-6 is loosely based on work completed with Francis Gardella and Brooke Nixon-Freidheim as part of the Graduate NYC! Curriculum Alignment Project. We are grateful for their collaboration on earlier versions of these introductory lessons. More information on this work is available at <http://gradnyc.com/curriculum-alignment-project/>

This textbook aligns to the 2016 CUNY Elementary Algebra Learning Outcomes. For more information about these learning outcomes and accompanying final exam, please visit <http://cuny.edu/testing>

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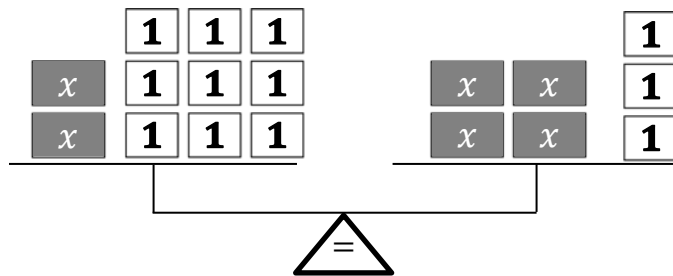
# Linear Equations and Inequalities

## GPS 1: Understanding Linear Equations

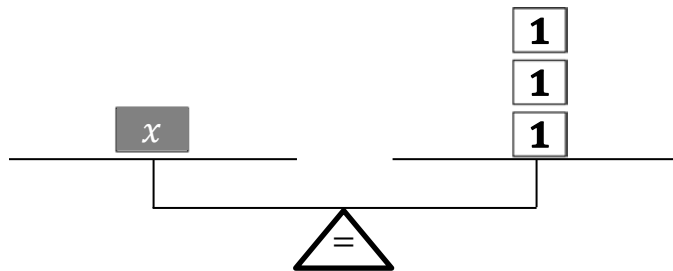
When we think about equations, we usually think of numbers, variables, and symbols.

$$2x + 9 = 4x + 3$$

The diagram below also represents the equation above. Here we visualize the equation as a balance with each side weighing the same.



It is likely that everyone has used a balance at some point. You may have used one in a science lab or when playing with a seesaw on a playground. Based on your experience using a balance, you know it would very easy to determine the weight of one  $x$  block if the picture looked like the one below.



Solving an equation is very similar the process of moving blocks on a balance to determine the weight of the unknown block. Here are a few observations you likely made about balances. These are very similar to the rules you may recall learning from working with equations.

- 1) The balance will only be level if both sides weigh exactly the same.
- 2) If you add weight to one side, then to keep it level you must add the same weight to the other side.
- 3) If you remove weight from one side, then to keep it level you must remove the same weight from the other side.

Let's see how we can perform a sequence of moves so that we are left with a single  $x$  on one side of the balance and a weight we know on the other side. We will show how each step is represented algebraically.

**Problem 1.1**Solve for  $x$ . Check your solution.

$$2x + 9 = 4x + 3$$

In each step, we draw the starting and ending picture of the balance. We also show the corresponding starting and ending equation.

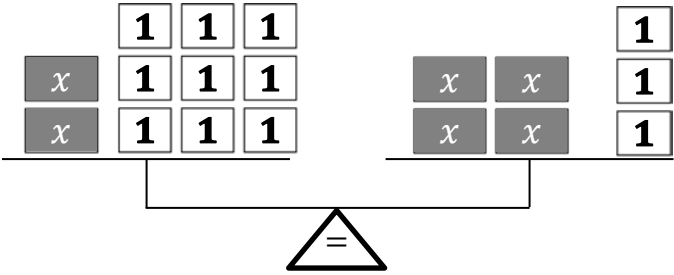
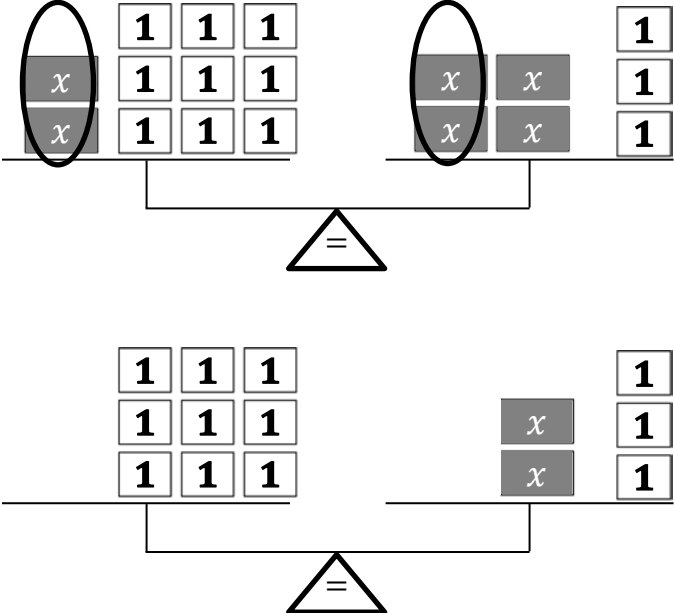
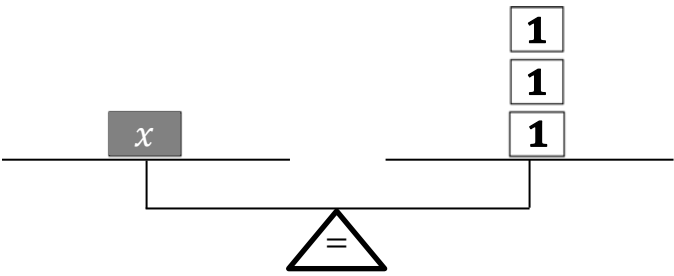
Diagram	Explanation	Equation
	<p>This is our starting point.</p> <p>Our first goal is to get all the <math>x</math> on one side of the balance, and the unit blocks on the opposite side.</p>	$2x + 9 = 4x + 3$
	<p>Remove two <math>x</math> from each side of the balance.</p> <p>This is the same as subtracting <math>2x</math> from both sides of the equation.</p> <p>After this step, the <math>x</math> are only on one side of the balance.</p>	$2x + 9 = 4x + 3$ $2x - 2x + 9 = 4x - 2x + 3$ $9 = 2x + 3$

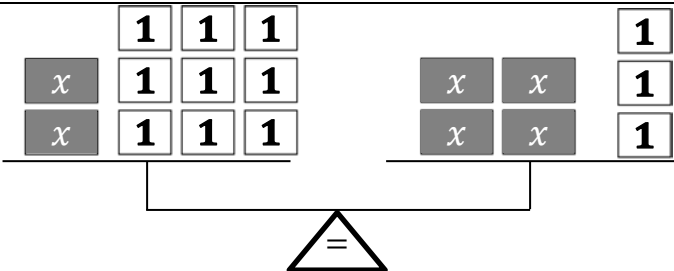
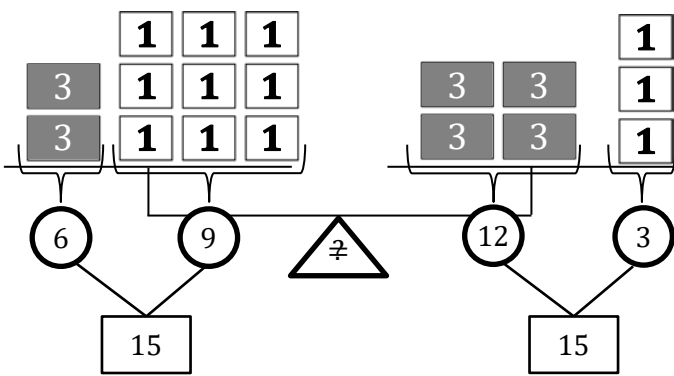
Diagram	Explanation	Equation
	<p>We must eliminate the 3 unit blocks on the right by removing three blocks from each side.</p> <p>One side of the balance has only <math>x</math> blocks, and the other side is only unit blocks.</p>	$9 = 2x + 3$ $9 - 3 = 2x + 3 - 3$ $6 = 2x$
	<p>We now know how much two <math>x</math> blocks weigh. But we would like to know how much only one <math>x</math> weighs. Since there are two <math>x</math> blocks that each weigh the same amount, each <math>x</math> must weigh half the total. To determine the weight of one <math>x</math>, we will divide each side into two groups and keep only one group.</p>	$6 = 2x$ $\frac{6}{2} = \frac{2x}{2}$ $3 = x$



Diagram	Explanation	Equation
	<p>We see that each <math>x</math> weighs 3 units. We ordinarily write our solutions as <math>x = 3</math>, rather than <math>3 = x</math>. We see this is like exchanging the left and right sides of the balance.</p>	$3 = x$ $x = 3$

We think each  $x$  weighs 3. We can check that our solution is correct by going back to the original balance and calculating the weight of each side. If we are correct, then both sides will weigh the same.

**Check:**

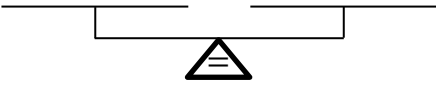
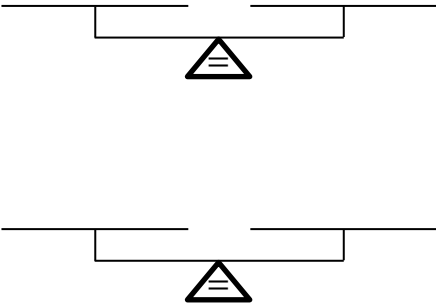
Diagram	Explanation	Equation
	<p>We start with our <i>original</i> balance. We then change the label of each block from <math>x</math> to 3, and then determine the total weight of each side.</p>	$2x + 9 = 4x + 3$ $2(3) + 9 \neq 4(3) + 3$ $6 + 9 \neq 12 + 3$ $15 = 15 \quad \checkmark$
	<p>We see that each side of the balance weighs the same. We know our solution is correct.</p>	

In the following problems, you will solve and check equations using these ideas. At each step, draw the starting and ending diagram. Also write what you are doing in words.

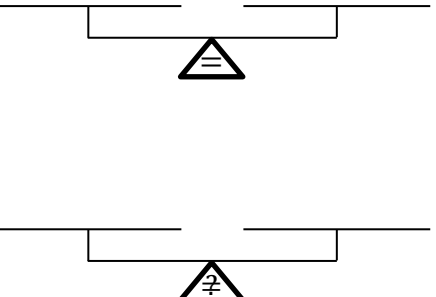
**Problem 1.2**

Solve for  $x$ . Check your solution.

$$3x = 12$$

Diagram	Explanation	Equation
	<p>Draw the starting diagram. Then draw changes to your picture. Do this for each step.</p>	
		

Check:

Diagram	Explanation	Equation
	<p>Start by redrawing your original diagram. Then change the labels on the second balance, and determine the total weight of each side.</p>	

■

 **Problem 1.3**

Solve for  $x$ . Check your solution.

$$3x = x + 10$$

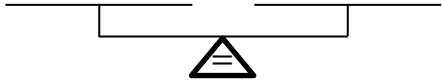
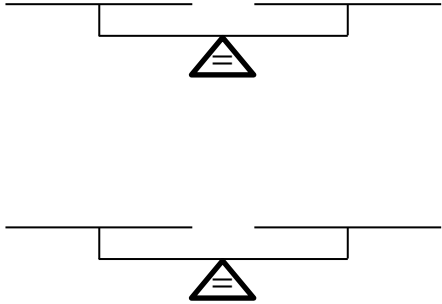
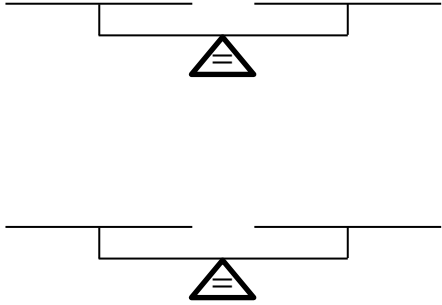
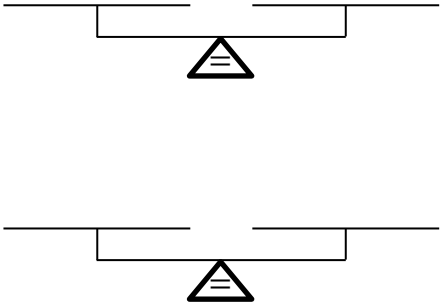
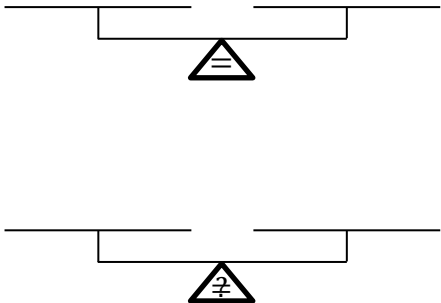
Diagram	Explanation	Equation
	<p>Draw the starting diagram. Then draw changes to your picture. Do this for each step.</p>	
		
		

Diagram	Explanation	Equation
		

Check:

Diagram	Explanation	Equation
	<p>Start by redrawing your original diagram. Then change the labels on the second balance, and determine the total weight of each side.</p>	



 **Problem 1.4**

Solve for  $x$ . Check your solution.

$$6x = 2x + 8$$

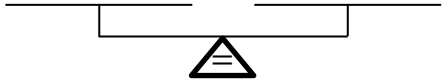
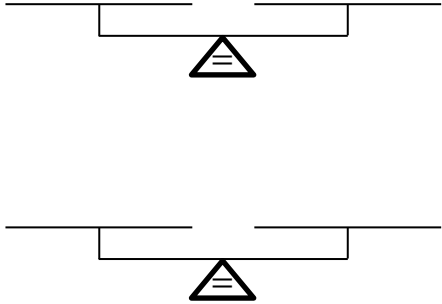
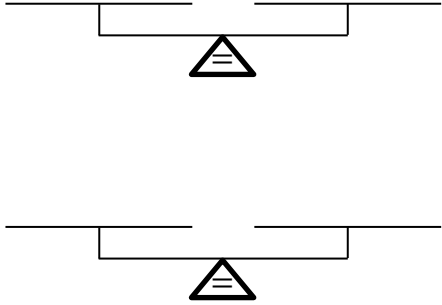
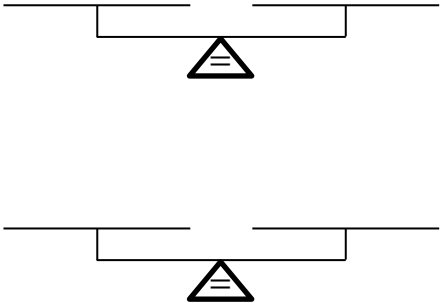
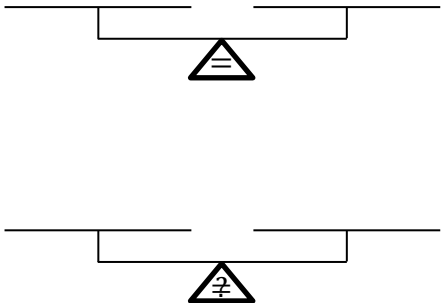
Diagram	Explanation	Equation
	<p>Draw the starting diagram. Then draw changes to your picture. Do this for each step.</p>	
		
		

Diagram	Explanation	Equation
		

Check:

Diagram	Explanation	Equation
	<p>Start by redrawing your original diagram. Then change the labels on the second balance, and determine the total weight of each side.</p>	



**Problem 1.5**

Solve for  $x$ . Check your solution.

$$5 = x + 2$$

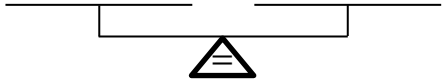
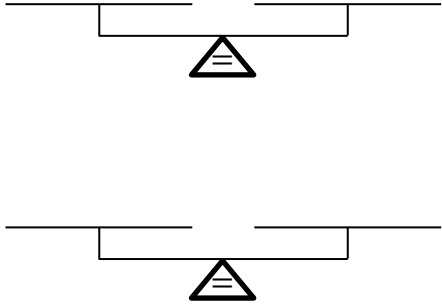
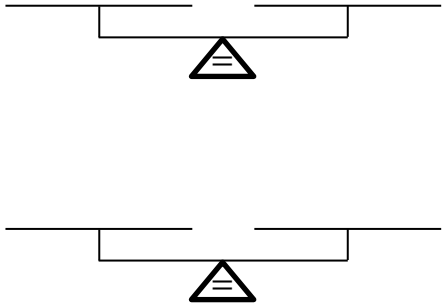
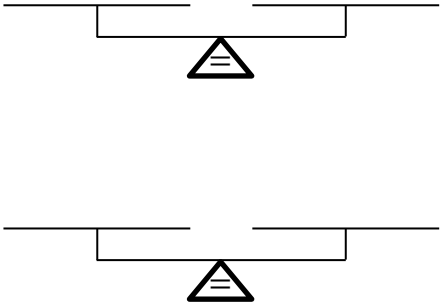
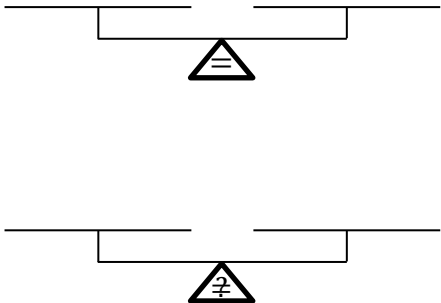
Diagram	Explanation	Equation
	<p>Draw the starting diagram. Then draw changes to your picture. Do this for each step.</p>	
		
		

Diagram	Explanation	Equation
		

Check:

Diagram	Explanation	Equation
	<p>Start by redrawing your original diagram. Then change the labels on the second balance, and determine the total weight of each side.</p>	

■



 **Problem 1.6**

Solve for  $x$ . Check your solution.

$$12 = 3x + 3$$

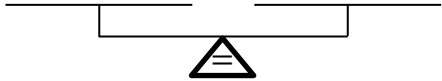
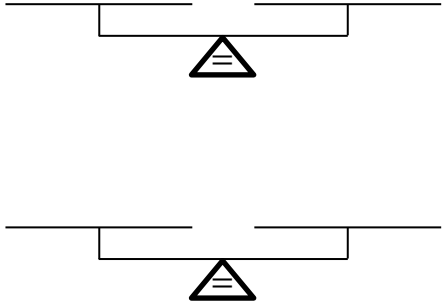
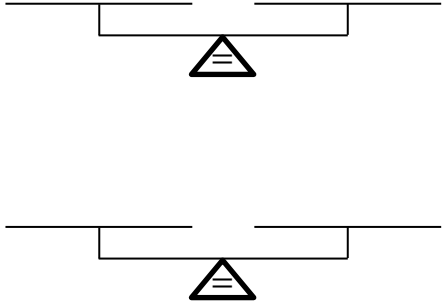
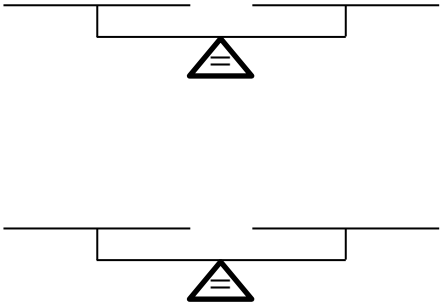
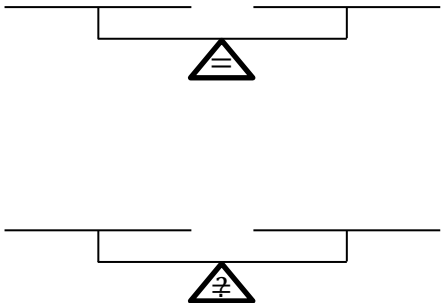
Diagram	Explanation	Equation
	<p>Draw the starting diagram. Then draw changes to your picture. Do this for each step.</p>	
		
		

Diagram	Explanation	Equation
		

Check:

Diagram	Explanation	Equation
	<p>Start by redrawing your original diagram. Then change the labels on the second balance, and determine the total weight of each side.</p>	



 **Problem 1.7**

Solve for  $x$ . Check your solution.

$$3x + 6 = 5x + 2$$

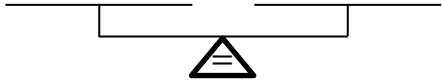
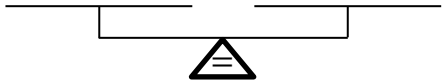

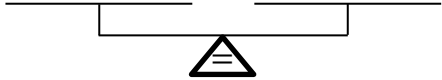
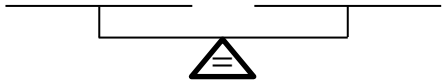
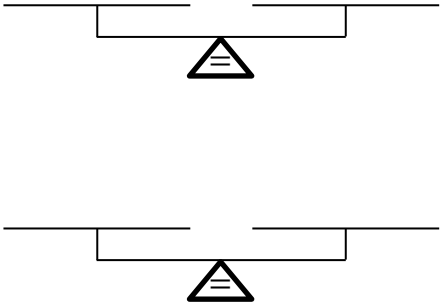
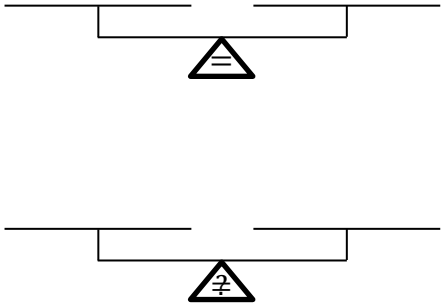
Diagram	Explanation	Equation
	<p>Draw the starting diagram. Then draw changes to your picture. Do this for each step.</p>	
 		
 		

Diagram	Explanation	Equation
		

Check:

Diagram	Explanation	Equation
	<p>Start by redrawing your original diagram. Then change the labels on the second balance, and determine the total weight of each side.</p>	

■



## GPS 2: Solving Linear Equations with Whole Numbers

In the last section we learned a different way to think about the process of solving and checking linear equations. In this section, we will continue to practice solving equations without drawing the pictures.

### Problem 2.1

Solve for  $x$ . Check your solution.

$$3x + 14 = 8x + 4$$

Our goal is to move all the variable terms to one side, and all the terms without variables to the opposite side. We have four choices how to start this problem. But after we make our first choice, we do not have any more reasonable choices of what to do for the remainder of the problem. We will revisit each of the three remaining choices next. For now, we will move the  $3x$  to the right side with the  $8x$ .

$3x + 14 = 8x + 4$	This is the original problem.
$3x - 3x + 14 = 8x - 3x + 4$	Subtract $3x$ from both sides.
$14 = 5x + 4$	Simplify.
$14 - 4 = 5x + 4 - 4$	Subtract 4 from both sides from both sides.
$10 = 5x$	Simplify.
$\frac{10}{5} = \frac{5x}{5}$	Divide both sides by 5.
$2 = x$	Simplify.
$x = 2$	Reverse the order of the equality.

### Check:

To check, we will substitute our solution into the *original equation*. If we are correct, then we will get a true statement.

$$\begin{aligned}
 3x + 14 &= 8x + 4 \\
 3(2) + 14 &\neq 8(2) + 4 \\
 6 + 14 &\neq 16 + 4 \\
 20 &= 20 \quad \checkmark
 \end{aligned}$$

Since we arrived at the true statement  $20 = 20$ , we know that our answer is correct.

■

In the previous problem, we commented that we had four choices how to start this problem. We elected to start by subtracting  $3x$  from each side. The result of this was that our variables were moved to the right of the equal sign. Once we choose to move the variables to the right, we had no reasonable choice except to move the terms without variables to the left. Let's see the three other choices now before doing one on your own.

First subtract 4 from both sides.

$$\begin{aligned}
 3x + 14 &= 8x + 4 \\
 3x + 14 - 4 &= 8x + 4 - 4 \\
 3x + 10 &= 8x \\
 3x - 3x + 10 &= 8x - 3x \\
 10 &= 5x \\
 \frac{10}{5} &= \frac{5x}{5} \\
 2 &= x \\
 x &= 2
 \end{aligned}$$

First subtract  $8x$  from both sides.

$$\begin{aligned}
 3x + 14 &= 8x + 4 \\
 3x - 8x + 14 &= 8x - 8x + 4 \\
 -5x + 14 &= 4 \\
 -5x + 14 - 14 &= 4 - 14 \\
 -5x &= -10 \\
 \frac{-5x}{-5} &= \frac{-10}{-5} \\
 x &= 2
 \end{aligned}$$

First subtract 14 from both sides.

$$\begin{aligned}
 3x + 14 &= 8x + 4 \\
 3x + 14 - 14 &= 8x + 4 - 14 \\
 3x &= 8x - 10 \\
 3x - 8x &= 8x - 8x - 10 \\
 -5x &= -10 \\
 \frac{-5x}{-5} &= \frac{-10}{-5} \\
 x &= 2
 \end{aligned}$$

As you can observe, in each of the four choices, we arrived at the same correct answer.

Which of the four ways do you prefer? If they all arrived at the correct answer, how will you decide which one you to do first? To help answer these questions, it may be helpful to attempt drawing the balance diagrams that would accompany each solution method.

### Problem 2.2

Solve for  $n$ . Check your solution.

$$8n + 6 = 3n + 21$$

### Problem 2.3

Solve for  $t$ . Check your solution.

$$7t + 1 = t + 19$$

 **Problem 2.4**Solve for  $n$ . Check your solution.

$$5n = 45$$

 **Problem 2.5**Solve for  $x$ . Check your solution.

$$3x + 5 = 14$$

 **Problem 2.6**Solve for  $y$ . Check your solution.

$$23 = 6y + 11$$



## The Distributive Property

Often equations involve parentheses that you must understand to solve the problem. For example,

$$2(3x + 4) = 11x + 3$$

What does  $2(3x + 4)$  mean? One way to think about this is that you have 2 groups of  $3x + 4$ . Let us return to the diagrams we were drawing in the previous section. In this section we learned we could represent  $3x + 4$  as follows

$$3x + 4 \quad \begin{array}{|c|} \hline x \\ \hline \end{array} \begin{array}{|c|} \hline x \\ \hline \end{array} \begin{array}{|c|} \hline x \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

If we have 2 groups of  $3x + 4$ , it would thus look like this

$$2(3x + 4) \quad \begin{array}{|c|} \hline x \\ \hline \end{array} \begin{array}{|c|} \hline x \\ \hline \end{array} \begin{array}{|c|} \hline x \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \\ \begin{array}{|c|} \hline x \\ \hline \end{array} \begin{array}{|c|} \hline x \\ \hline \end{array} \begin{array}{|c|} \hline x \\ \hline \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \quad 6x + 8$$

The outcome is that there are 2 times the number of  $x$ 's and 2 times the number of 1 blocks. This is an example of the Distributive Property.

### The Distributive Property

$$a(b + c) = ab + ac$$

In words, this property says that to multiply a quantity by multiple quantities added (or subtracted!) inside parentheses, multiply each piece inside the parentheses by the piece outside the parentheses.

A common diagram for this property is as follows:

$$a(b + c) = ab + ac$$

**Problem 2.7**

Solve for  $x$ . Check your solution.

$$2(3x + 4) = 11x + 3$$

Our first goal when solving equations is often to simplify each side. In this case, this means applying the Distributive Property. We then proceed as usual.

$2(3x + 4) = 11x + 3$	This is the original problem.
$2(3x) + 2(4) = 11x + 3$	Apply the Distributive Property.
$6x + 8 = 11x + 3$	Simplify.
$6x - 6x + 8 = 11x - 6x + 3$	Subtract $6x$ from both sides.
$8 = 5x + 3$	Simplify
$8 - 3 = 5x + 3 - 3$	Subtract 3 from both sides from both sides.
$5 = 5x$	Simplify.
$\frac{5}{5} = \frac{5x}{5}$	Divide both sides by 5.
$1 = x$	Simplify.
$x = 1$	Reverse the order of the equality.

**Check:**

To check, we will still substitute our solution into the *original equation*. If we are correct, then we will get a true statement.

$$\begin{aligned}
 2(3x + 4) &= 11x + 3 \\
 2(3(1) + 4) &\stackrel{?}{=} 11(1) + 3 \\
 2(3 + 4) &\stackrel{?}{=} 11 + 3 \\
 2(7) &\stackrel{?}{=} 14 \\
 14 &= 14 \quad \checkmark
 \end{aligned}$$



Recall that the Order of Operations requires you to first simplify inside the parentheses as we did while checking the last problem.

**Problem 2.8**Solve for  $n$ . Check your solution.

$$3(4 + n) = 4n + 10$$

**Problem 2.9**Solve for  $x$ . Check your solution.

$$3(x + 7) = 8(x + 2)$$

**Problem 2.10**Solve for  $x$ . Check your solution.

$$7 + 4(x + 2) = 3(2x + 1) + 2$$

### GPS 3: Adding Signed Numbers

In the previous sections, you were likely able to avoid doing any calculations with negative numbers. This will not be the case any longer. We must revisit signed numbers and arithmetic.

Much of the confusion with addition and subtraction of signed numbers stems from the symbols  $+$  and  $-$  and their multiple uses with different, although related, meanings.

The  $+$  symbol can mean either the *operation* addition or that the *sign of the following number* is positive.

**This is what we write**

$$5 + 3$$

**This is what we mean**

$$(+5) + (+3)$$

The 5 and 3 are both positive numbers, but we do not ordinarily write the positive symbol since by customary notation, numbers without a sign are positive. The  $+$  between the numbers indicates that we are adding them. Finally, we complicate the notation by putting parentheses around the numbers to group the sign with the number.

The  $-$  symbol can mean either the *operation* subtraction or that the *sign of the following number* is negative. In typing, the symbol is sometimes longer  $-$  or shorter  $-$  depending on the context, but you cannot depend on this distinction to determine meaning.

**This is what we write**

$$6 - 2$$

**This is what we mean**

$$(+6) - (+2)$$

In this example, the 6 and 2 are again both positive numbers. The  $-$  symbol between them means subtraction. We will see shortly that this  $-$  symbol can also have an alternative meaning.

Here is a sequence of examples, their meaning, and results.

#### Problem 3.1

This is what we write	This is what we mean	Sign of First Number	Sign of Second Number	Operation	Result
$5 + 3$	$(+5) + (+3)$	Positive	Positive	Addition	+8
$6 - 2$	$(+6) - (+2)$	Positive	Positive	Subtraction	+4
$-7 + 4$	$(-7) + (+4)$	Negative	Positive	Addition	-3
$-1 - 2$	$(-1) - (+2)$	Negative	Positive	Subtraction	-3
$-5 + (-2)$	$(-5) + (-2)$	Negative	Negative	Addition	-7
$-6 - (-9)$	$(-6) - (-9)$	Negative	Negative	Subtraction	+3
$4 + (-6)$	$(+4) + (-6)$	Positive	Negative	Addition	-2
$3 - (-7)$	$(+3) - (-7)$	Positive	Negative	Subtraction	+10

### Problem 3.2

Fill in the table.

This is what we write	This is what we mean	Sign of First Number	Sign of Second Number	Operation	Result
$8 + 4$					
$-3 + 8$					
$7 + (-4)$					
$-3 + (-8)$					
$2 - 7$					
$-2 - 6$					
$1 - (-8)$					
$-8 - (-2)$					

Almost certainly, the last four rows were the most problematic to determine the result. This is because subtraction combined with signed numbers offers an opportunity to showcase confused ideas, partially recalled rules and wrong calculations all at once. We will show you how to think differently about subtraction soon, but first we will look more closely at addition.

### Addition of Two Signed Numbers

There are many ways to think about addition, and ideally you have many, flexible ways of understanding addition. In the problems below, we will use the concept of addition as combining quantities and finding the total.

For these problems, we can imagine a positive number represents money in our pocket (or bank account), and a negative number represents money we owe someone (a debt or a bill). The first number in the addition represents how much money we have at the beginning. The second number can represent what we are combining—it may be more money (a positive number) or a new debt. Let's see how this can help us think about addition of signed numbers.

#### Problem 3.3

$$3 + 2$$

We have already seen that this really means

$$(+3) + (+2)$$

The scenario we can imagine goes as follows:

- 1) I have 3 dollars in my pocket.
- 2) I then earn 2 dollars more.
- 3) Altogether, I have 5 dollars in my pocket.

This leads us to the answer.

$$3 + 2 = 5$$

#### Problem 3.4

$$-5 + 8$$

This means

$$(-5) + (+8)$$

Here the scenario changes:

- 1) I start out in debt 5 dollars.
- 2) I then earn 8 dollars.
- 3) After paying my 5 dollar debt with my 8 dollars, I have 3 dollars in my pocket.

The answer is

$$-5 + 8 = 3$$

**Problem 3.5**

$$-6 + 4$$

This means

$$(-6) + (+4)$$

We can think of this as

- 1) I start out in debt 6 dollars.
- 2) I then earn 4 dollars.
- 3) After paying the 4 dollars toward my 6 dollar debt, I still owe 2 dollars. ☹

The answer:

$$-6 + 4 = -2$$

**Problem 3.6**

$$-9 + (-7)$$

This means

$$(-9) + (-7)$$

This is the worst scenario:

- 1) I start out in debt 9 dollars.
- 2) I go shopping and get 7 dollars more debt.
- 3) After the new 7 dollar debt is combined with my 9 dollar debt, I am a total of 16 dollars in debt. ☹

The answer:

$$-9 + (-7) = -16$$

Complete the problems below by following a similar way of thinking. Write out the scenario you can use to determine the answer.

 **Problem 3.7**

$$6 + 7$$

This means

A helpful scenario:

- 1)
- 2)
- 3)

The answer is

 **Problem 3.8**

$$-5 + (-2)$$

This means

A helpful scenario:

- 1)
- 2)
- 3)

The answer is

**Problem 3.9**

$-9 + 4$

This means

A helpful scenario:

- 1)
- 2)
- 3)

The answer is

**Problem 3.10**

$7 + (-3)$

This means

A helpful scenario:

- 1)
- 2)
- 3)

The answer is

Complete the problems below without explicitly writing out all the details.

**Problem 3.11**  
 $-9 + (-3)$

**Problem 3.12**  
 $-6 + 2$

**Problem 3.13**  
 $4 + (-4)$

**Problem 3.14**  
 $-5 + 9$

**Problem 3.15**  
 $-8 + (-6)$

**Problem 3.16**  
 $-5 + (-7)$

**Problem 3.17**  
 $5 + (-9)$

**Problem 3.18**  
 $7 + (-5)$

**Problem 3.19**  
 $-1 + 6$

**Problem 3.20**  
 $-8 + 3$

**Problem 3.21**  
 $-6 + (-5)$

**Problem 3.22**  
 $10 + (-7)$

**Problem 3.23**  
 $-6 + (-8)$

**Problem 3.24**  
 $-8 + 12$

**Problem 3.25**  
 $-6 + 6$

**Problem 3.26**  
 $-18 + (-15)$

### Addition of Multiple Terms

Often there are more than two terms that we must add together. We still add only two numbers at a time.

#### Problem 3.27

$$-3 + (-5) + 9 + (-12)$$

When there are multiple terms to add, it is important to organize yourself. Here, I will work left to right and ensure I do only one operation at a time. At each step, as annoying as it is continually write the same thing over, is best that you bring everything down and rewrite the problem.

$\underline{-3 + (-5)} + 9 + (-12)$	We start with the first computation on the left
$-8 + 9 + (-12)$	We replace the underlined part with the result of the computation, and carefully bring everything else down.
$\underline{-8 + 9} + (-12)$	We again complete the first computation on the left.
$1 + (-12)$	We replace the underlined part with the result of the computation, and carefully bring everything else down.
$\underline{1 + (-12)}$	There is only one computation remaining.
$-11$	This is the answer.

■



Students often make mistakes due to some bad habits.

1. **Don't perform more than one operation per step.** While this is possible to do correctly, much care must be taken. It is best to start out more slowly, and get the correct answer. You probably hate it when your instructor "skips steps," so don't do it yourself!
2. **Bring down the entire problem after each step.** It is extremely easy to lose track of the problem, and make many mistakes because you don't take care to rewrite at each step. In this book, there is ample room to write each step. Not using the space is wasting the paper.

#### Problem 3.28

$$-8 + (-6) + 15$$

#### Problem 3.29

$$-4 + 10 + (-7)$$



 **Problem 3.30**

$$-6 + (-3) + 17$$

 **Problem 3.31**

$$-12 + 7 + (-2)$$

 **Problem 3.32**

$$6 + (-5) + (-4) + (-3)$$

 **Problem 3.33**

$$9 + (-12) + (-8) + 11$$

### GPS 4: Subtracting Signed Numbers

In the last section we increased our understanding of addition of signed numbers. Using what we learned about addition, we can also perform subtraction. It all comes down to this one fact.



**Subtraction**

To subtract  $b$  from  $a$ , add the *opposite* of  $b$  to  $a$ .

$$a - b = a + (-b)$$

This is a different interpretation of subtraction than you might customarily think about. Instead of thinking about subtraction as “taking away” or “removing”, we think of it as adding the opposite. We need to know how to find the opposite. Simply put, the opposite of a positive number is the negative number of the same magnitude, and the opposite of a negative number is the positive number with the same magnitude.

**Problem 4.1**

- |                                |                                |
|--------------------------------|--------------------------------|
| 1) The opposite of 3 is $-3$ . | 2) The opposite of 2 is $-2$ . |
| 3) The opposite of $-5$ is 5.  | 4) The opposite of $-7$ is 7.  |

**Problem 4.2**


- |                                 |                                    |
|---------------------------------|------------------------------------|
| 1) The opposite of 50 is _____. | 2) The opposite of $-3$ is _____.  |
| 3) The opposite of 12 is _____. | 4) The opposite of $-16$ is _____. |


**Subtraction of Two Signed Numbers**


Now we will apply these ideas to subtraction.


	<b>Problem 4.3</b>	<b>Problem 4.4</b>
	$4 - 9$	$6 - (-8)$
Rewrite the problem without changing it.	$(+4) - (+9)$	$(+6) - (-8)$
Do both of these steps at once:		
1) Change the <i>operation</i> to addition.	$(+4) + (-9)$	$(+6) + (+8)$
2) Change the <i>second number</i> to its <i>opposite</i> .		
Calculate the answer as we learned in the previous section.	$-5$	$14$
	<b>Problem 4.5</b>	<b>Problem 4.6</b>
	$-24 - 18$	$-15 - (-9)$
Rewrite the problem without changing it.	$(-24) - (+18)$	$(-15) - (-9)$
Do both of these steps at once:		
1) Change the <i>operation</i> to addition.	$(-24) + (-18)$	$(-15) + (+9)$
2) Change the <i>second number</i> to its <i>opposite</i> .		
Calculate the answer as we learned in the previous section.	$-42$	$-6$





 **Problem 4.7**  
 $-19 - (-3)$


 **Problem 4.8**  
 $-13 - 12$


 **Problem 4.9**  
 $4 - (-4)$


 **Problem 4.10**  
 $4 - 4$


 **Problem 4.11**  
 $-12 - (-16)$


 **Problem 4.12**  
 $-15 - (-7)$


 **Problem 4.13**  
 $8 - (-7)$


 **Problem 4.14**  
 $12 - (-9)$


 **Problem 4.15**  
 $-10 - 6$


 **Problem 4.16**  
 $-6 - 3$


 **Problem 4.17**  
 $-4 - (-8)$

 **Problem 4.18**  
 $-6 - (-7)$

 **Problem 4.19**  
 $-7 - (-8)$

 **Problem 4.20**  
 $-1 - 12$

 **Problem 4.21**  
 $-6 - (-6)$

 **Problem 4.22**  
 $-22 - (-15)$

### Addition and Subtraction of Multiple Terms

Just as in the previous section, we must often add and subtract multiple terms at once.



The ***Order of Operations*** dictates that when performing more than one addition and subtraction, you should complete them from *left to right*. While it is possible to disobey this strict order, it requires more care than many students have mastered. For now, stick to this requirement!

**Problem 4.23**

$$3 - 5 - (-8) + 6 - 20$$

We will work left to right and perform only one operation at a time.

$\underline{3 - 5} - (-8) + 6 - 20$	We start with the first computation on the left. While you may not explicitly write this, recall that $3 - 5 = 3 + (-5)$
$-2 - (-8) + 6 - 20$	We replace the underlined part with the result of the computation, and carefully bring everything else down.
$\underline{-2 - (-8)} + 6 - 20$	In this case, you would be well advised to make the substitution of subtraction for addition, before performing any computations.
$\underline{-2 + (+8)} + 6 - 20$	We did not perform any calculations, but we rewrote the expression to make it easier to complete the problem correctly.
$6 + 6 - 20$	We performed the calculation and brought everything else down.
$\underline{6 + 6} - 20$	We now work left to right and perform the next operation.
$12 - 20$	We replace the underlined part with the result of the computation, and carefully bring everything else down.
$\underline{12 - 20}$	There is only one computation left. Again, you may not explicitly write this, but recall that $12 - 20 = 12 + (-20)$
$-8$	This is the answer.

■

 **Problem 4.24**

$$6 + (-5) - 2$$

 **Problem 4.25**

$$-16 + 13 - (-3)$$

 **Problem 4.26**

$$16 + (-24) - (-17) - 2$$

 **Problem 4.27**

$$-2 + (-10) - (-14)$$

 **Problem 4.28**

$$9 - (-12) + 17 + (-9)$$

 **Problem 4.29**

$$-15 + 13 - 11 - (-4)$$

## GPS 5: Multiplying and Dividing Signed Numbers

After learning about addition and subtraction, it should be obvious that multiplication and division are next.

### Multiplication and Division of Two Signed Numbers

Unlike addition and subtraction of signed numbers, multiplication and division of signed numbers only differs from working with only positive numbers when determining the sign of the answer, rather than the magnitude.

### Steps for Performing Multiplication and Division of Signed Numbers

- 1) Ignore all signs and perform the operation with the positive numbers.
- 2) Determine the correct sign of the answer.

While the steps above are simple enough to state, the second step requires further explanation. For this step, you may find it helpful to think of a negative sign as meaning “take the opposite of the number.” If there are multiple negative signs, then you must take the opposite multiple times.

### Problem 5.1

Problem	Explanation
$5 \cdot 3 = 15$	There are no negatives so the answer is positive.
$(-6) \cdot 4 = -24$	$6 \times 4 = 24$ . There is one negative number. This means we must take the opposite of 24 once to find the final answer. The opposite of 24 is $-24$ .
$7 \cdot (-8) = -56$	$7 \times 8 = 56$ . There is one negative number. We take the opposite of 56 one time and the final answer is $-56$ .
$(-6)(-8) = 48$	$6 \times 8 = 48$ . There are two negative numbers. We must take the opposite of 48 two times. The opposite of 48 is $-48$ , but we must take the opposite again. The opposite of $-48$ is 48.
$\frac{18}{9} = 2$	There are no negatives so the answer is positive.
$\frac{-36}{12} = -3$	$36 \div 12 = 3$ . There is one negative number. We must take the opposite of 3 one time and arrive at $-3$ .
$\frac{63}{-9} = -7$	$63 \div 9 = 7$ . There is one negative. We must take the opposite of 7 once to get $-7$ .
$\frac{-40}{-5} = 8$	$40 \div 5 = 8$ . There are two negative numbers. We take the opposite two times. The opposite of 8 is $-8$ , and the opposite of $-8$ is 8.

You may like to take a few moments to look for patterns about the sign of the the answer when the two numbers have the same sign, and the sign of the answer when the two numbers have the opposite sign.

 **Problem 5.2**

Fill in the answers in the first column. In the second column, use complete sentences to explain the sign of the answer.


Problem	Explanation
$7 \cdot 6 =$	
$(-8) \cdot 9 =$	
$12 \cdot (-4) =$	
$(-16)(-2) =$	


Problem	Explanation
$\frac{45}{5} =$	
$\frac{72}{-9} =$	
$\frac{-24}{3} =$	
$\frac{-30}{-6} =$	





Complete the next problems without writing out your explanation, but keep repeating the explanation in your mind as you do each problem.


 **Problem 5.3**  
 $6(-9)$


 **Problem 5.4**  
$$\frac{-81}{-9}$$


 **Problem 5.5**  
 $(-5)(-6)$

 **Problem 5.6**  
$$\frac{63}{-7}$$

 **Problem 5.7**  
 $8 \cdot (-4)$

 **Problem 5.8**  
$$\frac{-48}{8}$$

 **Problem 5.9**  
 $(12)(-7)$

 **Problem 5.10**  
$$\frac{-120}{-10}$$

### Multiple Operations of Signed Numbers

Often we must perform combinations of multiple operations rather than just do a single multiplication or a single division. You have already done this a few times when checking solutions in previous problems. We will practice a few more here.



#### Order of Operations for Addition, Subtraction, Multiplication and Division<sup>†</sup>

- 1) First, complete all operations inside any parentheses. You should continue to work inside the parentheses until there is only one number remaining.
- 2) Second, complete multiplication and division. Perform all multiplications and divisions from left to right.
- 3) Finally, complete addition and subtraction. Perform all additions and subtractions from left to right.

<sup>†</sup> Note that we did not include exponents in this order since we will not encounter exponents until later.

**Problem 5.11**

$$-3(2(-4) + 6) - 7 + 9$$

$-3(2(-4) + 6) - 7 + 9$	We start working with the inside parentheses. Our first goal will be to completely simplify $2(-4) + 6$ but we will do this one operation at a time. We will also carefully bring down the entire problem at each step.
$-3(\underline{2(-4) + 6}) - 7 + 9$	We first complete the multiplication.
$-3(-\underline{8} + 6) - 7 + 9$	We replace the underlined part with the result of the computation, and carefully bring everything else down.
$-3(\underline{-8 + 6}) - 7 + 9$	Next we perform the addition inside the parentheses.
$-3(\underline{-2}) - 7 + 9$	We replace the underlined part with the result of the computation, and carefully bring everything else down.
$\underline{-3(-2)} - 7 + 9$	Inside the parentheses is only a single number with no operations. We now look outside the parentheses. We complete the multiplication first.
$6 - 7 + 9$	We replace the underlined part with the result of the computation, and carefully bring everything else down.
$\underline{6 - 7} + 9$	At this point we perform the subtraction because it is first when reading from left to right. Recall that $6 - 7 = 6 + (-7)$
$-1 + 9$	We replace the underlined part with the result of the computation, and carefully bring everything else down.
8	There is only one operation remaining, and we perform it now. This is the answer.

 **Problem 5.12**

$$-3(-2) - 5$$

 **Problem 5.13**

$$-6(4(-8) - 9)$$

 **Problem 5.14**

$$\frac{-2(4) - 6}{-7}$$

 **Problem 5.15**

$$2 \cdot (-9) + 8 \cdot (-3) - (-6) \cdot (-3)$$

 **Problem 5.16**

$$-3(-6) - 7(2(-6) - 1)$$

 **Problem 5.17**

$$45 - 3(4(-2) + 1)$$

## GPS 6: Solving Linear Equations with Signed Numbers

This section does not contain anything new. However, you will not be able to avoid performing arithmetic with negative numbers in this section.

### Problem 6.1

Solve for  $x$ . Check your solution.

$$-3x + 4 = 2x + 19$$

We proceed as in the previous sections.

$-3x + 4 = 2x + 19$	This is the original problem.
$-3x + 3x + 4 = 2x + 3x + 19$	Add $3x$ to both sides.
$4 = 5x + 19$	Simplify.
$4 - 19 = 5x + 19 - 19$	Subtract 19 from both sides from both sides.
$-15 = 5x$	Simplify.
$\frac{-15}{5} = \frac{5x}{5}$	Divide both sides by 5.
$-3 = x$	Simplify.
$x = -3$	Reverse the order of the equality.

### Check:

To check, we will substitute our solution into the *original equation*. If we are correct, then we will get a true statement.

$$\begin{aligned}
 -3x + 4 &= 2x + 19 \\
 -3(-3) + 4 &\stackrel{?}{=} 2(-3) + 19 \\
 9 + 4 &\stackrel{?}{=} -6 + 19 \\
 13 &= 13 \quad \checkmark
 \end{aligned}$$

Since we arrived at the true statement  $13 = 13$ , we know that our answer is correct.

■

**Problem 6.2**Solve for  $x$ . Check your solution.

$$2x + 5 = -4x - 7$$

**Problem 6.3**Solve for  $n$ . Check your solution.

$$-3n = 21$$

**Problem 6.4**Solve for  $w$ . Check your solution.

$$7 = 35 - 4w$$

**Problem 6.5**Solve for  $q$ . Check your solution.

$$-3q - 5 = 19 + 3q$$

 **Problem 6.6**Solve for  $p$ . Check your solution.

$$-p + 5 = 37 - 9p$$

 **Problem 6.7**Solve for  $x$ . Check your solution.

$$-3(2x + 5) = -5 + 4x$$

 **Problem 6.8**Solve for  $x$ . Check your solution.

$$-4(1 - x) = 2(3x - 2) + 2$$

 **Problem 6.9**Solve for  $n$ . Check your solution.

$$-3n - 7(2n - 1) = 45 - 3(4n + 1)$$

**GPS 7: Translating Words into Expressions and Equations**

There are sometimes situations when we must translate words into equations before we can solve them. Take a few minutes to write down all the words you can think of that mean the same as the symbol to the left.

Symbol	Words
+	
-	
×	
÷	
=	
$x$ Variable	



**Problem 7.1**

If  $x$  represents a number, write an equation that is a correct translation of the sentence. Then find the number.

**The sum of a number and 5 is 13.**

We see several key words that indicate how to write the equation.

<b>The sum of</b>	<b>a number</b>	<b>and 5</b>	<b>is</b>	<b>13.</b>
+	$x$	5	=	13

We must put these symbols together in the proper order. Sometimes the order of the words is not the same as the order of the symbols. Here the word *sum* comes *before* the two symbols it should be placed *between*. We group the words into larger phrases to help us put the symbols together in the correct order.

<b>The sum of a number and 5</b>	<b>is</b>	<b>13.</b>
$x + 5$	=	13

The equation can be written down.

$$x + 5 = 13$$

Let's solve our equation.

$$\begin{aligned} x + 5 &= 13 \\ x + 5 - 5 &= 13 - 5 \\ x &= 8 \end{aligned}$$

**Check:**

We can check this answer by substituting the answer into the original sentence, and see if it makes sense.

**The sum of 8 and 5 is 13**

This sentence is correct, and so our answer is correct.

**Problem 7.2**

If  $x$  represents a number, write an equation that is a correct translation of the sentence. Then find the number.

**The sum of twice a number and 4 is 20.**

**Problem 7.3**

If  $n$  represents a number, write an equation that is a correct translation of the sentence. Then find the number.

**The product of a number and 7 is equal to the sum of the number and 12.**

Some words are trickier to get the order correct than others. This is particularly true with subtraction.

**Problem 7.4**

If  $y$  represents a number, write an equation that is a correct translation of the sentence. Then find the number.

**The product of a number and  $-5$  is equal to  $14$  less than twice the number.**

We start by breaking the sentence into smaller pieces and determine their corresponding symbols.

The product of	a number	and $-5$	is equal to	$14$	less than	twice the number
$\times$	$y$	$-5$	$=$	$14$	$-$	$2y$

When we recombine the pieces to put the operations in their proper order, we pay particular attention to the phrase “less than” which again reverses the order that the numbers appear.

The product of a number and $-5$	is equal to	$14$ less than twice the number
$-5y$	$=$	$2y - 14$

We now combine this into an equation, and solve for  $y$ .

$$\begin{aligned}
 -5y &= 2y - 14 \\
 -5y - 2y &= 2y - 2y - 14 \\
 -7y &= -14 \\
 \frac{-7y}{-7} &= \frac{-14}{-7} \\
 y &= 2
 \end{aligned}$$

■



Students often confuse the order with certain phrases that mean subtraction. Be careful with all subtractions and make sure you have the order correct. It is often helpful to substitute some actual numbers into the words to help.

**For example:**

$3$  less than  $5$  is  $2$  translates to  $5 - 3 = 2$ . You can recognize immediately that  $3 - 5 \neq 2$ .

**✎ Problem 7.5**

If  $x$  represents a number, write an equation that is a correct translation of the sentence. Then find the number.

**A number decreased by 6 is equal to 8.**

**✎ Problem 7.6**

If  $y$  represents a number, write an equation that is a correct translation of the sentence. Then find the number.

**8 subtracted from 5 times a number is  $-33$ .**

**✎ Problem 7.7**

If  $x$  represents a number, write an equation that is a correct translation of the sentence. Then find the number.

**12 less than triple a number is equal to the difference of double the number and 14.**

**✎ Problem 7.8**

If  $n$  represents a number, write an equation that is a correct translation of the sentence. Then find the number.

**The quotient of a number by 10 is three.**

## GPS 8: Solving Linear Inequalities, Part 1

Until this point we have worked only with equations where both sides are equal. Sometimes both sides are not equal, and one side is bigger or smaller than the other side. For these situations we use the idea of inequalities. We have new symbols to learn here as well.

$>$ Greater than  $4 > 3$ 4 is greater than 3	$\geq$ Greater than or equal to  $6 \geq 2$ and $4 \geq 4$ 6 is greater than or equal to 2.
$<$ Less than  $2 < 3$ 2 is less than 3	$\leq$ Less than or equal to  $1 \leq 3$ and $2 \leq 2$ 1 is less than or equal to 3

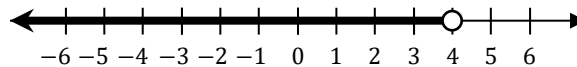
None of the examples above have any variables. Here are some examples of inequalities that include variables as well as how we can graph them.

### Problem 8.1

Graph the inequality.

$$x < 4$$

Here is the graph. We have shaded the part of the number line where the inequality is true.



Here is how we determined how to draw the graph.

- 1) Determine the endpoint. In the first few problems, this is easy to see. We see the  $x = 4$  is the endpoint. There is an open hollow (unshaded) circle at the endpoint  $x = 4$  because when we substitute  $x = 4$  into the inequality  $x < 4$ , we get  $4 < 4$  which is *not* true.
- 2) We shaded the left of 4. Here's how we figured this out.
  - a. Pick **any** point to the left of 4. Let's pick  $x = 1$ .
  - b. Substitute the point into the inequality  $x < 4$ . We get  $1 < 4$ . This is a true statement, and so we shaded to the left.
- 3) We did *not* shade the right of 4.
  - a. Pick **any** point to the right of 4. Let's pick  $x = 5$ .
  - b. Substitute the point into the inequality  $x < 4$ . We get  $5 < 4$ . This is **not** a true statement, and so we did **not** shade to the right.

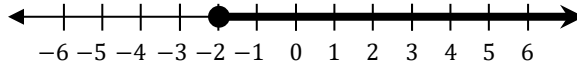
■

**Problem 8.2**

Graph the inequality.

$$-2 \leq x$$

Here is the graph.



Here is how we determined how to draw the graph. Again, our goal is to shade the points on the numberline where the inequality is true.

- 1) There is a closed shaded circle at the endpoint  $x = -2$  because when we substitute  $x = -2$  into the the original inequality  $-2 \leq x$ , we get  $-2 \leq -2$  which is true.
- 2) We did *not* shade the left of  $-2$ .
  - a. Pick **any** point to the left of  $-2$ . Let's pick  $x = -5$ .
  - b. Substitute the point into the inequality  $-2 \leq x$ . We get  $-2 \leq -5$ . This is **not** a true statement, and so we did **not** shade to the left.
- 3) We shaded the right of  $-2$ .
  - a. Pick **any** point to the right of  $-2$ . Let's pick  $x = 0$ .
  - b. Substitute the point into the inequality  $-2 \leq x$ . We get  $-2 \leq 0$ . This is a true statement, and so we shaded to the right.

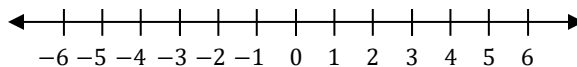
**Problem 8.3**

Graph the inequality. Then answer the questions below.

$$x > 1$$

- 1) The endpoint is  $x = \underline{\hspace{2cm}}$ . Will you use an open (hollow) or closed (shaded) circle at the endpoint? Why?
- 2) Will you shade to the left of the endpoint? Why? To answer this completely, do the following.
  - a. Pick any point to the left of the endpoint. What number did you pick?  $x = \underline{\hspace{2cm}}$
  - b. Substitute the number you picked into the original inequality. What is the inequality you get after substituting?  $\underline{\hspace{2cm}}$  Is this inequality true?  $\underline{\hspace{2cm}}$ . Will you shade to the left?  $\underline{\hspace{2cm}}$
- 3) Will you shade to the right of the endpoint? Why? Follow a similar procedure as in part 2). This time, write your answer in complete sentences explaining why you will or will not shade to the right.

- 4) Graph the inequality.

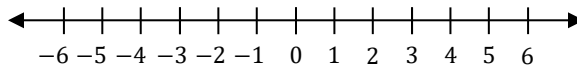


 **Problem 8.4**

Answer the questions below. Then graph the inequality.

$$-1 \geq x$$

- 1) Will you use an open (hollow) or closed (shaded) circle at the endpoint? Why? Answer in complete sentences.
- 2) Will you shade to the left of the endpoint? Why? Answer in complete sentences.
- 3) Will you shade to the right of the endpoint? Why? Answer in complete sentences.
- 4) Graph the inequality.



All of the inequalities we have graphed thus far had the variable  $x$  all by itself on one side. This made it very easy to determine the endpoint of our graphs. We now add an additional type of problem where the inequality is not so simplified.

**Problem 8.5**

Graph the solution to the inequality.

$$2x + 5 \leq 11$$

We will follow the same four step approach as in the previous problems, but we must do additional work in the first step to determine the endpoint of our graph.

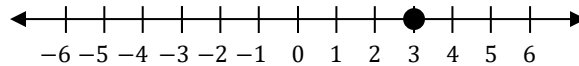
- 1) Determine the endpoint by changing the inequality symbol to an equals symbol and solve.

$$\begin{aligned} 2x + 5 &= 11 \\ 2x + 5 - 5 &= 11 - 5 \\ 2x &= 6 \\ \frac{2x}{2} &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

We see our endpoint is at  $x = 3$ . To determine if we use an open or closed circle at the endpoint, we substitute  $x = 3$  into our original inequality.

$$\begin{aligned} 2x + 5 &\leq 11 \\ 2(3) + 5 &\leq 11 \\ 6 + 5 &\leq 11 \\ 11 &\leq 11 \quad \checkmark \end{aligned}$$

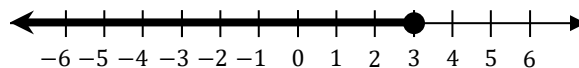
Since  $11 \leq 11$  is a true statement, we will shade the endpoint. At this point our graph looks like this.



- 2) Check to see if we shade to the left. As before, pick any point to the left of our endpoint. Let's use  $x = 0$ . Substitute  $x = 0$  into the original inequality.

$$\begin{aligned} 2x + 5 &\leq 11 \\ 2(0) + 5 &\leq 11 \\ 0 + 5 &\leq 11 \\ 5 &\leq 11 \quad \checkmark \end{aligned}$$

Since  $5 \leq 11$  is a true statement, we must shade to the left. At this point, our graph looks like this.

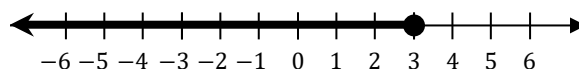


- 3) Finally, check to see if we shade to the right. Pick any point to the right of our endpoint. Let's use  $x = 4$ . Substitute  $x = 4$  into the original inequality.

$$\begin{aligned} 2x + 5 &\leq 11 \\ 2(4) + 5 &\leq 11 \\ 8 + 5 &\leq 11 \\ 13 &\leq 11 \quad \times \end{aligned}$$

Since  $13 \leq 11$  is not a true statement, we do not shade to the right.

- 4) Our final solution is shown below.



This is the graph of  $x \leq 3$ .

■

 **Problem 8.6**

Graph the solution to the inequality.

$$3x - 7 > 5$$

Follow the same four step approach as in the previous problems.

- 1) Determine the endpoint by changing the inequality symbol to an equals symbol and solve.

The endpoint is  $x =$ \_\_\_\_\_.

Do we use an open or closed circle at the endpoint? Why?

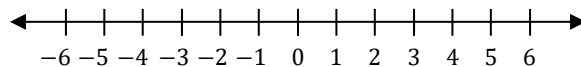
- 2) Check to see if we shade to the left by picking *any* point to the left of the endpoint and checking the original inequality.

Do we shade to the left? Why?

- 3) Check to see if we shade to the right by picking *any* point to the right of the endpoint and checking the original inequality.

Do we shade to the right? Why?

- 4) Finally, graph the solution, and fill in the blank below the graph.



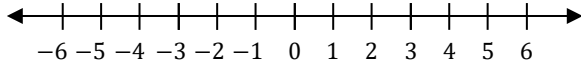
Write the solution as a simple inequality. \_\_\_\_\_



**Problem 8.7**

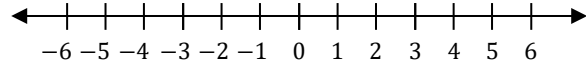
Graph the solution to the inequality.

$$-4 > 5x + 6$$

**Problem 8.8**

Graph the solution to the inequality.

$$6x + 2 \geq 20$$



## GPS 9: Solving Linear Inequalities, Part 2

In the previous section we learned to solve inequalities by first replacing the inequality with an equals symbol. While this has some advantages (most notably, this method works even with many more complicated equations you may learn in later mathematics), it is sometimes desirable to solve inequalities without switching out for an equals symbol. We can do this by making note of the following.



### Multiplying or Dividing by Negative Numbers Reverses the Inequality

When multiplying or dividing by a negative number, you must reverse the direction of the inequality.

#### For example:

$1 < 5$  but after multiplying both sides by  $-2$  we get  $-2 > -10$ .

### Problem 9.1

Solve the inequality.

$$-3x + 1 > 13$$

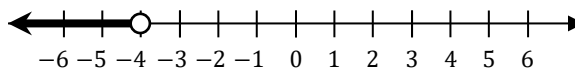
Unlike in the last section, we will solve this inequality without first replacing the  $>$  with an equals sign.

$$\begin{aligned} -3x + 1 &> 13 \\ -3x + 1 - 1 &> 13 - 1 \\ -3x &> 12 \\ \frac{-3x}{-3} &< \frac{12}{-3} \\ x &< -4 \end{aligned}$$

The first step is to subtract 1 from both sides. Addition and subtraction keeps the same inequality.

Our next step is to divide both sides by  $-3$ . We must also **reverse the order of the inequality**, because we are dividing by a negative number.

The solution to the inequality is  $x < -4$ . We have graphed the solution below.



You should still pick points and test that you have found the correct solution.

#### Check the endpoint.

Substitute  $x = -4$  into the original inequality.

$$\begin{aligned} -3x + 1 &> 13 \\ -3(-4) + 1 &> 13 \\ 12 + 1 &> 13 \\ 13 &> 13 \end{aligned}$$

Both sides are equal confirms that we have the correct endpoint. But this is not a true inequality so we leave an open circle at the endpoint.



#### Check the left.

Substitute  $x = -5$  into the original inequality.

$$\begin{aligned} -3x + 1 &> 13 \\ -3(-5) + 1 &> 13 \\ 15 + 1 &> 13 \\ 16 &> 13 \end{aligned}$$

This is a true inequality. This confirms that shading to the left was correct.

#### Check the right.

Substitute  $x = 0$  into the original inequality.

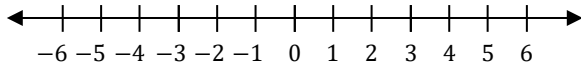
$$\begin{aligned} -3x + 1 &> 13 \\ -3(0) + 1 &> 13 \\ 0 + 1 &> 13 \\ 1 &> 13 \end{aligned}$$

This is NOT a true inequality. This confirms that not shading to the right was correct.

**Problem 9.2**

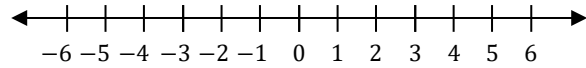
Solve the inequality. Then graph the solution

$$-5x + 8 \leq -12$$

**Problem 9.3**

Solve the inequality. Then graph the solution

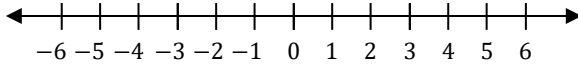
$$3x + 14 < 5x + 24$$



**Problem 9.4**

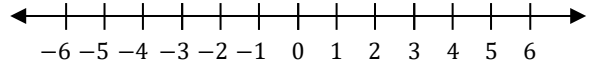
Solve the inequality. Then graph the solution

$$-5x - 9 \geq -4$$

**Problem 9.5**

Solve the inequality. Then graph the solution

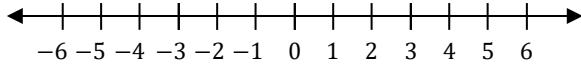
$$-2x < -14 + 5x$$



**Problem 9.6**

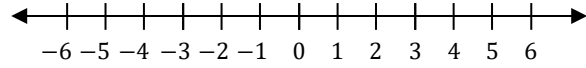
Solve the inequality. Then graph the solution

$$4x \leq 5x - 5$$

**Problem 9.7**

Solve the inequality. Then graph the solution

$$11x + 7 > 3x + 31$$



## GPS 10: Solving Linear Equations with Fractions

Oh no! Not fractions. Yes. Fractions! In this section, we will learn *how to make fractions go away*.

### Problem 10.1

Solve for  $x$ .

$$\frac{1}{3}x + \frac{5}{6} = \frac{17}{6}$$

While you can correctly solve this equation by performing the same sequence of steps you would if this problem did not have any fractions, the fraction arithmetic involved would make you much more likely to make mistakes.

Our first goal is to **eliminate** all fractions. In order to eliminate the fractions, we look at the denominators of all the fractions. In this case, the denominators are 3 and 6. We want to find a *common multiple* of 3 and 6. We see a common multiple of 3 and 6 is 6.

Multiply both sides of the equation by a common multiple of the denominators: 6.

$$6\left(\frac{1}{3}x + \frac{5}{6}\right) = 6\left(\frac{17}{6}\right)$$

We will distribute the 6 on the left.

$$6\left(\frac{1}{3}x\right) + 6\left(\frac{5}{6}\right) = 6\left(\frac{17}{6}\right)$$

Before multiplying the numbers, take a moment to recall our first goal is to eliminate the fractions. If you have correctly found a common multiple, you should be able to divide the common multiple by each denominator to get a whole number.

The diagram illustrates the simplification of the equation  $6\left(\frac{1}{3}x\right) + 6\left(\frac{5}{6}\right) = 6\left(\frac{17}{6}\right)$ . Two boxes above the equation show the simplification steps:  $\frac{6}{3} = 2$  and  $\frac{6}{6} = 1$ . Arrows point from these boxes to the corresponding terms in the equation, which are circled.

This will simplify our equation by eliminating all fractions.

$$\begin{aligned} 2(1x) + 1(5) &= 1(17) \\ 2x + 5 &= 17 \end{aligned}$$

This new equation is easier to solve.

$$\begin{aligned} 2x + 5 &= 17 \\ 2x + 5 - 5 &= 17 - 5 \\ 2x &= 12 \\ \frac{2x}{2} &= \frac{12}{2} \\ x &= 6 \end{aligned}$$

■

We will guide you through your first problem with fractions, and then you can work an additional problem alone.

 **Problem 10.2**

Solve for  $x$ .

$$\frac{1}{2}x = \frac{9}{14}x + \frac{1}{7}$$

Our first goal is to eliminate the denominators.

- 1) List the three denominators of fractions. \_\_\_\_\_, \_\_\_\_\_, and \_\_\_\_\_.
- 2) Find a common multiple of the denominators. Note that a common multiple *cannot* be *smaller* than the *largest* number. There are many ways to find a common multiple. You can always just multiply the denominators together to find a common multiple, but that's only a good idea if you are really stuck. A common multiple of the denominators is \_\_\_\_\_.
- 3) Multiply both sides of the equation by the common multiple. Fill in the box below with the common multiple. Note that we drew the parentheses around each side for you—you will need to draw your own in the next problem.

$$\boxed{\phantom{000}} \left( \frac{1}{2}x \right) = \boxed{\phantom{000}} \left( \frac{9}{14}x + \frac{1}{7} \right)$$

- 4) You must distribute the common multiple to every term on the right side of the equation.

$$\boxed{\phantom{000}} \left( \frac{1}{2}x \right) = \boxed{\phantom{000}} \left( \frac{9}{14}x \right) + \boxed{\phantom{000}} \left( \frac{1}{7} \right)$$

- 5) Simplify each term by dividing the term in the box by the denominator of the fraction. The new equation should not have any fractions. Solve the new equation without fractions below.

■

**Problem 10.3**Solve for  $x$ .

$$\frac{9}{5} = \frac{3}{4}x - \frac{6}{5}$$

**Problem 10.4**Solve for  $n$ .

$$\frac{4}{6} = \frac{n}{9}$$



**Problem 10.5**Solve for  $p$ .

$$\frac{p-2}{3} = 1 - \frac{p}{2}$$

**Problem 10.6**Solve for  $y$ .

$$\frac{3}{5}y + \frac{2}{3} = \frac{y}{2} + \frac{13}{15}$$

**GPS 11: More Linear Equations with Fractions**

Here, we will solve different types of linear equations with fractions, using the rules discussed in GPS 10.

** Problem 11.1**

Solve for  $x$ .

$$\frac{3x}{4} + \frac{13}{24} = \frac{5}{3}$$

** Problem 11.2**

Solve for  $n$ .

$$\frac{x-5}{3} - \frac{7}{12} = -\frac{23}{12}$$

**Problem 11.3**Solve for  $p$ .

$$\frac{p+1}{5} = \frac{p+2}{6}$$

**Problem 11.4**Solve for  $y$ .

$$\frac{2y}{5} + \frac{149}{60} = \frac{11}{4}$$

 **Problem 11.5**Solve for  $t$ .

$$\frac{t+1}{5} + \frac{7}{15} = \frac{22}{15}$$

 **Problem 11.6**Solve for  $y$ .

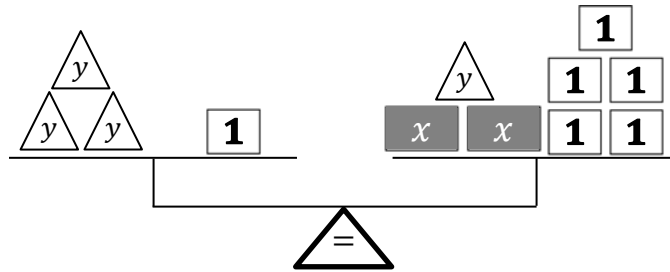
$$\frac{y+5}{4} = \frac{y+2}{5}$$



### GPS 12: Solving Literal Linear Equations

Sometimes equations have more than one variable. We will see soon that this is very common when working with lines, for example. While there are some important differences between an equation with one variable and an equation with more than one variable, you could still imagine these equations as a balance. Here’s an example.

$$3y + 1 = 2x + y + 5$$



The operations we used to solve equations until this point still work the same way.

#### Problem 12.1

Solve for  $y$ .

$$3y + 1 = 2x + y + 5$$

We follow the same strategies as before.

**First goal:** Get all the terms involving the variable we are solving for on one side of the equation, and every term without the variable on the opposite side.

$3y + 1 = 2x + y + 5$	This is our starting equation.
$3y - y + 1 = 2x + y - y + 5$	Subtract $y$ from each side.
$2y + 1 = 2x + 5$	Simplify.
$2y + 1 - 1 = 2x + 5 - 1$	Subtract 1 from both sides the equation.
$2y = 2x + 4$	Simplify.

We have accomplished our first goal. All the  $y$ 's are on the left and there are no  $y$ 's on the right.

**Second Goal:** Divide both sides by the coefficient of the variable we are solving.

$\frac{2y}{2} = \frac{2x + 4}{2}$	Divide both sides by 2.
-----------------------------------	-------------------------

$$y = \frac{2x + 4}{2}$$

$$y = \frac{2x}{2} + \frac{4}{2}$$

$$y = x + 2$$

Our final answer:  $y = x + 2$ .



Since there is a fraction on the right side, we can actually simplify this equation a bit more by breaking the fraction into two (since the denominators are the same) and reducing the fractions. You may not always have to do this step depending on the problem. Sometimes you'll want to leave the answer as one big fraction.

**Problem 12.2**Solve for  $y$ .

$$2x + 3y + 5 = 8x + 14$$

**Problem 12.3**Solve for  $w$ .

$$200 = 2w + 7\ell$$

**Problem 12.4**Solve for  $y$ .

$$2x + 3y - 5 = 16$$

**Problem 12.5**Solve for  $x$ .

$$z = 4x + y$$

**Problem 12.6**Solve for  $x$ .

$$z = 3x - 7y$$

**Problem 12.7**Solve for  $\ell$ .

$$P = 2\ell + 2w$$

---

Occasionally the problems will require us to divide by the variables and not just numbers.

**Problem 12.8**Solve for  $h$ .

$$A = \frac{1}{2}bh$$

We first note that there are fractions in this equation. As we learned in previous sections, our first step is to eliminate denominators by multiplying both sides by 2.

$$2A = 2\left(\frac{1}{2}bh\right)$$

$$2A = bh$$

To solve for  $h$ , we must divide by the coefficient of  $h$ . The coefficient in this case is the variable  $b$ .

$$\frac{2A}{b} = \frac{bh}{b}$$

$$\frac{2A}{b} = h$$

$$h = \frac{2A}{b}$$

That's it. We have successfully solved for  $h$ .





**Problem 12.9**Solve for  $r$ .

$$C = 2\pi r$$

 **Problem 12.10**Solve for  $m$ .

$$y = mx + b$$

 **Problem 12.11**Solve for  $C$ .

$$F = \frac{9}{5}C + 32$$

**Problem 12.12**Solve for  $B$ . Note that there is a lower case  $b$  and an upper case  $B$  and they are *not* the same variable.

$$A = \frac{1}{2}bh + \frac{1}{2}Bh$$

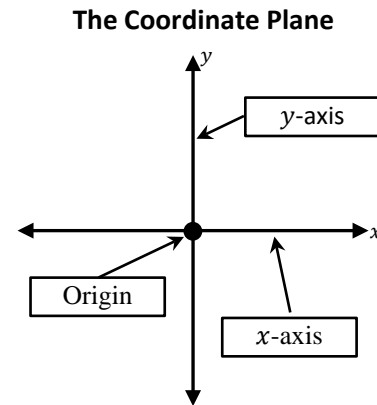
# The Coordinate Plane and Lines

## GPS 13: Introduction to the Coordinate Plane

In this section we will begin our study of graphing. We will recall how to graph points, and then begin a study of graphing lines.

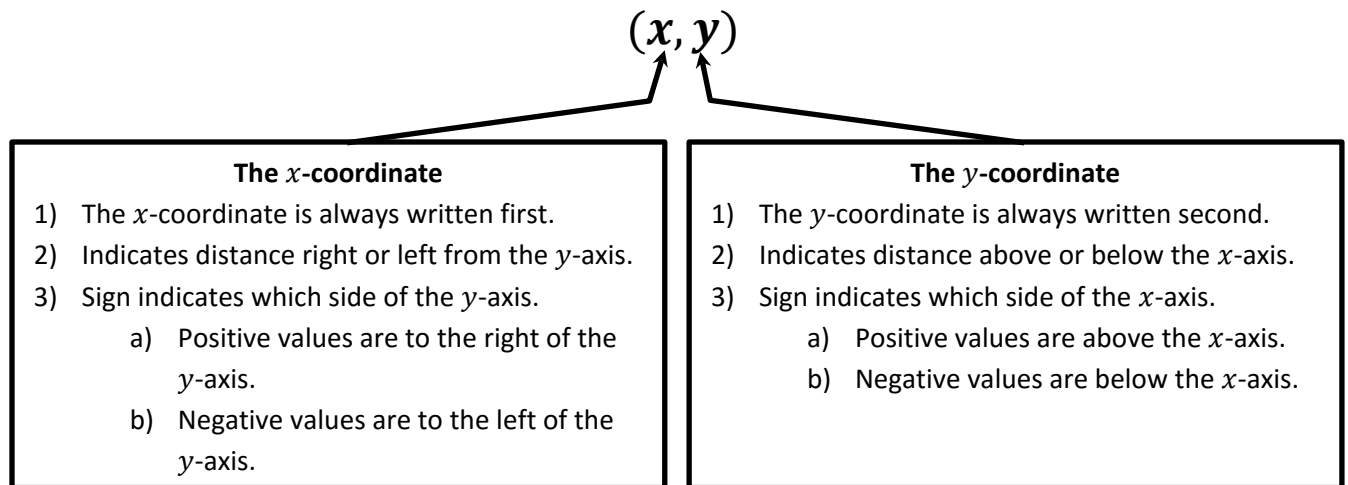
### Important Parts of the Coordinate Plane

1. There are two axes.
  - a. The  $x$ -axis is horizontal. The left side has negative values. The right side has positive values.
  - b. The  $y$ -axis is vertical. The bottom has negative values. The top has positive values.
2. The Origin is the point where the two axes intersect. The coordinates of the origin are  $(0,0)$ .



### Plotting Points

A point has two coordinates. We write the coordinates in parentheses with a comma between them. Sometimes the coordinates are called an ordered pair.

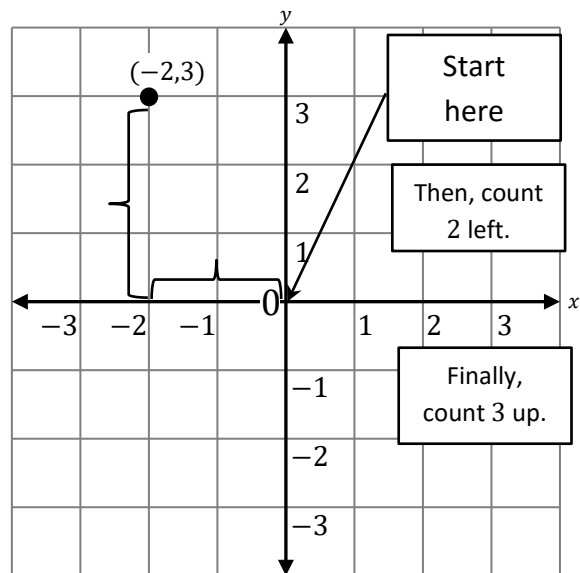


**Problem 13.1**

Plot the point  $(-2,3)$ . Label it with the coordinates.

Here's how we will plot this point.

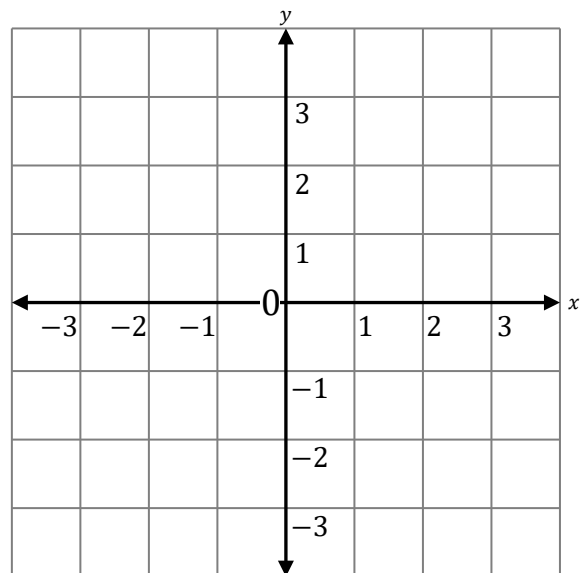
- 1) Do not draw anything yet, but start by placing your pencil at the origin.
- 2) Look at the  $x$ -coordinate of your point.
  - a) If the  $x$  is positive, count (but do not draw) that many spaces right.
  - b) If the  $x$  is negative count (but do not draw) that many spaces left.
- 3) Look at the  $y$ -coordinate of your point.
  - a) If the  $y$  is positive, count (but do not draw) that many spaces up.
  - b) If the  $y$  is negative count (but do not draw) that many spaces down.
- 4) Finally, draw your point at your final position. Write the coordinates of the point next to it.



 **Problem 13.2**

Plot and label the points.

- A)  $(3,4)$
- B)  $(3,-2)$
- C)  $(-1,-2)$
- D)  $(0,4)$
- E)  $(-3,0)$



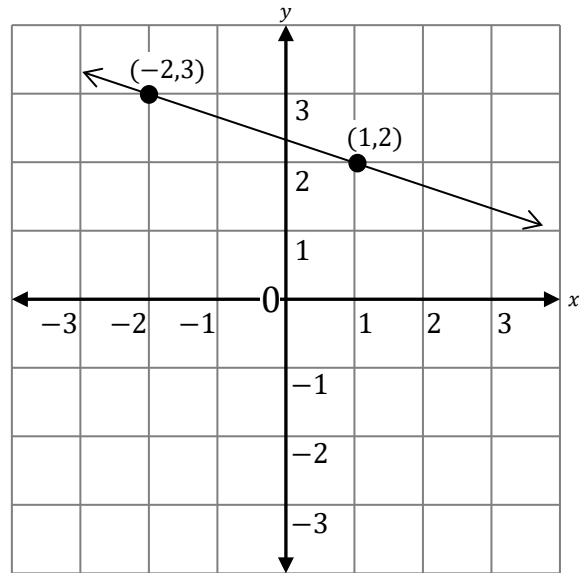
## Drawing Lines from Given Points

After learning to plot points, the next shape we will draw is a line.

### Problem 13.3

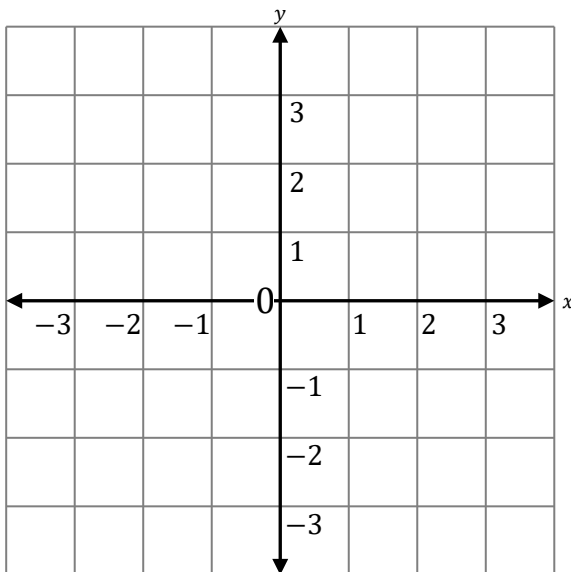
Plot the two points  $(-2,3)$  and  $(1,2)$ . Then draw the line containing the points.

- 1) Plot the first point.
- 2) Plot the second point.
- 3) Using a straight ruler, your ID card, credit card or some other stiff straight edge, connect the two points.
- 4) Make sure to note that the line does not start and stop at the two points. The line goes through the two points, and keeps going in both directions.



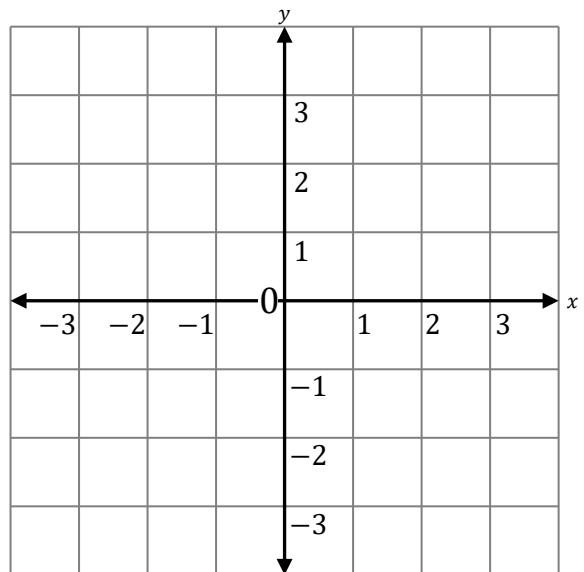
### Problem 13.4

Plot the two points  $(0,3)$  and  $(2,-1)$ . Then draw the line containing the points.



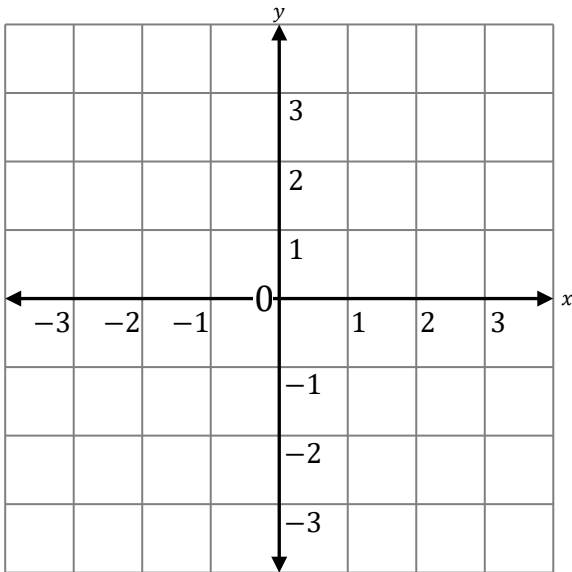
### Problem 13.5

Plot the two points  $(-2,-1)$  and  $(1,4)$ . Then draw the line containing the points.

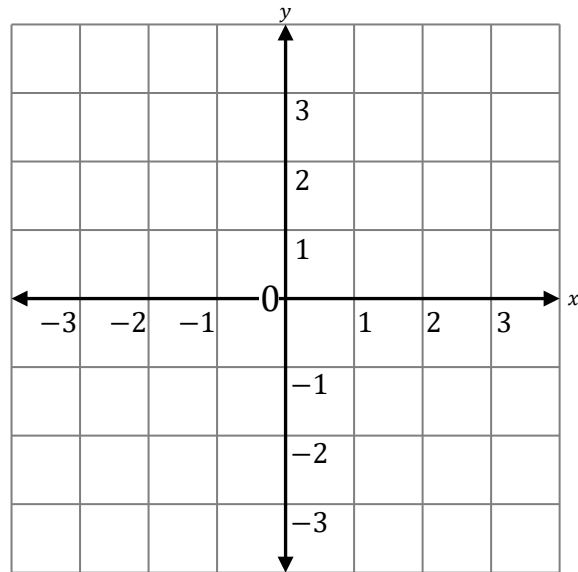


**Problem 13.6**

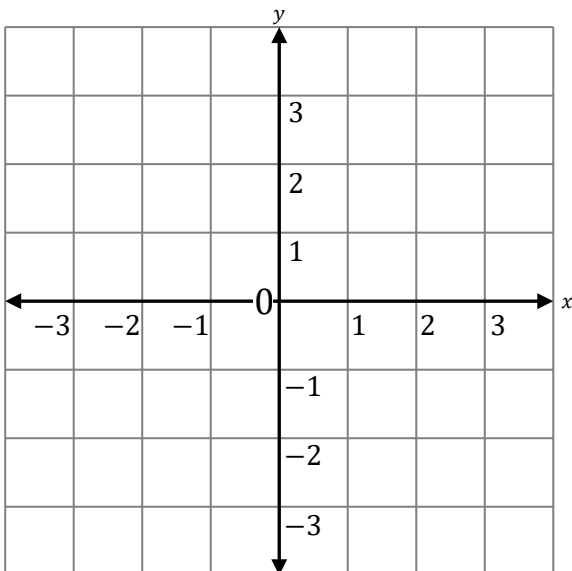
Plot the three points  $(-3, -2)$ ,  $(0, 1)$ , and  $(3, 4)$ . Then draw the line containing the points.

**Problem 13.7**

Plot the three points  $(0, 3)$ ,  $(-2, 3)$ , and  $(2, 3)$ . Then draw the line containing the points.

**Problem 13.8**

Plot the three points  $(-2, 3)$ ,  $(-1, 1)$ , and  $(3, 2)$ . Can you draw a line containing all three? Explain your answer. What happens when you try to connect any two of the points?



## Drawing Lines from Given Equation

In the last few problems, we started with points and graphed a line containing them. In the next few problems, we will begin an exploration of graphing lines by starting with an equation for the line. You observed that given any two different points, you can draw one line containing them.

### Problem 13.9

Fill in the missing coordinates for the line whose equation is given. Then graph the line.

Equation	Points with missing coordinates						
$-7x - 4y = 1$	<table border="1" style="display: inline-table; border-collapse: collapse;"> <thead> <tr> <th style="padding: 5px;"><math>x</math></th> <th style="padding: 5px;"><math>y</math></th> </tr> </thead> <tbody> <tr> <td style="padding: 5px; text-align: center;">-3</td> <td style="padding: 5px; text-align: center;">???</td> </tr> <tr> <td style="padding: 5px; text-align: center;">???</td> <td style="padding: 5px; text-align: center;">-2</td> </tr> </tbody> </table>	$x$	$y$	-3	???	???	-2
$x$	$y$						
-3	???						
???	-2						

Our first goal is to find the missing coordinates. For this we use a very important idea. When working with equations with two variables and only one known value, substitute the value you know and then solve for the one you don't.

#### Procedure

- 1) Substitute the value you know into the equation.
- 2) Simplify any arithmetic in the equation.
- 3) Solve for the variable you don't know.

#### First point

- 1) We know  $x = -3$ .  

$$-7x - 4y = 1$$

$$-7(-3) - 4y = 1$$
- 2) Simplify the multiplication.  

$$21 - 4y = 1$$
- 3) Solve for  $y$ .  

$$21 - 21 - 4y = 1 - 21$$

$$-4y = -20$$

$$\frac{-4y}{-4} = \frac{-20}{-4}$$

$$y = 5$$

The point is  $(-3, 5)$ .

#### Second point

- 1) We know  $y = -2$ .  

$$-7x - 4y = 1$$

$$-7x - 4(-2) = 1$$
- 2) Simplify the multiplication.  

$$-7x + 8 = 1$$
- 3) Solve for  $x$ .  

$$-7x + 8 - 8 = 1 - 8$$

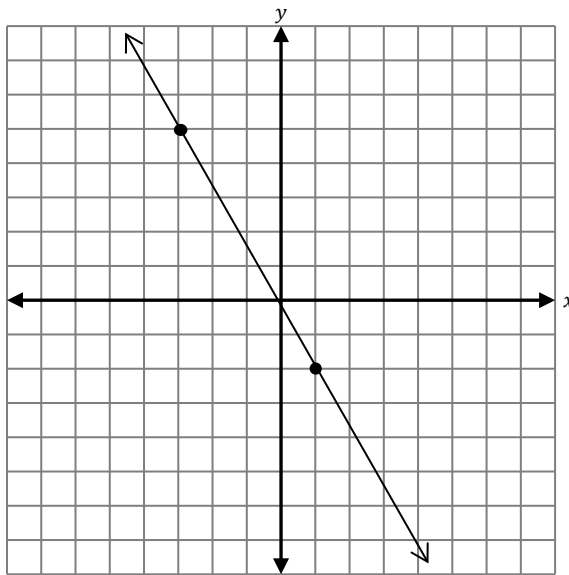
$$-7x = -7$$

$$\frac{-7x}{-7} = \frac{-7}{-7}$$

$$x = 1$$

The point is  $(1, -2)$ .

We plot the points (you may like to label the values on the axes), and then draw the line.



■

 **Problem 13.10**

Fill in the missing coordinates for the line whose equation is given. Then graph the line.

**Equation**      **Points with missing coordinates**

$$10x - 3y = 5$$

$x$	$y$
-1	???
???	5

**Procedure**

- 1) Substitute the value you know into the equation.
- 2) Simplify any arithmetic in the equation.
- 3) Solve for the variable you don't know.

**First point**

- 1) We know \_\_\_\_\_.  
 $10x - 3y = 5$

- 2) Simplify the multiplication.

- 3) Solve for the variable.

The point is \_\_\_\_\_.

**Second point**

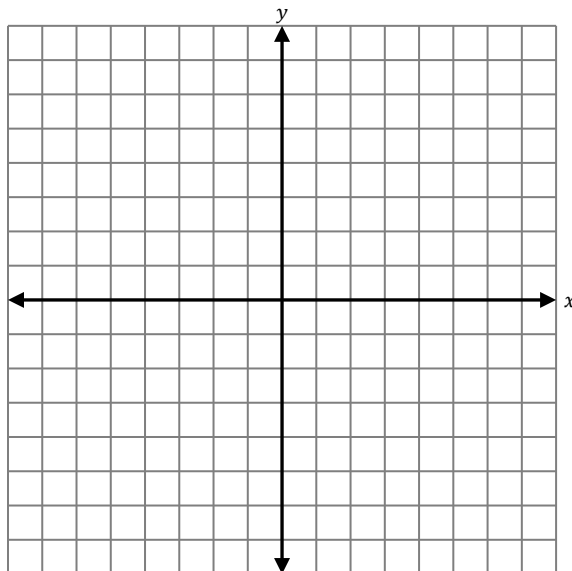
- 1) We know \_\_\_\_\_.  
 $10x - 3y = 5$

- 2) Simplify the multiplication.

- 3) Solve for the variable.

The point is \_\_\_\_\_.

Plot the points, and then draw the line.



■

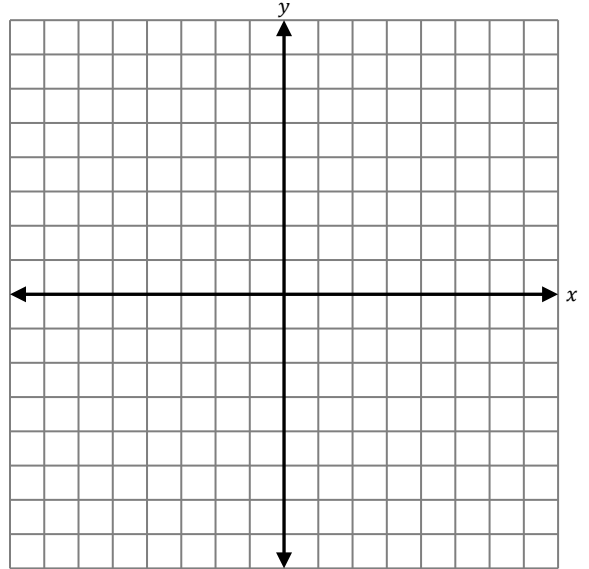
 **Problem 13.11**

Fill in the missing coordinates for the line whose equation is given. Then graph the line.

**Equation**      **Points with missing coordinates**

$$9x + 5y = 12$$

$x$	$y$
3	???
???	6





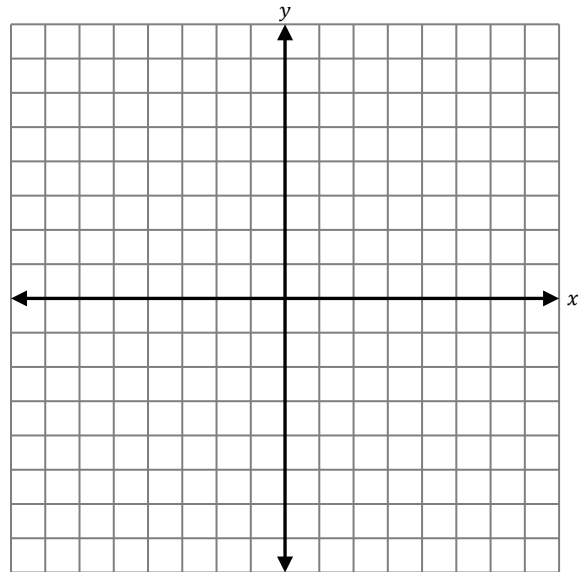
**Problem 13.12**

Fill in the missing coordinates for the line whose equation is given. Then graph the line.

**Equation**      **Points with missing coordinates**

$$3x + 3y = 6$$

$x$	$y$
-2	???
???	7
0	???



## GPS 14: Intercepts of a Line

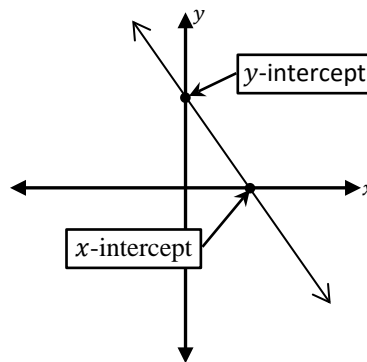
Throughout this section, we will learn more about lines by examining their equations and their graphs. You can always graph a line by first plotting *any* two points on the line, and then draw the line that passes through the two points. There are two special points that are often helpful to plot.

### The Intercepts

1. The ***y*-intercept** is the point on the line intersecting the *y*-axis.
2. The ***x*-intercept** is the point on the line intersecting the *x*-axis.

### Problem 14.1

Look at the sketch of a line and its intercepts.



Fill in the blanks below with *x* or *y*.

- 1) The \_\_\_\_\_-coordinate of the *y*-intercept is always 0.
- 2) The \_\_\_\_\_-coordinate of the *x*-intercept is always 0.



Based on these observations, we always know one coordinate of each intercept. Finding the other coordinate is just like in the previous section.

### The Intercepts

<i>x</i>	<i>y</i>	
0	???	<b><i>y</i>-intercept</b>
???	0	<b><i>x</i>-intercept</b>

**Problem 14.2**

Find the  $y$ -intercept of the line whose equation is given.

$$6x + 5y = 30$$

The  $y$ -intercept is the point on the line intersecting the  $y$ -axis. On the  $y$ -axis, the  $x$ -coordinate of every point is 0. We can use this observation to find the  $y$ -coordinate of the  $y$ -intercept.

- 1) Substitute  $x = 0$  into the equation of the line.

$$\begin{aligned} 6x + 5y &= 30 \\ 6(0) + 5y &= 30 \\ 5y &= 30 \end{aligned}$$

- 2) Solve for  $y$ .

$$\begin{aligned} 5y &= 30 \\ \frac{5y}{5} &= \frac{30}{5} \\ y &= 6 \end{aligned}$$

- 3) The  $y$ -intercept is the point  $(0,6)$ .



The  $y$ -intercept (and  $x$ -intercept) is a **point**. This means you must specify the  $x$ - and  $y$ -coordinate! Many students often say the  $y$ -intercept is 6 instead of  $(0,6)$ , and this is a very bad habit.

 **Problem 14.3**

Find the  $y$ -intercept of the line whose equation is given.

$$2x - 3y = 12$$

The  $y$ -intercept is \_\_\_\_\_.



**Problem 14.4**

Find the  $x$ -intercept of the line whose equation is given.

$$6x + 5y = 30$$

The  $x$ -intercept is the point on the line intersecting the  $x$ -axis. On the  $x$ -axis, the  $y$ -coordinate of every point is 0. We can use this observation to find the  $x$ -coordinate of the  $x$ -intercept.

- 1) Substitute  $y = 0$  into the equation of the line.

$$6x + 5y = 30$$

$$6x + 5(0) = 30$$

$$6x = 30$$

- 2) Solve for  $x$ .

$$6x = 30$$

$$\frac{6x}{6} = \frac{30}{6}$$

$$x = 5$$

- 3) The  $x$ -intercept is the point  $(5,0)$ .



 **Problem 14.5**

Find the  $x$ -intercept of the line whose equation is given.

$$2x - 3y = 12$$

The  $x$ -intercept is \_\_\_\_\_.



**Problem 14.6**

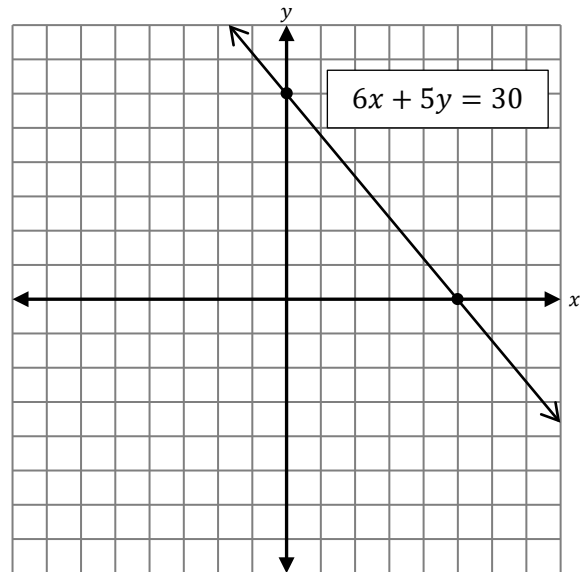
Plot the  $x$ - and  $y$ -intercept of the line whose equation is given. Then graph the line.

$$6x + 5y = 30$$

In Problem 14.2, we found the  $y$ -intercept is  $(0,6)$ .

In Problem 14.4, we found the  $x$ -intercept is  $(5,0)$ .

On the plane, we plotted both intercepts and graphed the line passing through them.



■

**Problem 14.7**

Plot the  $x$ - and  $y$ -intercept of the line whose equation is given. Then graph the line.

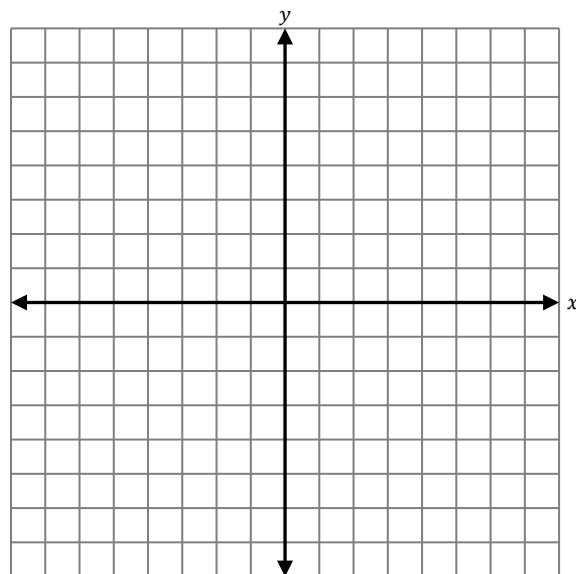
$$2x - 3y = 12$$

In Problem 14.3, you found the  $y$ -intercept: \_\_\_\_\_.

In Problem 14.5, you found the  $x$ -intercept: \_\_\_\_\_.

On the plane, plot both intercepts, and graph the line passing through them.

Label the equation of the line on the graph.



■

**Problem 14.8**

Find both intercepts, and then graph the line.

$$3x + 15y = 15$$

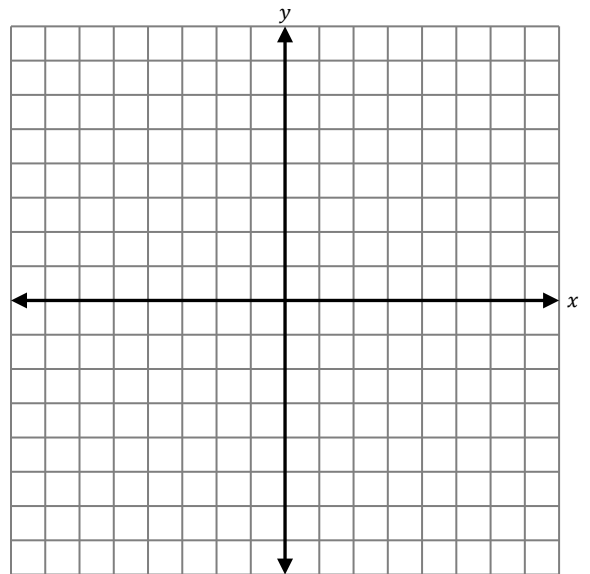
Find the  $y$ -intercept.

Find the  $x$ -intercept.

The  $y$ -intercept is \_\_\_\_\_

The  $x$ -intercept is \_\_\_\_\_

Plot both intercepts, and then graph the line.



■

**Problem 14.9**

Find both intercepts, and then graph the line.

$$-2x + y = -4$$

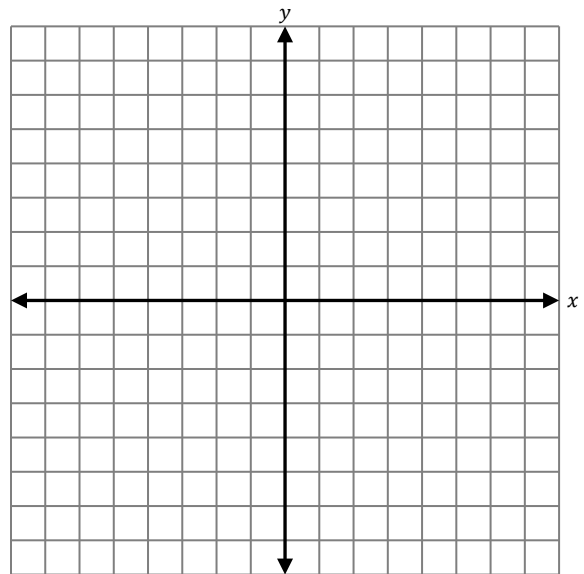
Find the  $y$ -intercept.

Find the  $x$ -intercept.

The  $y$ -intercept is \_\_\_\_\_

The  $x$ -intercept is \_\_\_\_\_

Plot both intercepts, and then graph the line.



■

We can graph a line using any two points. By looking at the graph we can then read off the intercepts.

**Problem 14.10**

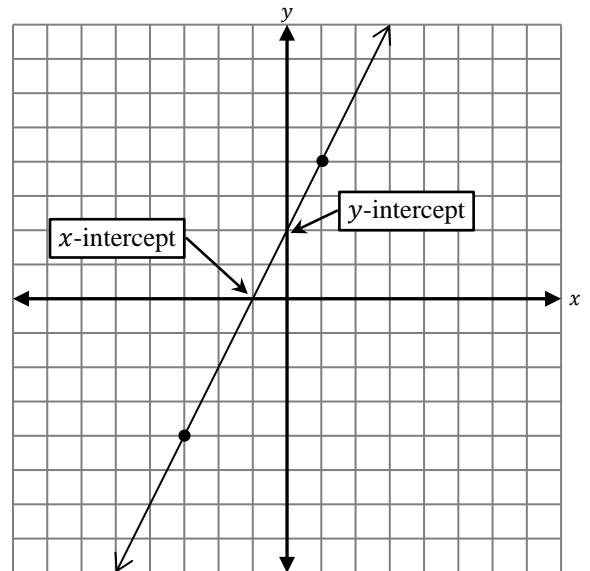
Graph the line passing through the points  $(-3, -4)$  and  $(1, 4)$ . Then determine the  $x$ - and  $y$ -intercepts of the line.

We can graph this line by plotting the two points and then connecting them.

By looking at the graph of this line, we can also determine the  $x$ - and  $y$ -intercept.

The  $y$ -intercept is  $(0, 2)$ .

The  $x$ -intercept is  $(-1, 0)$ .

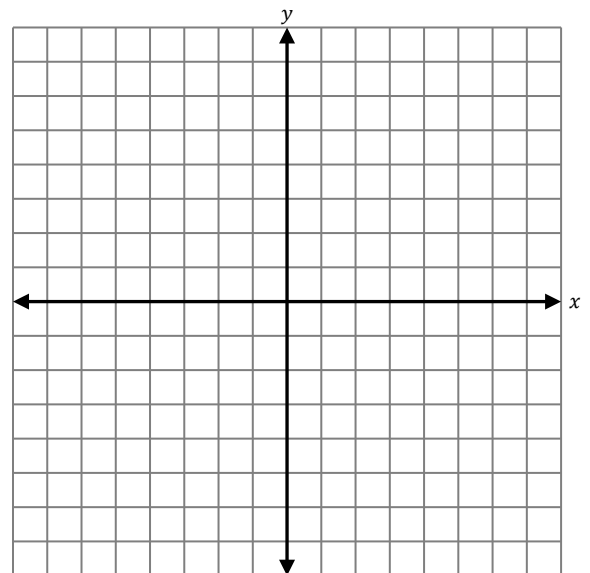


 **Problem 14.11**

Graph the line passing through the points  $(-1, -6)$  and  $(2, 3)$ . Then determine the  $x$ - and  $y$ -intercepts of the line.

The  $y$ -intercept is \_\_\_\_\_.

The  $x$ -intercept is \_\_\_\_\_.



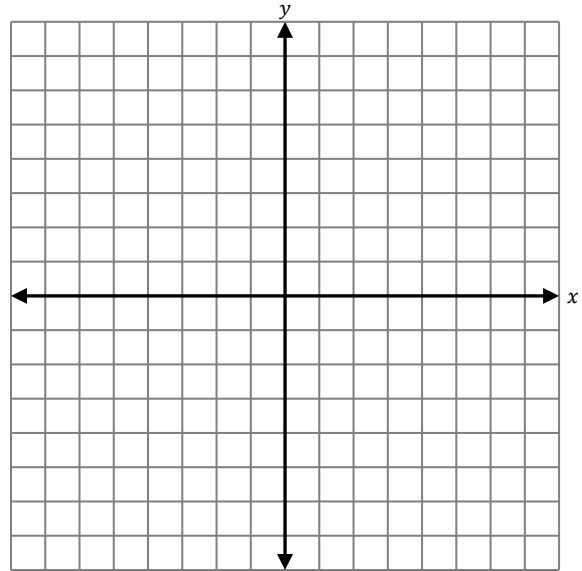


**Problem 14.12**

Graph the line passing through the points  $(-2, 6)$  and  $(4, -3)$ . Then determine the  $x$ - and  $y$ -intercepts of the line.

The  $y$ -intercept is \_\_\_\_\_.

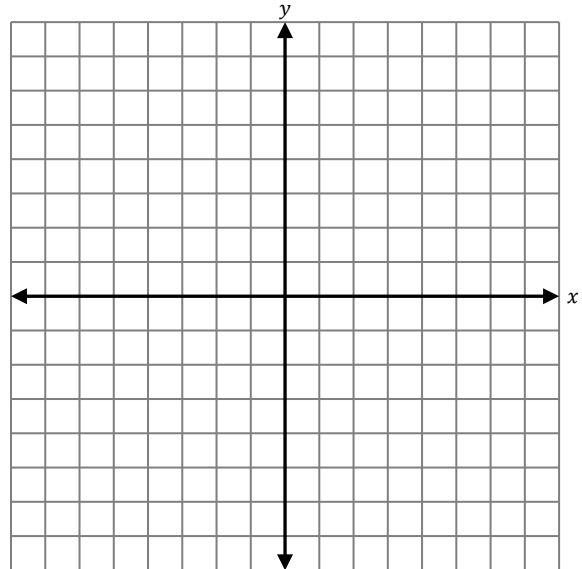
The  $x$ -intercept is \_\_\_\_\_.

**Problem 14.13**

Graph the line passing through the points  $(-4, 4)$  and  $(8, -2)$ . Then determine the  $x$ - and  $y$ -intercepts of the line.

The  $y$ -intercept is \_\_\_\_\_.

The  $x$ -intercept is \_\_\_\_\_.



## GPS 15: Slope and Equations of a Line

If you imagine a line is a road you are walking on, you would notice that some lines are steeper than others. The word we use to describe the steepness of a line is **slope**.

There are a variety of ways to express the formula to calculate the slope of a line. We start with this one.

### Definition of the Slope of a Line

The **slope** of the (nonvertical) line passing through the two points  $(x_1, y_1)$  and  $(x_2, y_2)$  is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

It is tradition to use the letter  $m$  to designate the slope.

The subscript 1 and 2 on the  $x$  and  $y$  are just a way of telling the 1<sup>st</sup> and 2<sup>nd</sup> points apart since they both have an  $x$ - and  $y$ -coordinate.

### Problem 15.1

Find the slope of the line passing through the points  $(-3, -4)$  and  $(1, 4)$ . Then graph the line.

$$x_1 = -3$$

$$y_1 = -4$$

By substituting into the formula, we calculate.

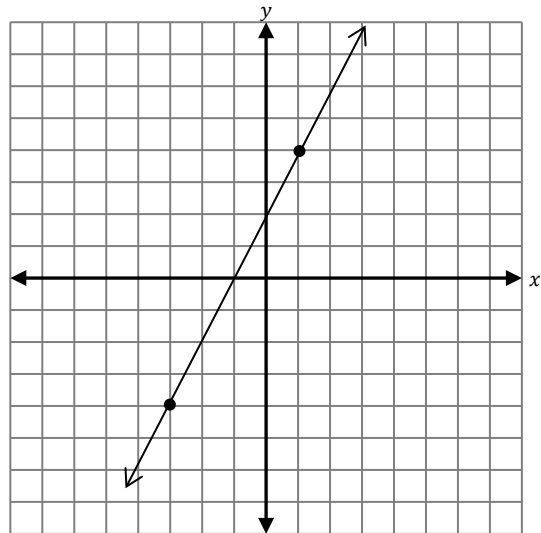
$$x_2 = 1$$

$$y_2 = 4$$

$$m = \frac{4 - (-4)}{1 - (-3)} = \frac{8}{4} = 2$$

The slope is 2.

To graph this line, we plot the two points and draw the line containing them as in the previous sections.



The subtraction often confuses students who forget that subtracting a negative is the same as adding the opposite (a positive!).

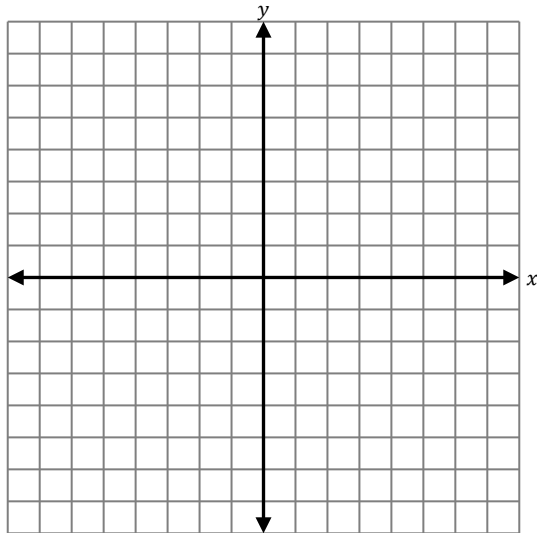
 **Problem 15.2**

Find the slope of the line passing through the points  $(-1, -6)$  and  $(2, 3)$ . Then graph the line.

$$x_1 = \underline{\hspace{2cm}} \quad y_1 = \underline{\hspace{2cm}}$$

$$x_2 = \underline{\hspace{2cm}} \quad y_2 = \underline{\hspace{2cm}}$$

The slope is  $\underline{\hspace{2cm}}$ .



■

Memorizing the previous formula is often very confusing since the formula has lots of weird subscripts. A better way to remember the formula is by thinking of it differently.

**Slope Formula Again**

The **slope** of the (nonvertical) line passing through any two points is

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{subtract } y - \text{coordinates}}{\text{subtract } x - \text{coordinates}}$$

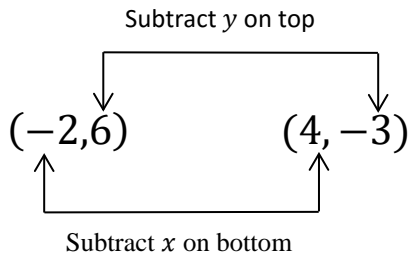


Be careful with the *order* of subtraction and do not change it between the  $x$ - and  $y$ -coordinates. Whatever point's  $y$ -coordinate you put first in the numerator, you must also put that point's  $x$ -coordinate first in the denominator.

**Problem 15.3**

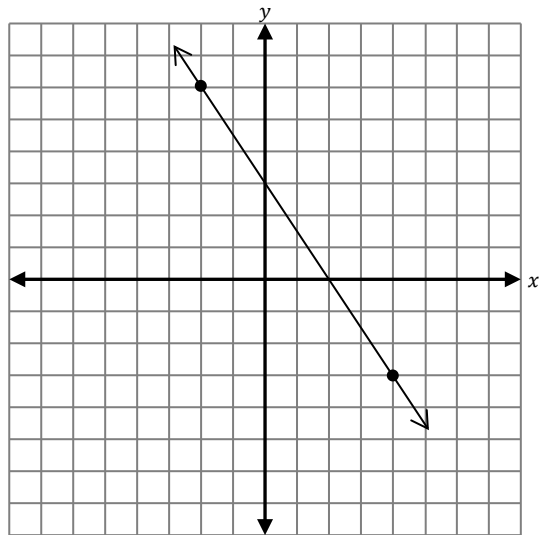
Find the slope of the line passing through the points  $(-2, 6)$  and  $(4, -3)$ .

We can subtract in any order we want, as long as we do not reverse the order of the subtraction between the top and bottom. We will put the first point first this time.



$$\begin{aligned} m &= \frac{6 - (-3)}{-2 - 4} \\ &= \frac{9}{-6} \\ &= -\frac{3}{2} \end{aligned}$$

The slope is  $m = -\frac{3}{2}$

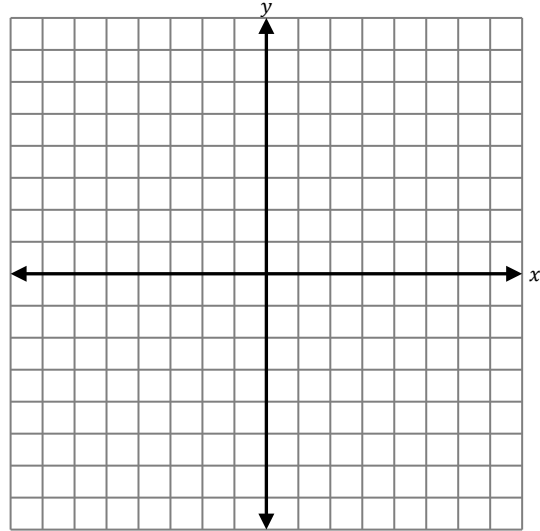


Recall with fractions we can write the negative in *any* one of three places.  
For example:

$$-\frac{3}{2} = \frac{-3}{2} = \frac{3}{-2}$$

**Problem 15.4**

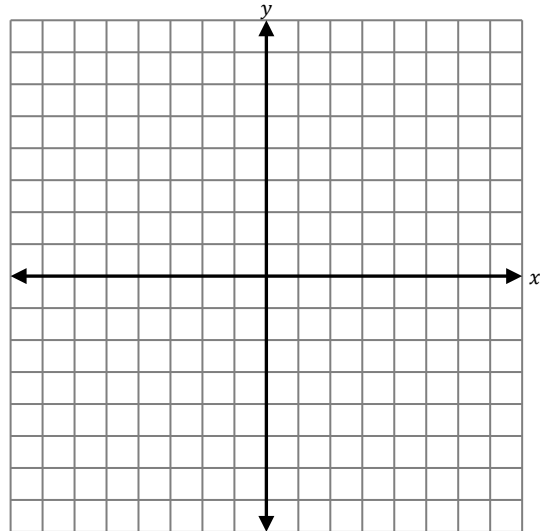
Find the slope of the line passing through the points  $(-4, 4)$  and  $(8, -2)$ . Then graph the line.



The slope is \_\_\_\_\_.

**Problem 15.5**

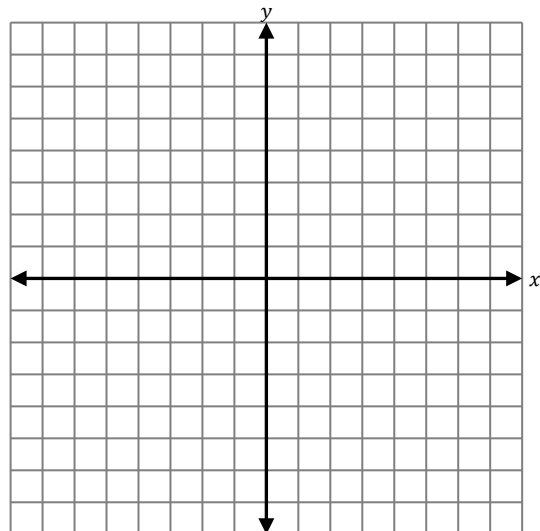
Find the slope of the line passing through the points  $(3, 2)$  and  $(6, 8)$ . Then graph the line.



The slope is \_\_\_\_\_.

**Problem 15.6**

Find the slope of the line passing through the points  $(-2, 7)$  and  $(2, -7)$ .

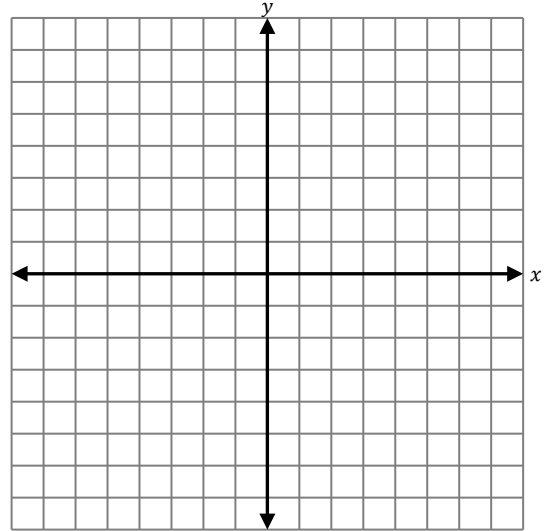


The slope is \_\_\_\_\_.

**Problem 15.7**

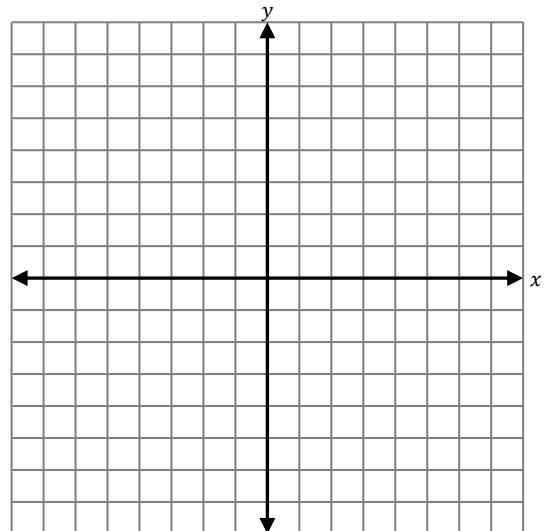
Find the slope of the line passing through the points  $(-4,0)$  and  $(0,6)$ . Then graph the line.

The slope is \_\_\_\_\_.

**Problem 15.8**

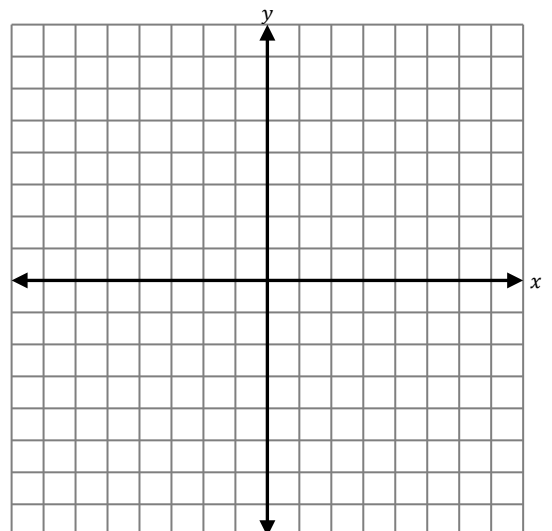
Find the slope of the line passing through the points  $(-2,2)$  and  $(4,7)$ . Then graph the line.

The slope is \_\_\_\_\_.

**Problem 15.9**

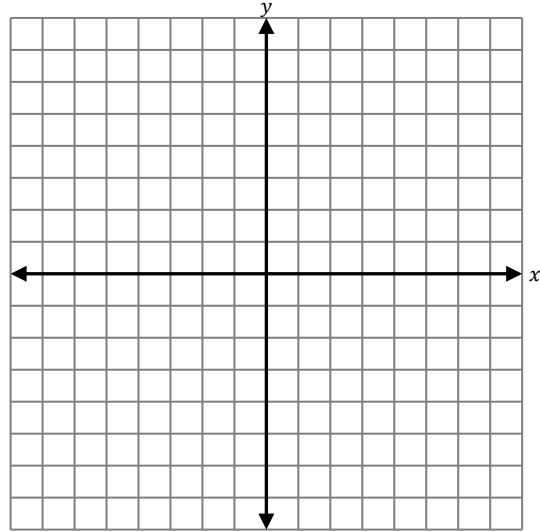
Find the slope of the line passing through the points  $(-4,-8)$  and  $(-6,-2)$ . Then graph the line.

The slope is \_\_\_\_\_.



**Problem 15.10**

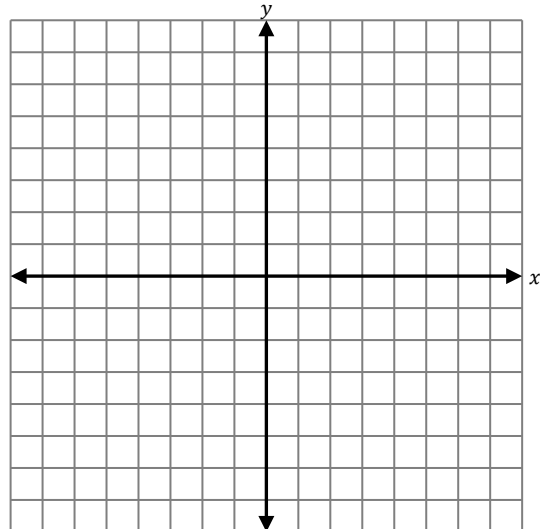
Find the slope of the line passing through the points  $(3,0)$  and  $(0,6)$ . Then graph the line.



The slope is \_\_\_\_\_.

**Problem 15.11**

Find the slope of the line passing through the points  $(0,-7)$  and  $(2,7)$ . Then graph the line.



The slope is \_\_\_\_\_.

## GPS 16: The Slope-Intercept Equation of a Line

### Problem 16.1

Next to each of the graphs in GPS 15, label the graph as one of the following.

1. Positive Slope
2. Negative Slope

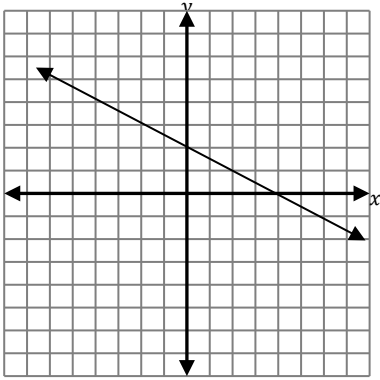
A) Below, describe and draw a sketch to indicate the general shape of lines with a *positive slope*.

B) Below, describe and draw a sketch to indicate the general shape of lines with a *negative slope*.

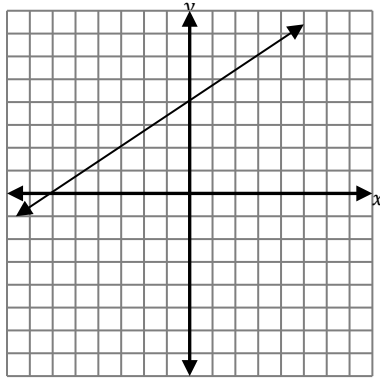


 **Problem 16.2**

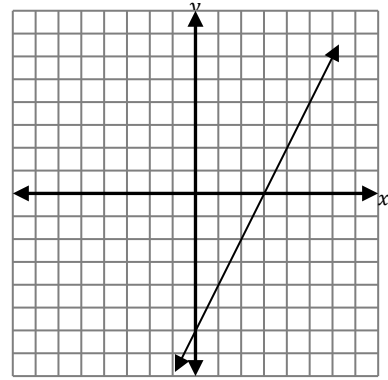
Below each graph, indicate whether the line has positive or negative slope.



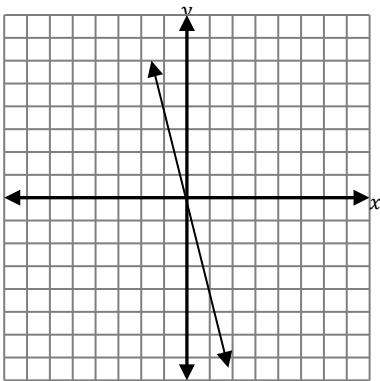
\_\_\_\_\_



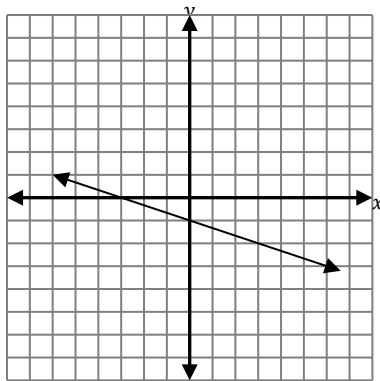
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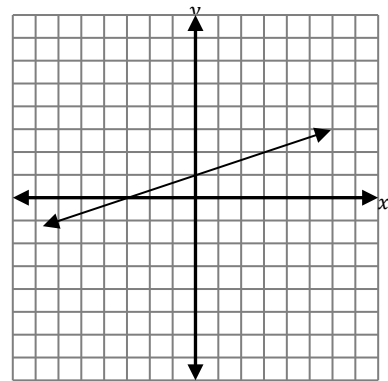
\_\_\_\_\_



\_\_\_\_\_



\_\_\_\_\_



\_\_\_\_\_

We still don't know an equation for our lines. We will learn one of the most common ways to find an equation for a line.

**The Slope-Intercept Equation of a Line**

If you know the slope  $m$  of a line and the  $y$ -intercept  $(0, b)$ , then the **slope-intercept equation** of the line is

$$y = mx + b$$

**Problem 16.3**

Find the equation of the line passing through the points  $(-4, 2)$  and  $(1, -8)$ . Write the equation in slope-intercept form.

Our goal is to write the equation in the form  $y = mx + b$ . For this, we must determine the values of  $m$  and  $b$ .

- 1) We know to use the slope formula to find  $m$ .

$$m = \frac{-8 - 2}{1 - (-4)} = \frac{-10}{5} = -2$$

- 2) To find  $b$ , we substitute the value of  $m$  and the values of one of our points into the equation  $y = mx + b$ . We can use either of the two points we want. Let's use the first point  $(-4, 2)$ .

$$\begin{array}{ll} y = mx + b & \\ 2 = -2(-4) + b & \text{We substituted } m = -2, x = -4 \text{ and } y = 2. \\ 2 = 8 + b & \text{Simplify the arithmetic.} \\ 2 - 8 = 8 - 8 + b & \text{Subtract 8 from both sides.} \\ -6 = b & \text{We now know the value of } b. \end{array}$$

- 3) We know  $m = -2$  and  $b = -6$ . We can write the equation as

$$y = -2x - 6$$

If we chose the second point  $(1, -8)$  in step 2), we would get the same value for  $b$  in the equation.

$$\begin{array}{ll} y = mx + b & \\ -8 = -2(1) + b & \text{We substituted } m = -2, x = 1 \text{ and } y = -8. \\ -8 = -2 + b & \text{Simplify the arithmetic.} \\ -8 + 2 = -2 + 2 + b & \text{Add 2 to both sides.} \\ -6 = b & \text{We now know the value of } b. \end{array}$$

We can write the equation as

$$y = -2x - 6$$

■

**Problem 16.4**

Find the slope of the line passing through the points  $(-4, -1)$  and  $(4, 3)$ .

The slope is \_\_\_\_\_.

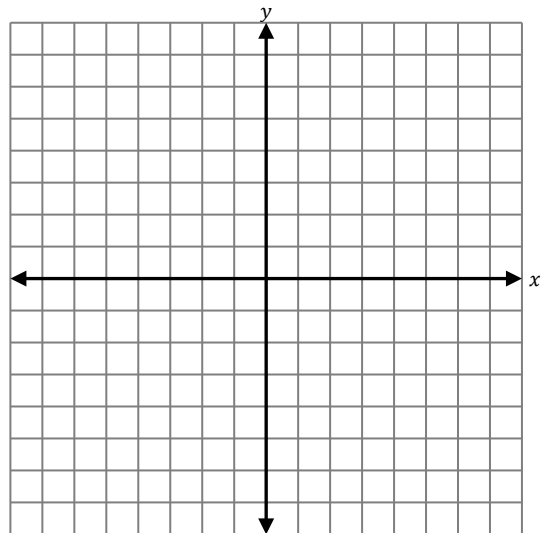
Write the equation of the line in slope-intercept form.

The equation is \_\_\_\_\_.

Graph the line and determine the intercepts.

The  $y$ -intercept is \_\_\_\_\_.

The  $x$ -intercept is \_\_\_\_\_.



**Problem 16.5**

Find the slope of the line passing through the points  $(2, -4)$  and  $(-1, 8)$ .

The slope is \_\_\_\_\_.

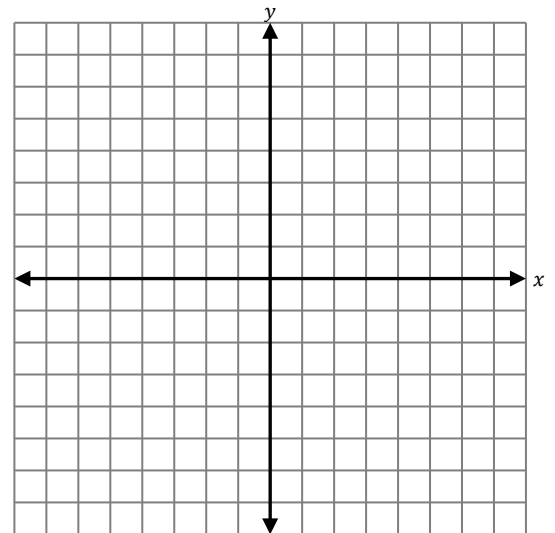
Write the equation of the line in slope-intercept form.

The equation is \_\_\_\_\_.

Graph the line and determine the intercepts.

The  $y$ -intercept is \_\_\_\_\_.

The  $x$ -intercept is \_\_\_\_\_.



Now, we restrict ourselves to just finding the equations of lines without graphing.

 **Problem 16.6**

Find the equation of the line passing through the points  $(1,2)$  and  $(-1,8)$ . Write the equation in slope-intercept form.

 **Problem 16.7**

Find the equation of the line passing through the points  $(2, -1)$  and  $(-2, -9)$ . Write the equation in slope-intercept form.

 **Problem 16.8**

Find the equation of the line passing through the points  $(5, -2)$  and  $(-3,6)$ . Write the equation in slope-intercept form.

 **Problem 16.9**

Find the equation of the line passing through the points  $(-4,8)$  and  $(1, -2)$ . Write the equation in slope-intercept form.

## GPS 17: Slope-Intercept Equation and Graphing

In the last section, we were introduced to the slope-intercept equation.

### The Slope-Intercept Equation of a Line

If you know the slope  $m$  of a line and the  $y$ -intercept  $(0, b)$ , then **the slope-intercept equation** of the line is

$$y = mx + b$$

Soon we will see that this equation makes graphing lines incredibly easy. We will first practice working with this equation.

### Problem 17.1

Find the slope and  $y$ -intercept for the graph of the equation.

$$y = 3x - 5$$

This equation is already written in slope-intercept form. As a result, we can easily read off the information.

The slope is the coefficient of the  $x$ , which is 3.

The  $y$ -coordinate of the  $y$ -intercept is  $-5$  which means the  $y$ -intercept is the point  $(0, -5)$ .

The slope is  $m = 3$ .

The  $y$ -intercept is  $(0, -5)$ .

### Problem 17.2

Find the slope and  $y$ -intercept for the graph of the equation.

$$y = -\frac{2}{3}x + 4$$

The slope is \_\_\_\_\_.

The  $y$ -intercept is \_\_\_\_\_.

### Problem 17.3

Find the slope and  $y$ -intercept for the graph of the equation.

$$y = \frac{3}{4}x - 7$$

The slope is \_\_\_\_\_.

The  $y$ -intercept is \_\_\_\_\_.

### Problem 17.4

Find the slope and  $y$ -intercept for the graph of the equation.

$$y = 4x$$

The slope is \_\_\_\_\_.

The  $y$ -intercept is \_\_\_\_\_.

### Problem 17.5

Find the slope and  $y$ -intercept for the graph of the equation.

$$y = -2x$$

The slope is \_\_\_\_\_.

The  $y$ -intercept is \_\_\_\_\_.

**Problem 17.6**

Find the slope and  $y$ -intercept for the graph of the equation.

$$5x - 10y = 30$$

This equation is **not** in slope-intercept form, but we can first convert it to the slope-intercept form by solving this equation for  $y$ .

$$5x - 10y = 30$$

$$5x - \mathbf{5x} - 10y = \mathbf{-5x} + 30 \quad \text{subtract } 5x \text{ from both sides}$$

$$-10y = -5x + 30$$

$$\frac{-10y}{\mathbf{-10}} = \frac{-5x}{\mathbf{-10}} + \frac{30}{\mathbf{-10}} \quad \text{divide every term by } -10$$

$$y = \frac{1}{2}x - 3$$

This equation is now in slope-intercept form, and we can easily read off the slope and  $y$ -intercept.

The slope is  $m = \frac{1}{2}$

The  $y$ -intercept is  $(0, -3)$ .



 **Problem 17.7**

Find the slope and  $y$ -intercept for the graph of the equation.

$$-6x - 5y = 15$$

The slope is  $m =$  \_\_\_\_\_.

The  $y$ -intercept is \_\_\_\_\_.



 **Problem 17.8**

Find the slope and  $y$ -intercept for the graph of the equation.

$$-6x + 4y - 12 = 0$$

The slope is  $m =$  \_\_\_\_\_.

The  $y$ -intercept is \_\_\_\_\_.



**Problem 17.9**

Find the slope and  $y$ -intercept for the graph of the equation.

$$-3x - 5y = 0$$

The slope is  $m =$  \_\_\_\_\_.

The  $y$ -intercept is \_\_\_\_\_.

**Problem 17.10**

Find the slope and  $y$ -intercept for the graph of the equation.

$$2x + 3y - 5 = 16$$

The slope is  $m =$  \_\_\_\_\_.

The  $y$ -intercept is \_\_\_\_\_.







## GPS 18: Graphing Lines in Slope-Intercept Form

To see why it is so easy to graph a line in slope-intercept form, we first revisit an earlier formulation of the slope.

### Slope Formula

The **slope** of the (nonvertical) line passing through any two points is

$$m = \frac{\text{subtract } y - \text{coordinates}}{\text{subtract } x - \text{coordinates}}$$

When we subtract the  $y$ -coordinates, geometrically we are calculating the distance between their values on the  $y$ -axis. This tells us how far up or down the  $y$ -axis we have moved. Similarly, when we subtract the  $x$ -coordinates, geometrically we are calculating the distance between their values on the  $x$ -axis. This leads to the final interpretation of the slope.

### Slope Formula Geometrically

The **slope** of the (nonvertical) line passing through any two points is

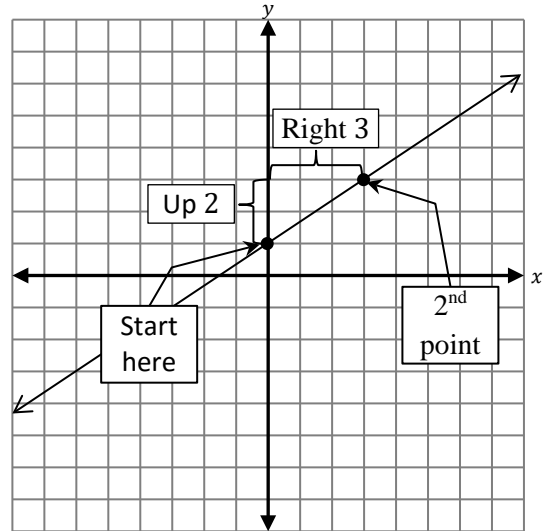
$$m = \frac{\text{subtract } y - \text{coordinates}}{\text{subtract } x - \text{coordinates}} = \frac{\text{rise}}{\text{run}} = \frac{\text{distance up/down}}{\text{distance right/left}}$$

**Problem 18.1**

Graph the line.

$$y = \frac{2}{3}x + 1$$

- 1) Identify the slope and y-intercept.
  - a. The slope is  $m = \frac{2}{3}$ .
  - b. The y-intercept is  $(0,1)$ .
- 2) Plot the y-intercept.
- 3) Use the slope to plot a 2<sup>nd</sup> point.
  - a. If the numerator (top) of the slope is **positive**, go **up** that many units. If it is **negative**, go **down** that many units.
  - b. If the denominator (bottom) of the slope is **positive**, go **right** that many units. If it is **negative**, go **left** that many units.



Our slope tells us

$$m = \frac{2}{3} = \frac{\text{up } 2}{\text{right } 3}$$

- 4) Draw the line passing through the two points.

Even though plotting two points is good enough to graph the line, it is often helpful to plot more points. To do this, we can simply repeat the moving up/down and moving right/left as directed by the slope.

It is easy to see how to use this technique to plot a 3<sup>rd</sup> point. But if we examine this line closely, we can spot 4<sup>th</sup> and 5<sup>th</sup> points on the line. It turns out that we can also find these two points by remembering a fact from arithmetic.

The slope can be written as

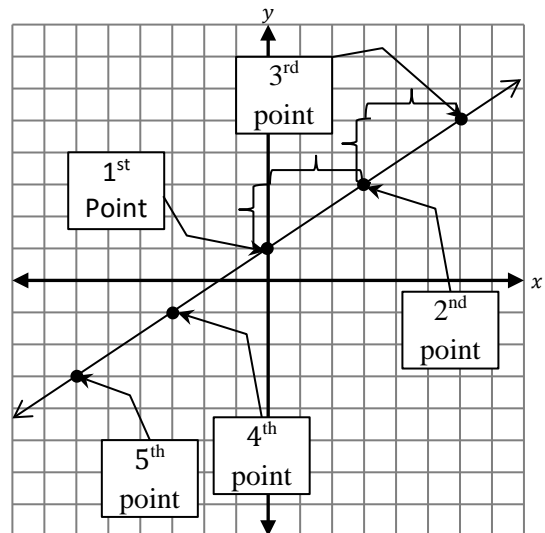
$$m = \frac{2}{3}$$

But since a negative divided by a negative is positive, we can also write the slope as

$$m = \frac{2}{3} = \frac{-2}{-3} = \frac{\text{down } 2}{\text{left } 3}$$

By doing this, we can easily plot the 4<sup>th</sup> and 5<sup>th</sup> points!

■





Recall with fractions we can write the negative in *any* one of three places.  
For example:

$$-\frac{3}{2} = \frac{-3}{2} = \frac{3}{-2}$$

**Problem 18.2**

Graph the line.

$$y = -\frac{3}{2}x + 4$$

The slope is  $m = -\frac{3}{2}$ .

The y-intercept is (0,4).

We start by plotting (0,4)

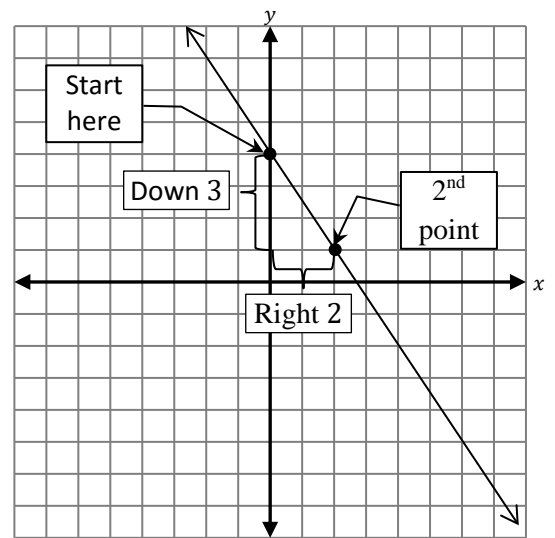
Using the slope, we find our directions

$$-\frac{3}{2} = \frac{-3}{2} = \frac{\text{down } 3}{\text{right } 2}$$

We can also write the negative in the denominator to get

$$-\frac{3}{2} = \frac{3}{-2} = \frac{\text{up } 3}{\text{left } 2}$$

Doing this repeatedly, we can plot more than 2 points quite easily.



**Problem 18.3**

Graph the line.

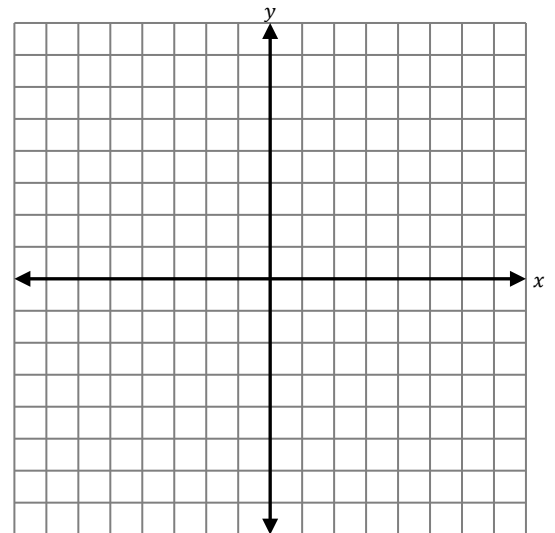
$$y = -\frac{1}{2}x + 3$$

The slope is  $m =$  \_\_\_\_\_.

Using the slope, write down the directions to follow to find additional points.

The y-intercept is \_\_\_\_\_.

Plot additional points on the line by repeatedly moving up/down and right/left as directed by the slope.





Whole numbers can be written as a fraction by rewriting with denominator 1.  
For example:

$$3 = \frac{3}{1}$$

**Problem 18.4**

Graph the line.

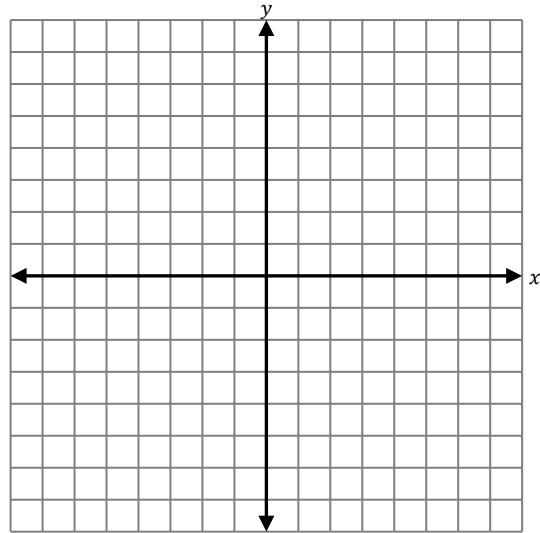
$$y = 3x - 2$$

The slope is  $m =$  \_\_\_\_\_.

Using the slope, write down the directions to follow to find additional points.

The y-intercept is \_\_\_\_\_.

Plot additional points on the line by repeatedly moving up/down and right/left as directed by the slope.



**Problem 18.5**

Graph the line.

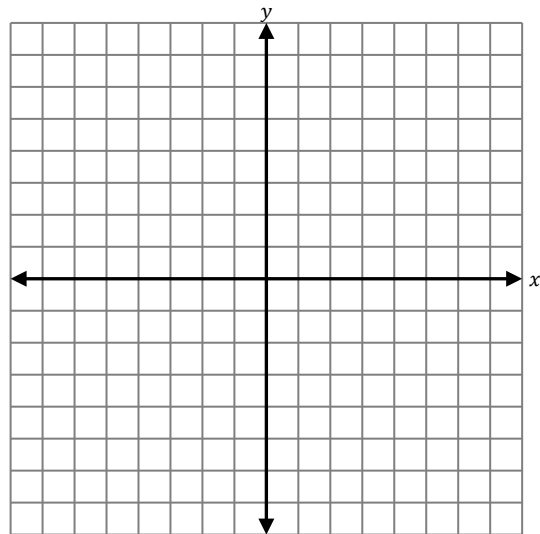
$$3x - 5y = 10$$

The slope is  $m =$  \_\_\_\_\_.

Using the slope, write down the directions to follow to find additional points.

The y-intercept is \_\_\_\_\_.

Plot additional points on the line by repeatedly moving up/down and right/left as directed by the slope.



**Problem 18.6**

Graph the line.

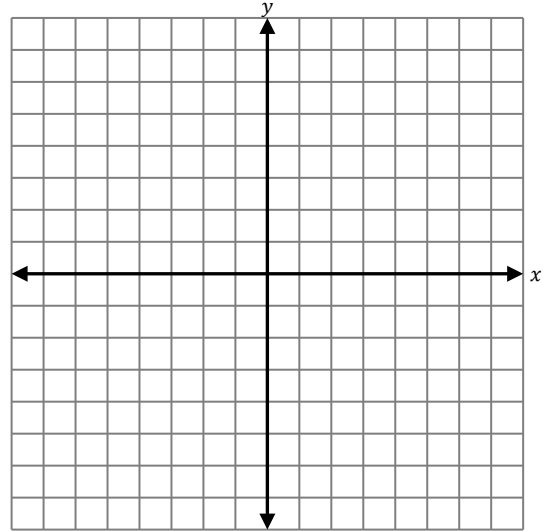
$$2x + 3y = 0$$

The slope is  $m =$  \_\_\_\_\_.

Using the slope, write down the directions to follow to find additional points.

The  $y$ -intercept is \_\_\_\_\_.

Plot additional points on the line by repeatedly moving up/down and right/left as directed by the slope.

**Problem 18.7**

Graph the line.

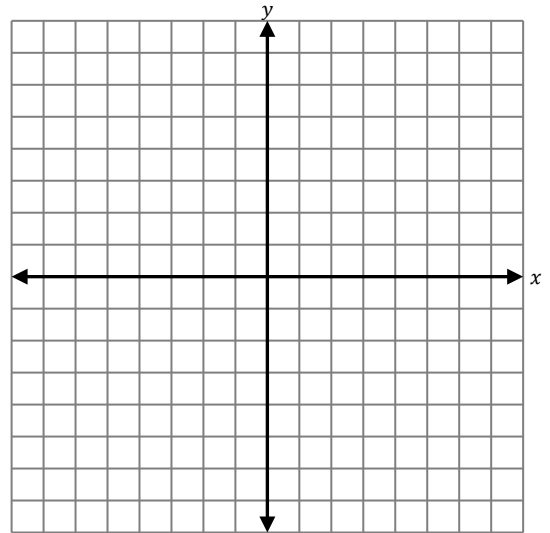
$$-x + 4y = 8$$

The slope is  $m =$  \_\_\_\_\_.

Using the slope, write down the directions to follow to find additional points.

The  $y$ -intercept is \_\_\_\_\_.

Plot additional points on the line by repeatedly moving up/down and right/left as directed by the slope.



**Problem 18.8**

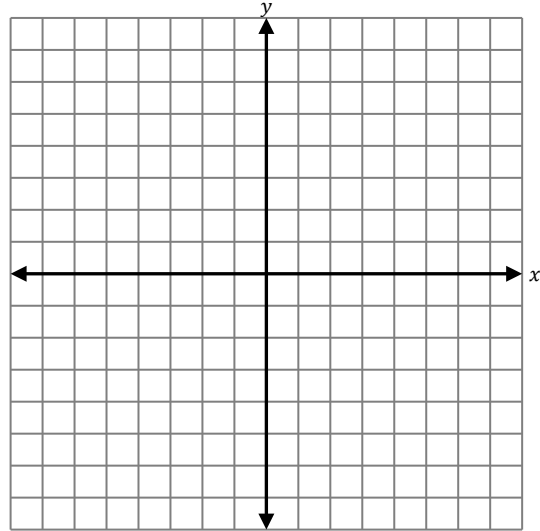
Graph the line.

$$-6x - 5y = 15$$

The slope is  $m =$  \_\_\_\_\_.

The y-intercept is \_\_\_\_\_.

Plot additional points on the line by repeatedly moving up/down and right/left as directed by the slope.

**Problem 18.9**

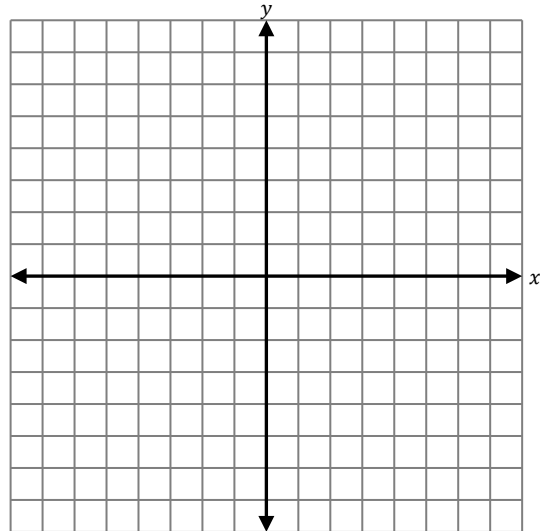
Graph the line.

$$y = 4x$$

The slope is  $m =$  \_\_\_\_\_.

The y-intercept is \_\_\_\_\_.

Plot additional points on the line by repeatedly moving up/down and right/left as directed by the slope.

**Problem 18.10**

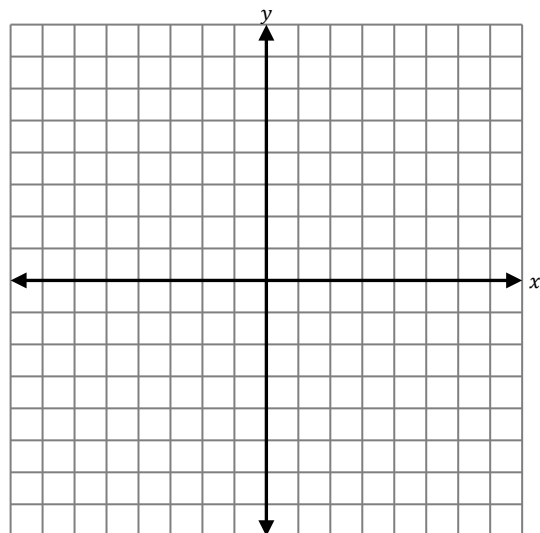
Graph the line.

$$x + y = 4$$

The slope is  $m =$  \_\_\_\_\_.

The y-intercept is \_\_\_\_\_.

Plot additional points on the line by repeatedly moving up/down and right/left as directed by the slope.



### GPS 19: Finding an Equation of a Line from its graph

Instead of graphing a line from the equation, we will find an equation of the line from its graph by using the slope and the  $y$ -intercept.

#### Problem 19.1

Find an equation of the line.

To find an equation, we need to find the slope and the  $y$ -intercept.

We first find the  $y$ -intercept by locating the point where the line cuts the  $y$ -axis.

We see that this point is  $(0,1)$ , and so  $b = 1$ .

Now, we locate another point on the line, and find the slope by counting how many units we need to travel up/down and right/left to reach the next point.

Starting at the  $y$ -intercept, we must go up 3.

Therefore, **the rise is +3**.

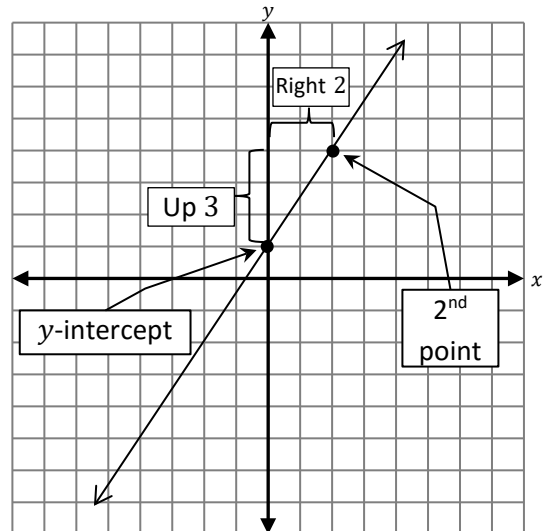
From there, we must go right 2. Therefore, **the run is +2**.

Combining this, our slope  $m = \frac{\text{rise}}{\text{run}} = \frac{+3}{+2} = \frac{3}{2}$ .

Finally, we can write the equation in  $y = mx + b$  form.

$$y = \frac{3}{2}x + 1$$

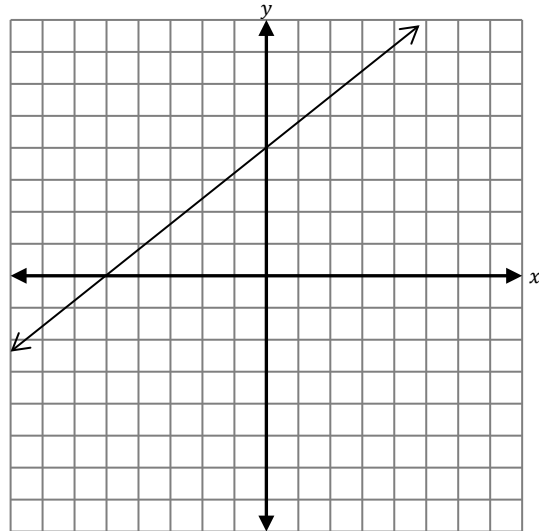
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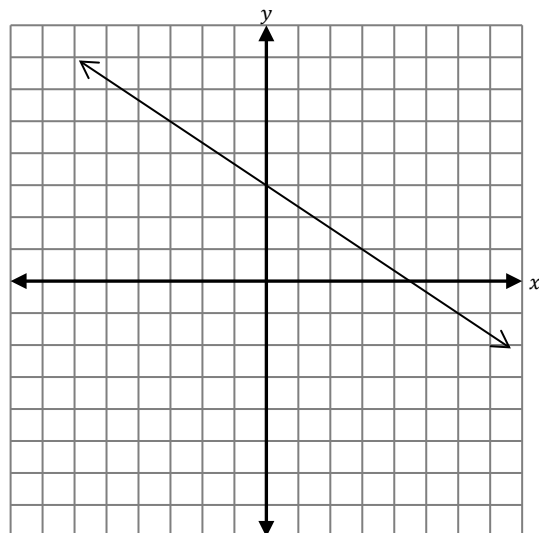


**Problem 19.2**

Find an equation of the line.

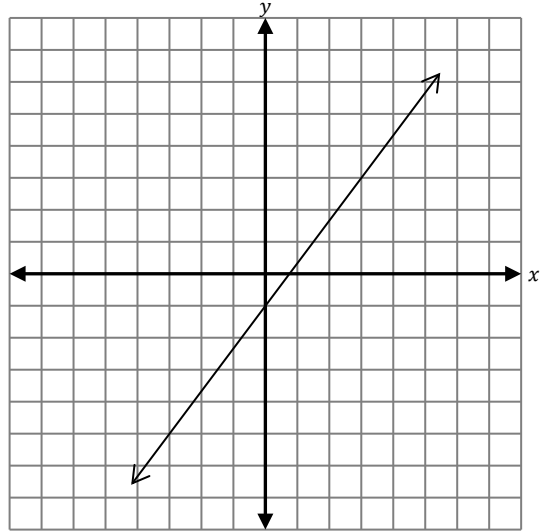
**Problem 19.3**

Find an equation of the line.

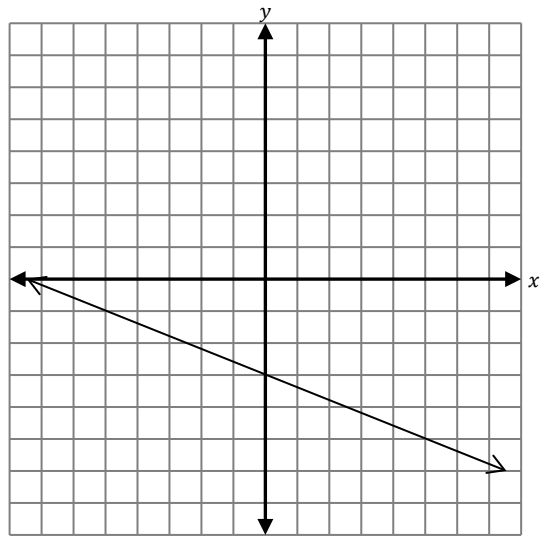


**Problem 19.4**

Find an equation of the line.

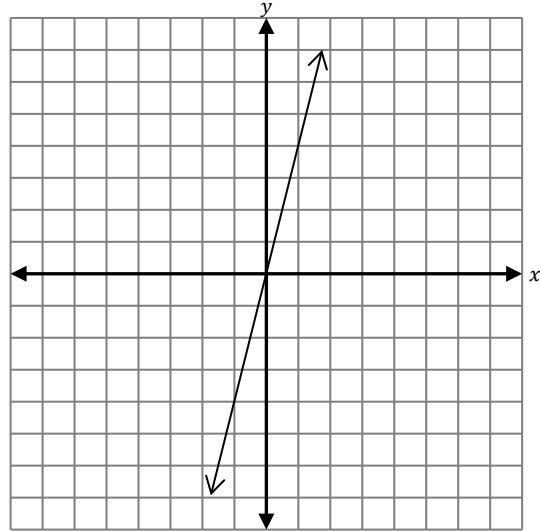
**Problem 19.5**

Find an equation of the line.

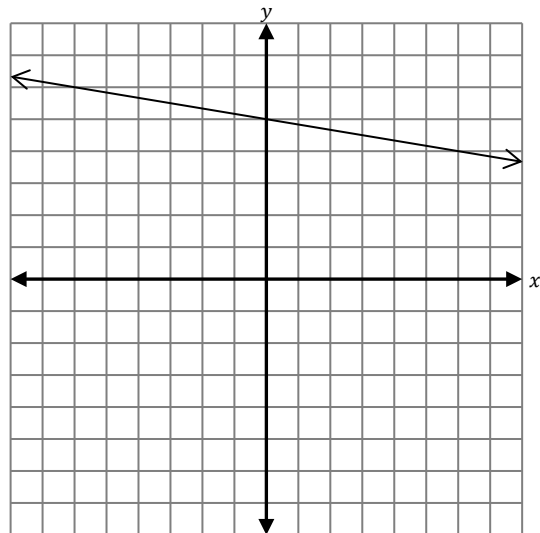


**Problem 19.6**

Find an equation of the line.

**Problem 19.7**

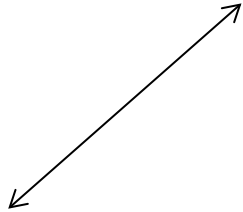
Find an equation of the line.



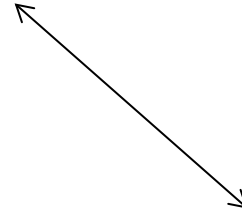
## GPS 20: Horizontal and Vertical Lines

So far all the lines we graphed looked like one of the two pictured below.

Lines with a **positive slope**.

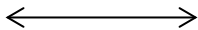


Lines with a **negative slope**



But there are two types of lines we have not yet considered.

**Horizontal Lines**



**Vertical Lines**



### Horizontal Lines

#### Graphs and Equations of Horizontal Lines

##### Problem 20.1

Graph the horizontal line passing through the point (2,3).

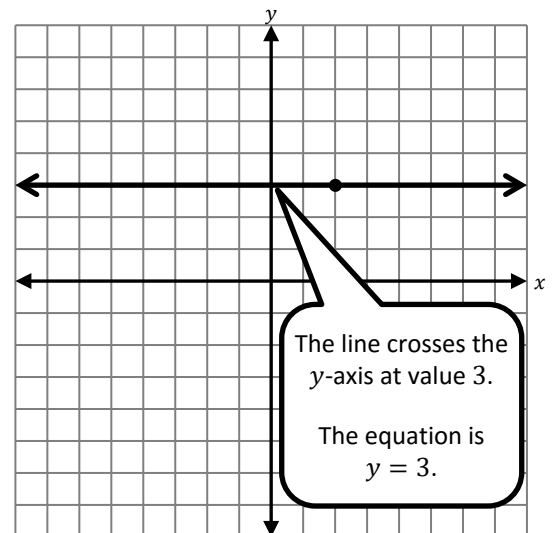
It is quite easy to graph a horizontal line since they all look the same.

We can easily find lots of points on this line. Here are a few.

(5,3)	(-1,3)	(0,3)
(4,3)	(-2,3)	(1.75,3)
(3,3)	(-3,3)	(-4.6,3)
(1,3)	(-4,3)	(2.48,3)

Do you notice a pattern? All the  $y$ -coordinates are the same as the  $y$ -coordinate of our original point! We can use this pattern to write the equation of the line.

The equation of the horizontal line passing through the point (2,3) is  $y = 3$ .



The pattern that all the  $y$ -coordinates on a horizontal line are the same as the  $y$ -coordinate of our original point will be true for all horizontal lines. This is because to draw **any** horizontal line, all we need to know is how far above or below the  $x$ -axis to draw it.

### Equation of a Horizontal Line

An equation of the horizontal line passing through the point  $(a, b)$  is

$$y = b$$

#### Problem 20.2

Graph the horizontal line passing through the point  $(1, 4)$ .

Find lots of points on this line.


The equation of the line is \_\_\_\_\_.



Let's look at the slope.

### Slope of Horizontal Lines

#### Problem 20.3

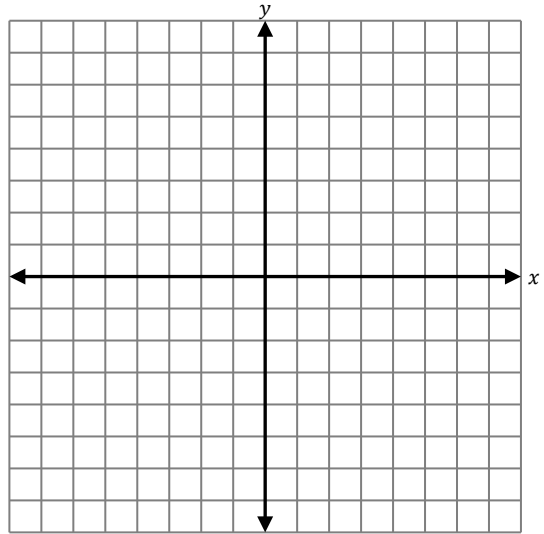
Find the slope of the horizontal line passing through the point  $(2, 3)$ .

We have been calculating the slope by using two points on the line. While graphing this line in Problem 20.1, we observed there were many additional points on the line we could easily spot. Let's pick one additional point and use the point  $(2, 3)$  from the problem to calculate the slope.

Using the points  $(2, 3)$  and  $(5, 3)$ , we calculate.

$$\begin{aligned} m &= \frac{3 - 3}{5 - 2} \\ &= \frac{0}{3} \\ &= 0 \end{aligned}$$

The slope is 0.



 **Problem 20.4**

Find the slope of the horizontal line passing through the point  $(1,4)$ .

■

The slopes of both lines are zero because the numerators of both fractions are zero. Recall that the numerator of the slope is the difference in the  $y$ -coordinates. On a horizontal line, we saw that the  $y$ -coordinates are *always the same*—this means their difference will always be zero.

**Slope of Horizontal Lines**

The slope of any horizontal line is **zero**.

## Vertical Lines

### Graphs and Equations of Vertical Lines

**Problem 20.5**

Graph the vertical line passing through the point  $(2,3)$ .

It is quite easy to graph a vertical line since they all look the same.

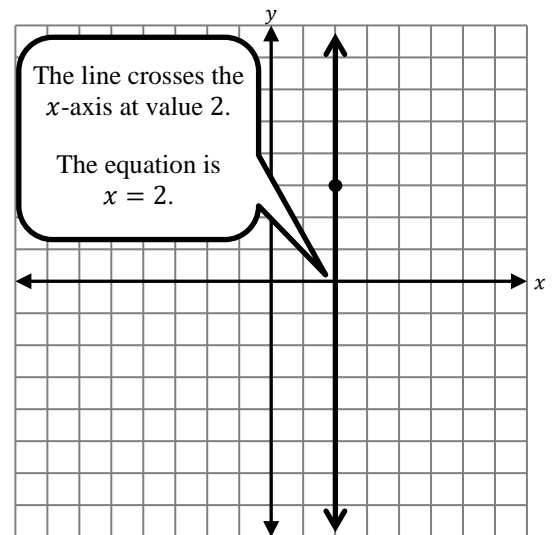
We can easily find lots of points on this line. Here are a few.

$(2,5)$	$(2,0)$	$(2,6.7)$
$(2,4)$	$(2,-1)$	$(2,1.74)$
$(2,2)$	$(2,-2)$	$(2,-4.2)$
$(2,1)$	$(2,-3)$	$(2,483)$

Do you notice a pattern? All the  $x$ -coordinates are the same as the  $x$ -coordinate of our original point! We can use this pattern to write the equation of the line.

The equation of the vertical line passing through the point  $(2,3)$  is  $x = 2$ .

■

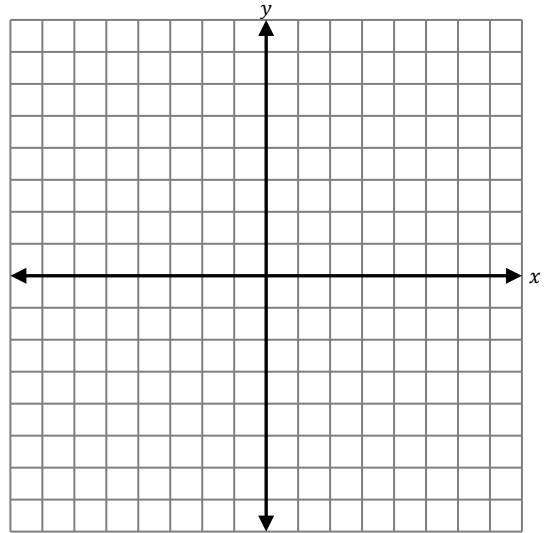


**Problem 20.6**

Graph the vertical line passing through the point (1,4).

Find lots of points on this line.


The equation of the line is \_\_\_\_\_.



The pattern that all the  $x$ -coordinates on a vertical line are the same as the  $x$ -coordinate of our original point will be true for all vertical lines. This is because to draw **any** vertical line, all we need to know is how far left or right of the  $y$ -axis to draw it.

**Equation of a Vertical Line**

An equation of the vertical line passing through the point  $(a, b)$  is

$$x = a$$

**Slope of Vertical Lines****Problem 20.7**

Find the slope of the vertical line passing through the point (2,3).

We have been calculating the slope by using two points on the line. While graphing this line in Problem 20.5, we observed there were many additional points on the line we could easily spot. Let's pick one additional point and use the point (2,3) from the problem to calculate the slope.

Using the points (2,3) and (2,5), we calculate.

$$\begin{aligned} m &= \frac{5 - 3}{2 - 2} \\ &= \frac{2}{0} \end{aligned}$$

There is a zero in the denominator, but we are not allowed to divide by zero! This means **the slope is not defined**. Sometimes people say a vertical line has **no slope**.

 **Problem 20.8**

Find the slope of the vertical line passing through the point  $(1,4)$ .

The slope is \_\_\_\_\_.



The slopes of both lines are not defined because the denominators of both fractions are zero. Recall that the denominator of the slope is the difference in the  $x$ -coordinates. On a vertical line, we saw that the  $x$ -coordinates are *always the same*—this means their difference will always be zero.

**Slope of Vertical Lines**

The slope of any vertical line is ***not defined (undefined)***. You may also say a vertical line has ***no slope***.



Many students confuse vertical lines having an *undefined slope* or (*no slope*) with having *zero* slope. The difference between zero and undefined can be a little tricky. Saying there is *no slope* means that the slope does not exist; it is not even a number. Zero is a number, and it does exist.

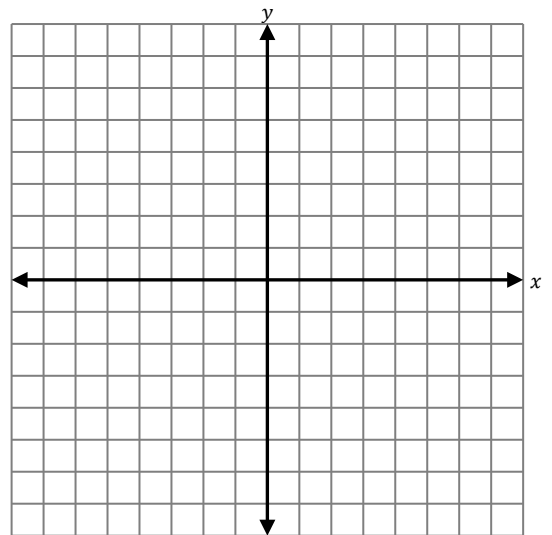
 **Problem 20.9**

Graph the vertical line passing through the point  $(-1,2)$ .

Find lots of points on this line.


The equation of the line is \_\_\_\_\_.

The slope is \_\_\_\_\_.





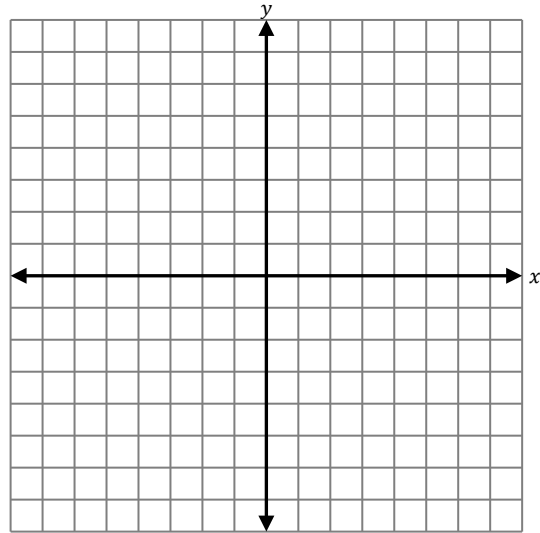
**Problem 20.10**

Graph the horizontal line passing through the point  $(-1, 2)$ .

Find lots of points on this line.


The equation of the line is \_\_\_\_\_.

The slope is \_\_\_\_\_.

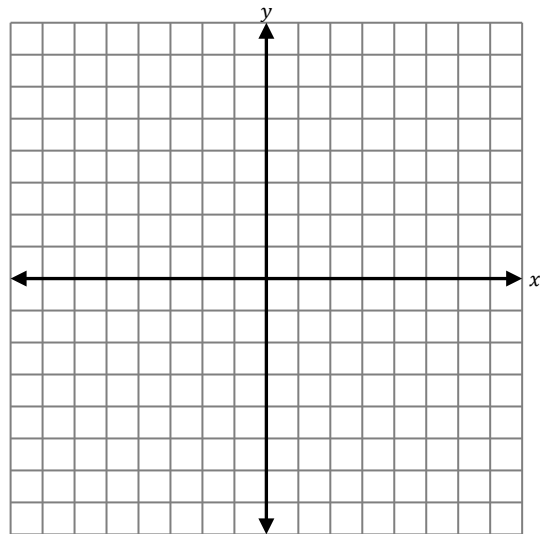
**Problem 20.11**

Graph the vertical line passing through the point  $(0, -3)$ .

Find lots of points on this line.


The equation of the line is \_\_\_\_\_.

The slope is \_\_\_\_\_.



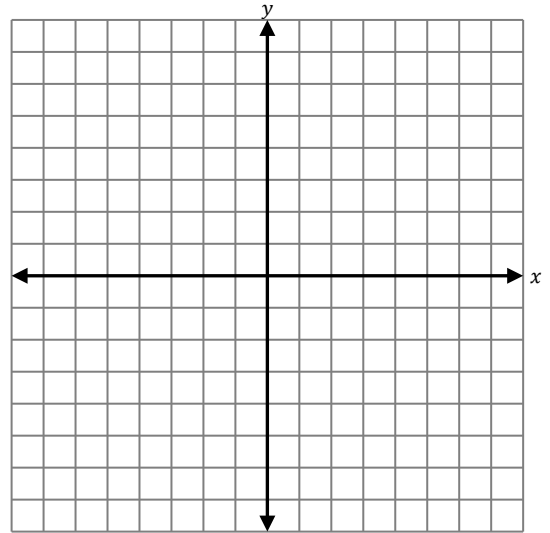
**Problem 20.12**

Graph the horizontal line passing through the point  $(-3,0)$ .

Find lots of points on this line.


The equation of the line is \_\_\_\_\_.

The slope is \_\_\_\_\_.

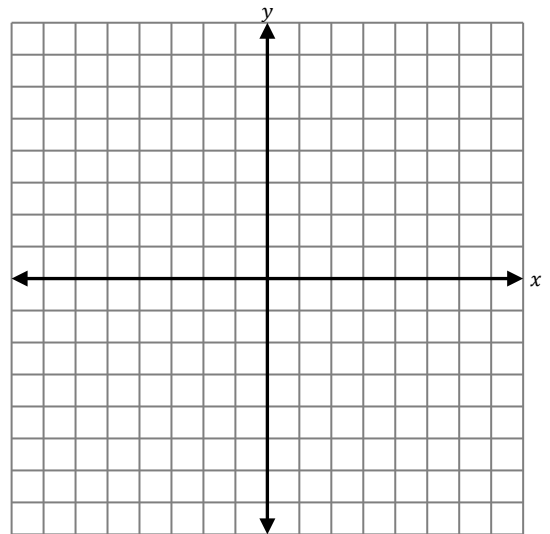


**Problem 20.13**

Graph the line with equation below.  
 $y = 3$

Find lots of points on this line.


The slope is \_\_\_\_\_.

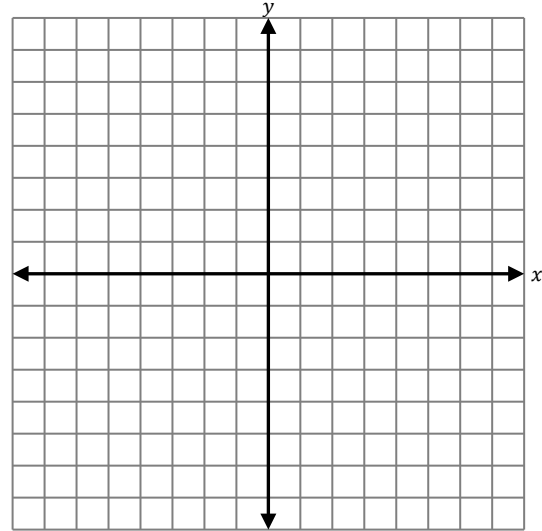


**Problem 20.14**

Graph the line with equation below.

$$x = -2$$

Find lots of points on this line.

The slope is \_\_\_\_\_.

# Systems of Linear Equations

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## GPS 21: Introduction to Systems of Linear Equations and Solving Graphically

### Introduction to Systems of Linear Equations

#### Problem 21.1

Consider the linear equation below.

$$2x + 5 = 9$$

Solve the equation.

Check your solution is correct by substituting back into the original equation.

■

Usually we present a worked problem before asking you to solve one on your own. But this problem is a review of a previous topic, and so you should know how to do it on your own. Just in case you need a refresher, we work a similar problem next.

#### Problem 21.2

Consider the linear equation below.

$$3x - 7 = 5$$

Solve the equation.

$$\begin{aligned} 3x - 7 &= 5 \\ 3x - 7 + 7 &= 5 + 7 && \text{add 7 to both sides} \\ 3x &= 12 \\ \frac{3x}{3} &= \frac{12}{3} && \text{divide both sides by 3} \\ x &= 4 \end{aligned}$$

■

Check your solution is correct by substituting back into the original equation.

$$\begin{aligned} 3x - 7 &= 5 \\ 3(4) - 7 &\neq 5 \\ 12 - 7 &\neq 5 \\ 5 &= 5 \quad \checkmark \end{aligned}$$

In both of the previous problems, we only had one linear equation with one variable. We are now going to extend these ideas to a system of linear equations. In a system of linear equations, we have more than one linear equation with more than one variable. We will see our first system of linear equations in the next problem.

In the next problem, there are two linear equations. There are two variables. The graph of each equation is a line. This is an example of a **system of linear equations**. Before we learn how to solve the system, we will start by learning how to check if a point is a solution to the system.

### Problem 21.3

Is the point (5,6) a solution to the system of equations?

$$\begin{aligned}2x - 2y &= -2 \\7x - y &= 5\end{aligned}$$

To answer this question, we must substitute the values  $x = 5$  and  $y = 6$  into *both* equations.

Substitute (5,6) into the first equation.

$$\begin{aligned}2x - 2y &= -2 \\2(5) - 2(6) &\stackrel{?}{=} -2 \\10 - 12 &\stackrel{?}{=} -2 \\-2 &= -2 \quad \checkmark\end{aligned}$$

Substitute (5,6) into the second equation.

$$\begin{aligned}7x - y &= 5 \\7(5) - 6 &\stackrel{?}{=} 5 \\35 - 6 &\stackrel{?}{=} 5 \\29 &\neq 5 \quad \times\end{aligned}$$

We see that the point (5,6) satisfies the first equation. However, it did *not* satisfy the second equation. Since it does not satisfy **both** equations, (5,6) is **not** a solution to the system of equations.

■

### Problem 21.4

Is the point  $(-2, -1)$  a solution to the system of equations?

$$\begin{aligned}-3x + y &= 5 \\6x + y &= -4\end{aligned}$$

Substitute  $(-2, -1)$  into the first equation.

Substitute  $(-2, -1)$  into the second equation.

Is the point  $(-2, -1)$  a solution to the system? Why or why not?

■

 **Problem 21.5**

Is the point  $(-1,2)$  a solution to the system of equations?

$$-3x + y = 5$$

$$6x + y = -4$$

Substitute  $(-1,2)$  into the first equation.

Substitute  $(-1,2)$  into the second equation.

Is the point  $(-1,2)$  a solution to the system? Why or why not?



The main idea of this section is summarized in the box below.

**Solution to a System of Equations**

A point that when substituted into *all* equations in a system of equations results in *all* true statements is a solution to the system of equations.

We often say that the point must *satisfy all equations* rather than the much longer sentence above.

## Solving Systems of Linear Equations Graphically

We will learn the first method for finding the solution to a system of linear equations. We spent a great deal of time learning to graph lines in previous sections, and we will now use graphing as a tool to help us find a solution. We return to the system of equations first introduced in Problem 21.3.

### Problem 21.6

Find the solution to the system of equations by graphing both lines and finding their point of intersection. Check your solution algebraically.

$$\begin{aligned} 2x - 2y &= -2 \\ 7x - y &= 5 \end{aligned}$$

In our work on lines, we recognized that the graph of each equation is a line. We will graph *both lines* on the *same* coordinate plane.

- 1) Put each equation into slope-intercept form ( $y = mx + b$ ) to make graphing easier.

$$\begin{aligned} 2x - 2y &= -2 \\ 2x - 2x - 2y &= -2x - 2 \\ -2y &= -2x - 2 \\ \frac{-2y}{-2} &= \frac{-2x}{-2} - \frac{2}{-2} \\ y &= x + 1 \end{aligned}$$

$$\begin{aligned} 7x - y &= 5 \\ 7x - 7x - y &= -7x + 5 \\ -y &= -7x + 5 \\ \frac{-y}{-1} &= \frac{-7x}{-1} + \frac{5}{-1} \\ y &= 7x - 5 \end{aligned}$$

- 2) Graph both lines on the *same* coordinate plane.  
 3) Find the point where the two lines intersect. Our two lines intersect at the point  $(1,2)$ .  
 4) Check the solution by substituting it back into both *original* equations.

#### Check:

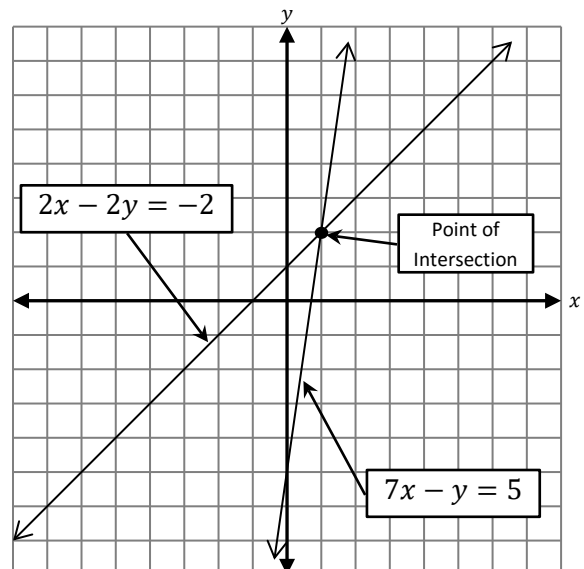
Substitute  $(1,2)$  into the first equation.

$$\begin{aligned} 2x - 2y &= -2 \\ 2(1) - 2(2) &\stackrel{?}{=} -2 \\ 2 - 4 &\stackrel{?}{=} -2 \\ -2 &= -2 \quad \checkmark \end{aligned}$$

Substitute  $(1,2)$  into the second equation.

$$\begin{aligned} 7x - y &= 5 \\ 7(1) - 2 &\stackrel{?}{=} 5 \\ 7 - 2 &\stackrel{?}{=} 5 \\ 5 &= 5 \quad \checkmark \end{aligned}$$

The solution is  $(1,2)$ .



Geometrically, finding the point of intersection of both lines (a common point on both lines) is the same as finding  $x$  and  $y$  values so that when we algebraically substitute into both equations, they are both true.

**Problem 21.7**

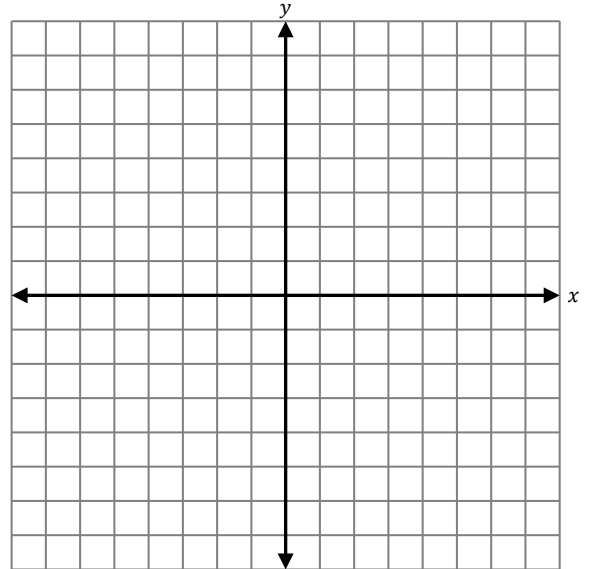
Find the solution to the system of equations by graphing both lines and finding their point of intersection. Check your solution algebraically.

$$\begin{aligned} -3x + y &= 5 \\ 6x + y &= -4 \end{aligned}$$

1) Write each equation in slope-intercept form  $y = mx + b$

$$-3x + y = 5$$

$$6x + y = -4$$



Slope is \_\_\_\_\_.  
y-intercept is \_\_\_\_\_.

Slope is \_\_\_\_\_.  
y-intercept is \_\_\_\_\_.

- 2) Graph both lines on the same coordinate plane.
- 3) The two lines intersect in the point \_\_\_\_\_.
- 4) Check that the point of intersection is a solution by substituting it into both equations.

**Check:**

Substitute into the first equation.

$$-3x + y = 5$$

Substitute into the second equation.

$$6x + y = -4$$

The solution is \_\_\_\_\_.





**Problem 21.8**

Find the solution to the system of equations by graphing both lines and finding their point of intersection. Check your solution algebraically.

$$3x + 6y = 12$$

$$6x - 2y = 10$$

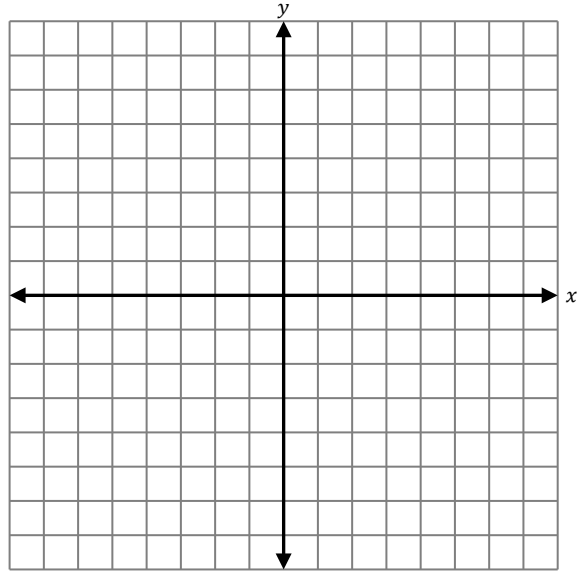
1) Write each equation in slope-intercept form  $y = mx + b$

$$3x + 6y = 12$$

$$6x - 2y = 10$$

Slope is \_\_\_\_\_.  
y-intercept is \_\_\_\_\_.

Slope is \_\_\_\_\_.  
y-intercept is \_\_\_\_\_.



2) Graph both lines on the same coordinate plane.

3) The two lines intersect in the point \_\_\_\_\_.

4) Check that the point of intersection is a solution by substituting it into both equations.

**Check:**

Substitute into the first equation.

$$3x + 6y = 12$$

Substitute into the second equation.

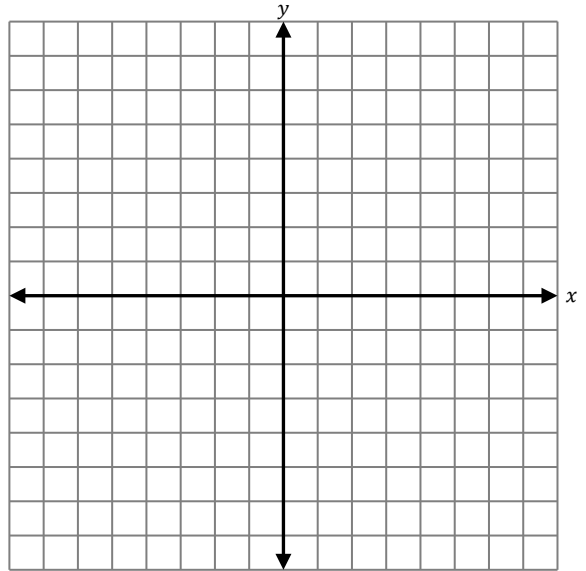
$$6x - 2y = 10$$

The solution is \_\_\_\_\_.

**Problem 21.9**

Find the solution to the system of equations by graphing both lines and finding their point of intersection. Check your solution algebraically.

$$\begin{aligned}x - y &= 6 \\x + y &= 2\end{aligned}$$



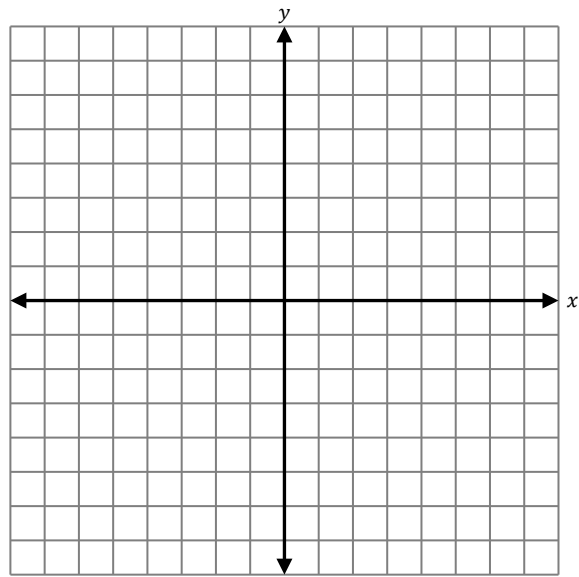
The solution is \_\_\_\_\_.

**Problem 21.10**

Find the solution to the system of equations by graphing both lines and finding their point of intersection. Check your solution algebraically.

$$3x + 2y = 6$$

$$3x + 4y = 0$$



The solution is \_\_\_\_\_.

## GPS 22: More on Solving Systems Graphically

### Problem 22.1

Find the solution to the system of equations by graphing both lines and finding their point of intersection. Check your solution algebraically.

$$3x - 2y = -8$$

$$6x - 4y = 8$$

We start by writing the equations in slope-intercept form.

$$3x - 2y = -8$$

$$3x - 3x - 2y = -3x - 8$$

$$-2y = -3x - 8$$

$$\frac{-2y}{-2} = \frac{-3x}{-2} - \frac{8}{-2}$$

$$y = \frac{3}{2}x + 4$$

$$6x - 4y = 8$$

$$6x - 6x - 4y = -6x + 8$$

$$-4y = -6x + 8$$

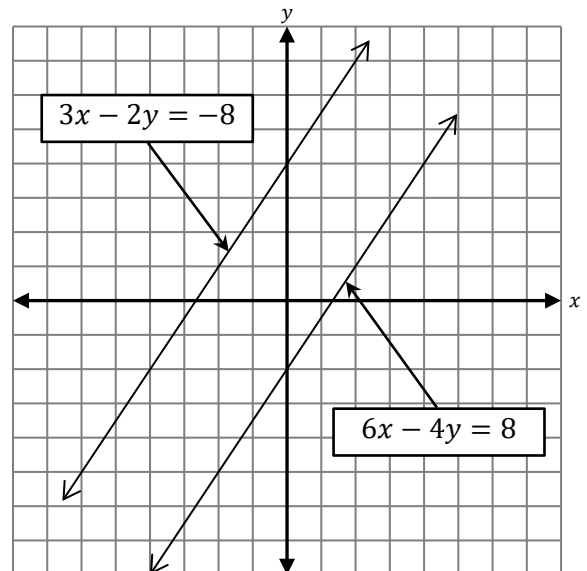
$$\frac{-4y}{-4} = \frac{-6x}{-4} + \frac{8}{-4}$$

$$y = \frac{3}{2}x - 2$$

We notice something different in this picture. There is no point where the lines intersect. We may worry that we made a mistake graphing our lines. Maybe our picture is too small, and they lines intersect far away?

Let's take a closer look. Notice the slope of each line is  $\frac{3}{2}$ . Both lines have the *same slope*. This means the lines are **parallel**, and parallel lines **never** intersect.

This system of equation has **no solution**.

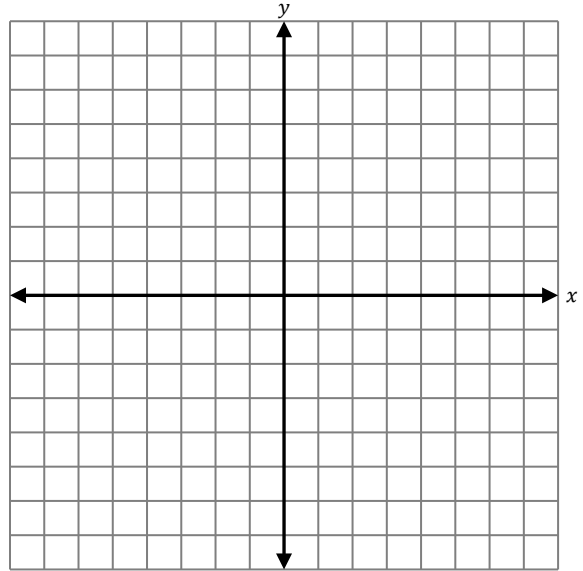


If two different lines have the same slope, then they are **parallel** and never intersect. The system of linear equations has **no solution**. A system of linear equations with no solution is sometimes called **inconsistent**.

**Problem 22.2**

Find the solution to the system of equations by graphing both lines and finding their point of intersection. Check your solution algebraically.

$$\begin{aligned}3x + y &= 3 \\6x + 2y &= -8\end{aligned}$$

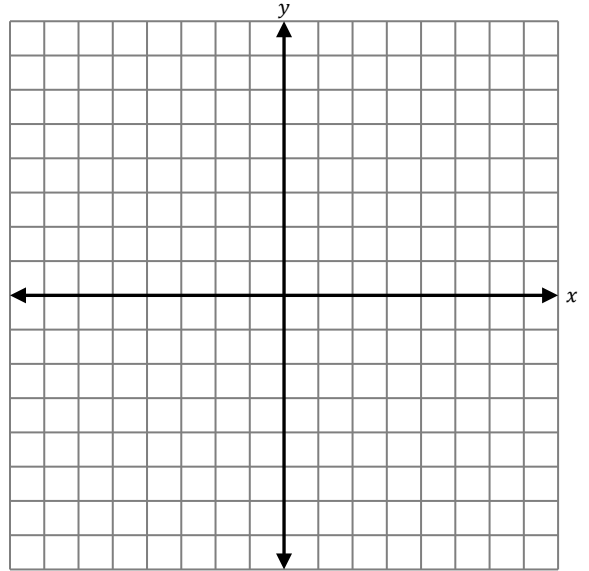


The solution is \_\_\_\_\_.

**Problem 22.3**

Find the solution to the system of equations by graphing both lines and finding their point of intersection. Check your solution algebraically.

$$\begin{aligned}2x - 6y &= -6 \\ -3x + 9y &= 36\end{aligned}$$



The solution is \_\_\_\_\_.

**Problem 22.4**

Find the solution to the system of equations by graphing both lines and finding their point of intersection. Check your solution algebraically.

$$\begin{aligned} 3x - y &= -2 \\ -6x + 2y &= 4 \end{aligned}$$

We start by writing the equations in slope-intercept form.

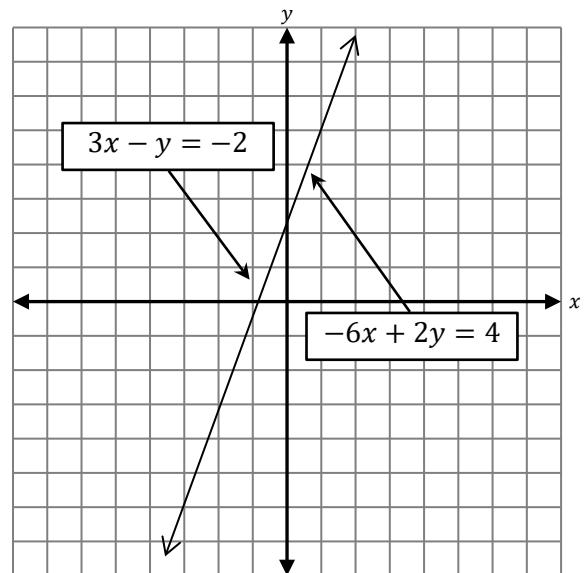
$$\begin{aligned} 3x - y &= -2 \\ 3x - 3x - y &= -3x - 2 \\ -y &= -3x - 2 \\ \frac{-y}{-1} &= \frac{-3x}{-1} - \frac{2}{-1} \\ y &= 3x + 2 \end{aligned}$$

$$\begin{aligned} -6x + 2y &= 4 \\ -6x + 6x + 2y &= 6x + 4 \\ 2y &= 6x + 4 \\ \frac{2y}{2} &= \frac{6x}{2} + \frac{4}{2} \\ y &= 3x + 2 \end{aligned}$$

We notice something different in this picture. We thought we were graphing two *different* lines, but it turns out that there is really *only one line*! Since a solution to a system is a point on both lines and there is only one line, every point on the line is a solution to the system.

This system of equation has ***infinitely many solutions***.

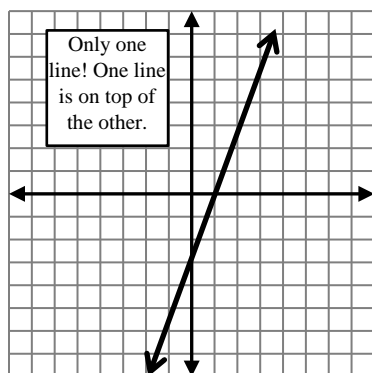
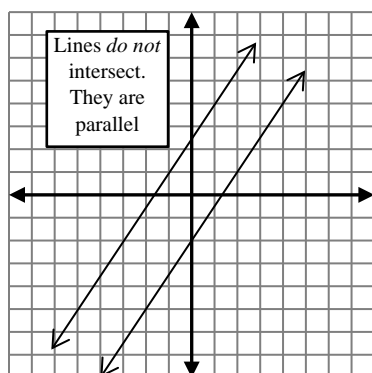
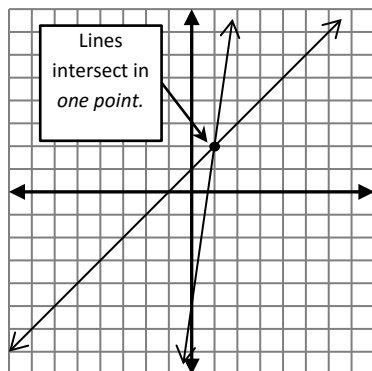
Look at the slope and  $y$ -intercept of each line. Notice that they have the same slope and the same  $y$ -intercept. When our lines were parallel in Problem 22.1, they also had the *same slope*, but their  $y$ -intercepts were *different*.



If the two equations are actually the same line (same slope *and*  $y$ -intercept), then the system has ***infinitely many solutions***; any point on the line is a solution. A system of equations with infinitely many solutions is sometimes called ***dependent***.

We have seen there are three different possibilities for systems of two linear equations. We take a moment to summarize them below.

### How the graph looks



### What it means

#### Unique Solution

This is the most common. The two lines intersect in a single point. The coordinates of this point is the one and only solution to the system of equations.

#### No Solution

If two different lines have the same slope, then they are **parallel** and never intersect. The system of linear equations has **no solution**. A system of linear equations with no solution is sometimes called **inconsistent**.

#### Infinitely Many Solutions

If the two equations are actually the same line (same slope *and* *y*-intercept), then the system has **infinitely many solutions**; any point on the line is a solution. A system of equations with infinitely many solutions is sometimes called **dependent**.

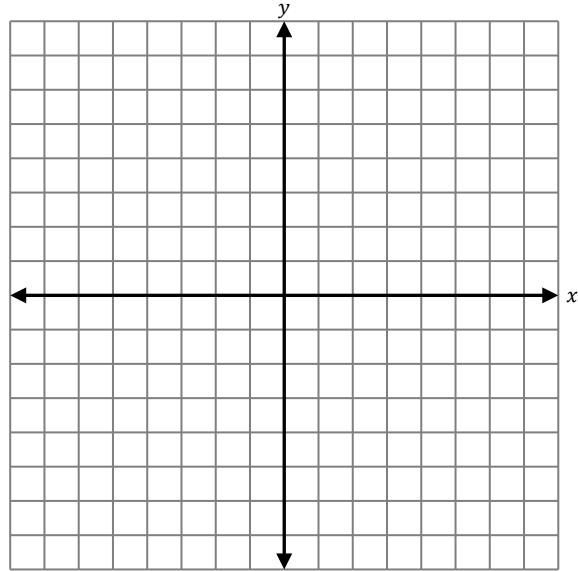


**Problem 22.5**

Find the solution to the system of equations by graphing both lines and finding their point of intersection. Check your solution algebraically.

$$4x + 2y = 8$$

$$6x + 3y = 12$$

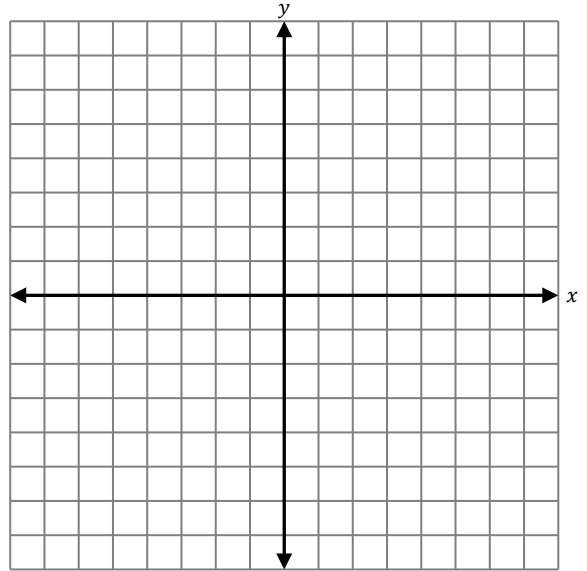


The solution is \_\_\_\_\_.

**Problem 22.6**

Find the solution to the system of equations by graphing both lines and finding their point of intersection. Check your solution algebraically.

$$\begin{aligned}4x + y &= 10 \\ 2x - 3y &= 12\end{aligned}$$



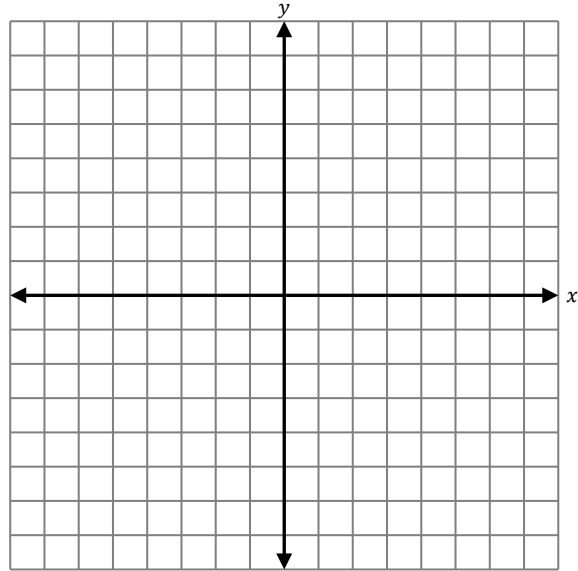
The solution is \_\_\_\_\_.

**Problem 22.7**

Find the solution to the system of equations by graphing both lines and finding their point of intersection. Check your solution algebraically.

$$2x - 3y = -18$$

$$2x + 3y = 6$$

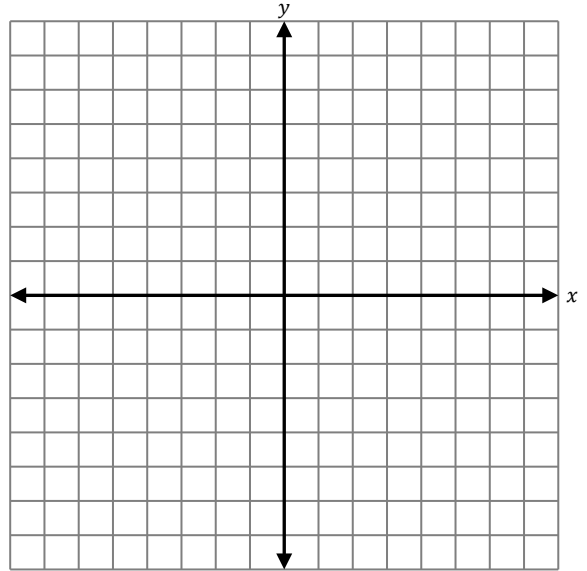


The solution is \_\_\_\_\_.

**Problem 22.8**

Find the solution to the system of equations by graphing both lines and finding their point of intersection. Check your solution algebraically.

$$\begin{aligned}3x - 2y &= -4 \\ -6x + 4y &= -8\end{aligned}$$



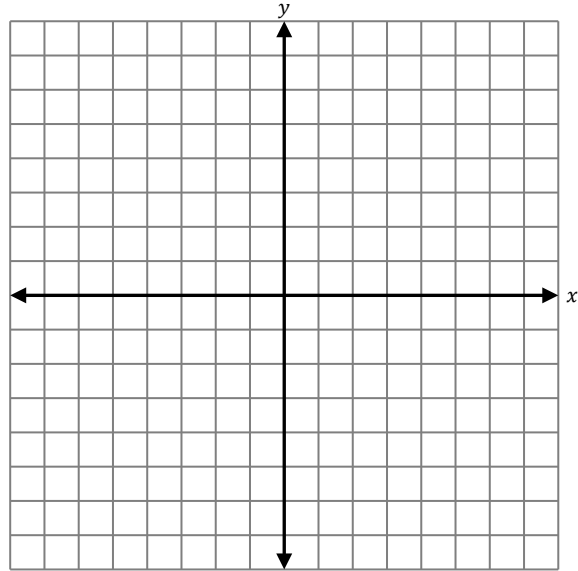
The solution is \_\_\_\_\_.

**Problem 22.9**

Find the solution to the system of equations by graphing both lines and finding their point of intersection. Check your solution algebraically.

$$-3x + y = 1$$

$$4x + y = -6$$

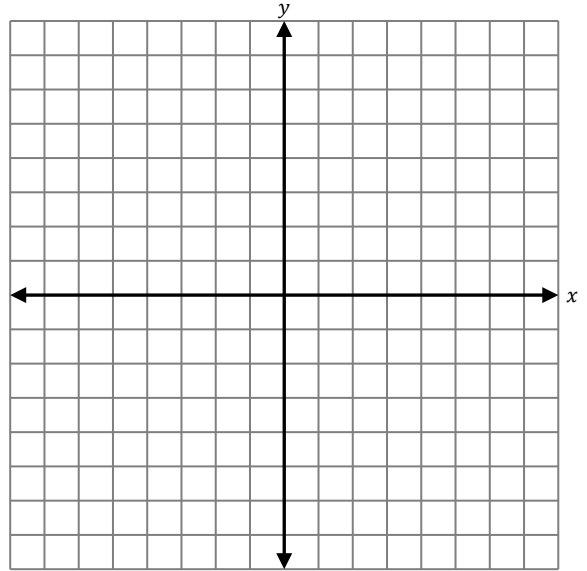


The solution is \_\_\_\_\_.

**Problem 22.10**

Find the solution to the system of equations by graphing both lines and finding their point of intersection. Check your solution algebraically.

$$\begin{aligned}x - 3y &= -6 \\ y &= \frac{1}{3}x + 2\end{aligned}$$



The solution is \_\_\_\_\_.



## GPS 23: Solving Systems of Linear Equations Algebraically

In the previous section, we learned to find a solution to a system of equations by graphing both lines on the same coordinate plane and looking for a point of intersection. While this method was visually nice, there are times when drawing a picture is just not realistic. In these cases, we must use an alternate method of solving a system. The method we are going to learn is often called the ***addition method***. Sometimes this method is also referred to as the ***elimination method***.

One difficulty with a system of equations is that there are too many variables and too many equations. We will see how we can use a technique to reduce our two equations with two variables to only one equation with only one variable.

We introduce you to the addition method by way of the next problem. You will see soon why it is called the addition method.



**Problem 23.1**

Use the addition method to solve the system of equations.

$$4x + 5y = 24 \quad (\text{Eq. 1})$$

$$6x + 7y = 34 \quad (\text{Eq. 2})$$

**Goal 1:** Eliminate one variable, then solve for the remaining variable.

We will accomplish our goal of eliminating one variable by multiplying each equation by a (nonzero) integer so that the coefficients of the  $x$ 's are *opposites*. In Eq. 1 the coefficient of  $x$  is 4. In Eq. 2, the coefficient of  $x$  is 6. The (least) common multiple of 4 and 6 is 12. If we multiply 4 by  $-3$ , the result will be  $-12$ . If we multiply 6 by 2, the result will be 12. Let's proceed to see how this helps.

We start by multiplying every term in Eq. 1 by  $-3$ .

$$4x + 5y = 24 \quad (\text{Eq. 1})$$

$$(-3)(4x) + (-3)(5y) = (-3)(24)$$

$$-12x - 15y = -72 \quad (\text{Eq. 3})$$

This gives us a new version of Eq. 1 which we have labeled Eq. 3. Notice that the coefficient of  $x$  is now  $-12$ . We now multiply every term in Eq. 2 by 2.

$$6x + 7y = 34 \quad (\text{Eq. 2})$$

$$(2)(6x) + (2)(7y) = (2)(34)$$

$$12x + 14y = 68 \quad (\text{Eq. 4})$$

This gives us a new version of Eq. 2 which we have labeled Eq. 4. Notice that the coefficient of  $x$  is now 12. We will now add Eq. 3. and Eq. 4.

$$-12x - 15y = -72 \quad (\text{Eq. 3})$$

$$+ \quad 12x + 14y = 68 \quad (\text{Eq. 4})$$

$$\hline -y = -4 \quad (\text{Eq. 5})$$

$$y = 4$$

Notice that Eq. 5 (which we attained by adding Eq. 3 and Eq. 4) has only one variable and we were able to easily solve it to determine  $y = 4$ .

**Goal 2:** Find the other variable.

Now that we know the value of  $y$ , we substitute  $y = 4$  into *either* of our first two *original equations*. We will demonstrate how you can substitute into either equation, but ordinarily you only need to do one or the other.

Substitute  $y = 4$  into Eq. 1

$$4x + 5y = 24 \quad (\text{Eq. 1})$$

$$4x + 5(4) = 24$$

$$4x + 20 = 24 \quad (\text{Eq. 6})$$

$$4x = 4$$

$$x = 1$$

Substitute  $y = 4$  into Eq. 2

$$6x + 7y = 34 \quad (\text{Eq. 2})$$

$$6x + 7(4) = 34$$

$$6x + 28 = 34 \quad (\text{Eq. 7})$$

$$6x = 6$$

$$x = 1$$

Once we substituted  $y = 4$  into Eq. 1, the new Eq. 6. had only  $x$ , and it was easy to solve to see that  $x = 1$ .

Similarly, we were able to substitute  $y = 4$  into Eq. 2 and get Eq 7 with only  $x$ , and it was easy to solve to see  $x = 1$ .

At this point, we hope we have reached our solution  $x = 1, y = 4$  which we write as an ordered pair  $(1,4)$ . But before moving on, we will check that we have the correct solution by substituting  $(1,4)$  into **both original equations**.

**Goal 3:** Check the answer is correct!

**Check:**

Substitute  $(1,4)$  into Eq. 1.

$$\begin{aligned} 4x + 5y &= 24 \\ 4(1) + 5(4) &\neq 24 \\ 4 + 20 &\neq 24 \\ 24 &= 24 \quad \checkmark \end{aligned}$$

Our solution is correct. The solution is  $(1,4)$ .

■

Substitute  $(1,4)$  into Eq. 2.

$$\begin{aligned} 6x + 7y &= 34 \\ 6(1) + 7(4) &\neq 34 \\ 6 + 28 &\neq 34 \\ 34 &= 34 \quad \checkmark \end{aligned}$$



**Always check your solution in both original equations!**

Solving a system of equations requires more steps than any other topic we will learn about all semester. This means there are even more opportunities to make mistakes. But by checking your answer every time, you can learn to avoid costly mistakes.

Before asking you to do one completely on your own, we will guide you through the procedure.

 **Problem 23.2**

Use the addition method to solve the system of equations.

$$3x + 7y = 26 \quad (\text{Eq. 1})$$

$$5x + 4y = 28 \quad (\text{Eq. 2})$$

We will guide you through solving your first system of equations algebraically.

**Goal 1:** Eliminate one variable, then solve for the remaining variable.

- 1) You must pick one of the variables  $x$  or  $y$  to focus on. Let's pick  $x$  so that we can work together.
- 2) You must determine the coefficient of  $x$  in each equation. Fill in the blanks below.
  - a. The coefficient of  $x$  in Eq. 1 is \_\_\_\_\_.
  - b. The coefficient of  $x$  in Eq. 2 is \_\_\_\_\_.
- 3) Find the (least) common multiple of the coefficients from the previous step. Recall that finding the (least) common multiple was used when finding a common denominator to add fractions. We also found this when we cleared denominators in previous sections. If you can't easily find the least common multiple, you can always just multiply the two numbers together and use this multiple.

The (least) common multiple is \_\_\_\_\_. The opposite value is \_\_\_\_\_.

- 4) We must now determine by what you will multiply each of the coefficients from Step 2 so that the product is the common multiple, or its opposite, from Step 3. Additionally, you must ensure that one of the products is positive and one is negative.
  - a. The coefficient of  $x$  in Eq. 1 is \_\_\_\_\_ (same blank from Step 2a). If I multiply it by \_\_\_\_\_, the result will be \_\_\_\_\_ (same blank from Step 3, or the opposite).
  - b. The coefficient of  $x$  in Eq. 2 is \_\_\_\_\_ (same blank from Step 2b). If I multiply it by \_\_\_\_\_, the result will be \_\_\_\_\_ (same blank from Step 3, or the opposite).



Before moving on, check again. Did you make sure you chose numbers to multiply by so that the results are **opposites**?

- 5) Multiply every term of each equation by the numbers you determined in Step 4. We put parentheses in to help you. All 3 parentheses in each *row* must be the *same* number.

$$( \quad )(3x) + ( \quad )(7y) = ( \quad )(26) \quad (\text{Eq. 1})$$

$$( \quad )(5x) + ( \quad )(4y) = ( \quad )(28) \quad (\text{Eq. 2})$$

Simplify each of the equations to get two new equations. Write them below being careful to line up the variables.

- 6) Add the two equations together. If you successfully made sure the coefficients of  $x$  were opposites, then  $x$  should now be *eliminated*!

- 7) Solve the equation from Step 6 to determine the  $y$  value.

$$y = \underline{\hspace{2cm}}$$

**Goal 2:** Find the other variable.

- 8) Substitute the  $y$  value from Step 7 into either of the original equations. So you can see it doesn't matter which you choose, there is space below for you to substitute into both equations. Usually you only do one or the other, however.

Substitute  $y = \underline{\hspace{2cm}}$  into Eq. 1. Fill in the  $y$  value in the parentheses to get started. Then solve the equation for  $x$ .

$$3x + 7y = 26$$

$$3x + 7(\ ) = 26$$

$$x = \underline{\hspace{2cm}}$$

Substitute  $y = \underline{\hspace{2cm}}$  into Eq. 2. Fill in the  $y$  value in the parentheses to get started. Then solve the equation for  $x$ .

$$5x + 4y = 28$$

$$5x + 4(\ ) = 28$$

$$x = \underline{\hspace{2cm}}$$

You should have found the same  $x$  value no matter which equation you chose.

**Goal 3:** Check the answer is correct!

- 9) We must now check our solution by substituting both  $x$  and  $y$  into **both original equations**.

We think our solution is  $x = \underline{\hspace{2cm}}$  and  $y = \underline{\hspace{2cm}}$ .

Substitute both values into Eq. 1.

$$3x + 7y = 26$$

$$3(\ ) + 7(\ ) \neq 26$$

Substitute both values into Eq. 2.

$$5x + 4y = 28$$

$$5(\ ) + 4(\ ) \neq 28$$

Write the solution as an ordered pair. The solution is  $\underline{\hspace{2cm}}$ .



If both equations didn't check in Step 9, then you made a mistake, and you must correct it. The first place to look for a mistake is in your arithmetic in Step 9. Make sure you substituted and calculated correctly! If you didn't make a mistake in Step 9, then take a look at previous steps. Step 4 is another common place students make mistakes, and we have issued a warning during that step to help you avoid it. Sometimes if you can't find the mistake in a reasonable time, it is best to turn to a *new* page and start again. *Don't erase your incorrect solution* because you can often find your mistake later after you work it correctly or after getting help from someone. It is always helpful to determine where the mistake was so you can avoid doing it again.

We summarize the procedure we just followed here.

### The Addition Method for Solving a System of Linear Equations

**Goal 1:** Eliminate one variable, then solve for the remaining variable.

- 1) Pick one of the variables you would like to eliminate. You may use whichever variable you would like. Sometimes one variable is easier than the other.
- 2) Determine the coefficients of the variable you picked.
- 3) Find the (least) common multiple of the coefficients.
- 4) Determine what you must multiply each coefficient by in order to ensure the products are **opposites**. This means the products must have *opposite* signs after you multiply—there should be one negative and one positive coefficient!
- 5) Multiply each term of the equations by the numbers you determined in the previous step. After simplifying the coefficients of the variable you picked, the new coefficients should be **opposites**.
- 6) Add the new equations together. If you successfully made sure the coefficients of your variable were opposites, then that variable should now be *eliminated*! There should only be one variable remaining.
- 7) Solve the new equation for the remaining variable.

**Goal 2:** Find the other variable.

- 8) Substitute the value you found in the previous step into either of the original equations. Solve this equation for the remaining variable.

**Goal 3:** Check the answer is correct!

- 9) Check your solution by substituting into both original equations.

Try one completely on your own.

 **Problem 23.3**

Use the addition method to solve the system of equations.

$$-3x + 5y = 19$$

$$9x - 2y = 8$$

The solution is \_\_\_\_\_.

 **Problem 23.4**

Use the addition method to solve the system of equations.

$$2x + y = -1$$

$$7x - 3y = -23$$

The solution is \_\_\_\_\_.

## GPS 24: More on Solving Systems Algebraically

### Problem 24.1

Use the addition method to solve the system of equations.

$$2x + 3y = 11 \quad (\text{Eq. 1})$$

$$4x + 6y = 22 \quad (\text{Eq. 2})$$

We start by multiplying every term in Eq. 1 by  $-2$ .

$$2x + 3y = 11 \quad (\text{Eq. 1})$$

$$(-2)(2x) + (-2)(3y) = (-2)(11)$$

$$-4x - 6y = -22 \quad (\text{Eq. 3})$$

This gives us a new version of Eq. 1 which we have labeled Eq. 3. We don't have to multiply Eq. 2 by anything (or you can say we multiplied by 1 which changes nothing).

We will now add Eq. 3. and Eq. 2.

$$\begin{array}{r} -4x - 6y = -22 \quad (\text{Eq. 3}) \\ + \quad 4x + 6y = 22 \quad (\text{Eq. 2}) \\ \hline 0 = 0 \end{array}$$

**Solution:** We have with a **true** statement  $0 = 0$ . This means there are *infinitely many solutions*, and the system is said to be *dependent*.

**Explanation:** Something different happened here. **Both** variables were eliminated! The good news is  $0 = 0$  is a true statement. Let's see what happens if we try the graphical method of solving this system. We start by putting each equation into slope-intercept form ( $y = mx + b$ ).

$$\begin{array}{l} 2x + 3y = 11 \\ 2x - 2x + 3y = -2x + 11 \\ 3y = -2x + 11 \\ \frac{3y}{3} = \frac{-2x}{3} + \frac{11}{3} \\ y = -\frac{2}{3}x + \frac{11}{3} \end{array}$$

$$\begin{array}{l} 4x + 6y = 22 \\ 4x - 4x + 6y = -4x + 22 \\ 6y = -4x + 22 \\ \frac{6y}{6} = \frac{-4x}{6} + \frac{22}{6} \\ y = -\frac{2}{3}x + \frac{11}{3} \end{array}$$

We see that both lines have the *same slope* **and** the *same y-intercept*. We have seen this before. Recall this meant we actually have only one line instead of two. In this case, we said there are infinitely many solutions.

■

### Problem 24.2

Use the addition method to solve the system of equations.

$$3x + 5y = -13$$

$$9x + 15y = -39$$

The solution is \_\_\_\_\_.



**Problem 24.3**

Use the addition method to solve the system of equations.

$$4x + 6y = 22 \quad (\text{Eq. 1})$$

$$2x + 3y = 5 \quad (\text{Eq. 2})$$

We start by multiplying every term in Eq. 2 by  $-2$ .

$$2x + 3y = 5 \quad (\text{Eq. 2})$$

$$(-2)(2x) + (-2)(3y) = (-2)(5)$$

$$-4x - 6y = -10 \quad (\text{Eq. 3})$$

This gives us a new version of Eq. 2 which we have labeled Eq. 3. We don't have to multiply Eq. 1 by anything.

We will now add Eq. 1. to Eq. 3.

$$\begin{array}{r} 4x + 6y = 22 \quad (\text{Eq. 1}) \\ + \quad -4x - 6y = -10 \quad (\text{Eq. 3}) \\ \hline 0 \neq 12 \end{array}$$

**Solution:** We have with a **false** statement  $0 = 12$ . This means there are *no solutions*, and the system is said to be *inconsistent*.

**Explanation:** Something different happened here. Again, both variables were eliminated! The bad news is we are left with a false statement since 0 does not equal 12. Let's try the graphical method of solving this system.

We start by putting each equation into slope-intercept form ( $y = mx + b$ ).

$4x + 6y = 22$ $4x - 4x + 6y = -4x + 22$ $6y = -4x + 22$ $\frac{6y}{6} = \frac{-4x}{6} + \frac{22}{6}$ $y = -\frac{2}{3}x + \frac{11}{3}$		$2x + 3y = 5$ $2x - 2x + 3y = -2x + 5$ $3y = -2x + 5$ $\frac{3y}{3} = \frac{-2x}{3} + \frac{5}{3}$ $y = -\frac{2}{3}x + \frac{5}{3}$
---	--	--

We see that both lines have the *same slope* **but** they have *different y-intercepts*. Recall, this means we have two *parallel* lines that never intersect. In this case, there is *no solution*, and the system is said to be *inconsistent*.

■

**Problem 24.4**

Use the addition method to solve the system of equations.

$$10x + 12y = -1$$

$$15x + 18y = 4$$

The solution is \_\_\_\_\_.

■

**What does it mean when both variables are eliminated?**

In the last few problems we saw an unusual occurrence when both variables were eliminated after adding the equations. We highlight what this meant.

- 1) **When we are left with a true statement, usually  $0 = 0$ :**  
This means the two lines are actually the same line. The system has *infinitely many solutions*, and we say the system is *dependent*.
- 2) **When we are left with a false statement, such as  $0 = 12$ :**  
This means the two lines are parallel and never intersect. The system has *no solution*, and we say the system is *inconsistent*.

 **Problem 24.5**

Use the addition method to solve the system of equations.

$$2x + 5y = 32$$

$$x - y = 2$$

The solution is \_\_\_\_\_.

 **Problem 24.6**

Use the addition method to solve the system of equations.

$$3x + 4y = -16$$

$$9x + 2y = -38$$

The solution is \_\_\_\_\_.

 **Problem 24.7**

Use the addition method to solve the system of equations.

$$\begin{aligned}x + 2y &= 5 \\3x + 6y &= 15\end{aligned}$$

The solution is \_\_\_\_\_.

 **Problem 24.8**

Use the addition method to solve the system of equations.

$$-6x + 5y = 48$$

$$9x + 2y = -15$$

The solution is \_\_\_\_\_.

 **Problem 24.9**

Use the addition method to solve the system of equations.

$$12x + 16y = -1$$

$$9x + 12y = 9$$

The solution is \_\_\_\_\_.

** Problem 24.10**

Use the addition method to solve the system of equations.

$$-3x + 10y = 25$$

$$2x - 7y = -18$$

The solution is \_\_\_\_\_.

 **Problem 24.11**

Use the addition method to solve the system of equations.

$$3x + 5y = -1$$

$$2x - 11y = 28$$

The solution is \_\_\_\_\_.



 **Problem 24.12**

Use the addition method to solve the system of equations.

$$\begin{aligned}7x - 4y &= 16 \\ -3x + 2y &= -6\end{aligned}$$

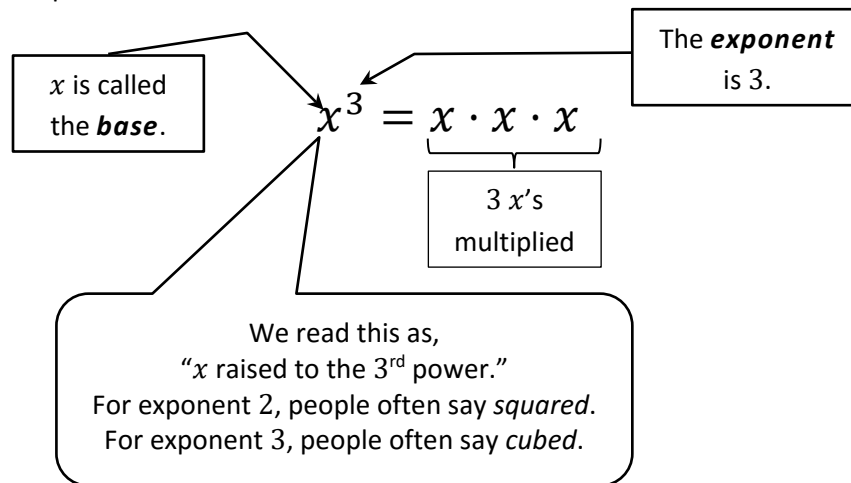
The solution is \_\_\_\_\_.

# Exponents

## GPS 25: Rules of Exponents, Part 1

### The Product Rule

Recall some basics of exponents.



### Problem 25.1

Multiply.

$$(x^2)(x^3)$$

2 $x$ 's	3 $x$ 's
----------	----------

$$(x^2)(x^3) = \underbrace{(x \cdot x)}_{2 \text{ } x\text{'s}} \underbrace{(x \cdot x \cdot x)}_{3 \text{ } x\text{'s}} = x^5$$

5 $x$ 's <b>total</b>
-----------------------

In simplified form,  $(x^2)(x^3) = x^5$ .

■

### Problem 25.2

Multiply.

$$(y^5)(y^4)$$

■

### Problem 25.3

Multiply.

$$(a^9)(a^6)$$

■

After working these problems, we observe a pattern.

**The Product Rule**

To multiply two expressions with the same base, keep the same base and add the exponents.

$$x^n \cdot x^m = x^{n+m}$$

In the next problem there are coefficients in front of the variables.

**Problem 25.4**

Multiply.

$$(3x^5)(6x^7)$$

We rearrange the factors being multiplied.

$$\begin{aligned} (3x^5)(6x^7) &= (3 \cdot 6)(x^5 \cdot x^7) \\ &= 18x^{5+7} \\ &= 18x^{12} \end{aligned}$$

In simplified form,  $(3x^5)(6x^7) = 18x^{12}$

■

 **Problem 25.5**

Multiply.

$$(8b^2)(7b^9)$$

 **Problem 25.6**

Multiply.

$$(-5x^3)(6x^9)$$

■

■

It is no more difficult when there are more variables in the problem; simply group each variable separately.

**Problem 25.7**

Multiply.

$$(2x^5y^4)(9x^7y^3)(3xy^8)$$

We rearrange the factors being multiplied.

$$\begin{aligned} (2x^5y^4)(9x^7y^3)(3xy^8) &= (2 \cdot 9 \cdot 3)(x^5 \cdot x^7 \cdot x)(y^4 \cdot y^3 \cdot y^8) \\ &= 54x^{5+7+1}y^{4+3+8} \\ &= 54x^{13}y^{15} \end{aligned}$$

In simplified form,  $(2x^5y^4)(9x^7y^3)(3xy^8) = 54x^{13}y^{15}$ .

■

In the last problem, we made use of a fact we have not mentioned yet.



When no exponent is written, it is understood to be 1. When we write  $x$  without an exponent, we can see that there is *one*  $x$ , and so this makes sense.

$$x = x^1$$

 **Problem 25.8**

Multiply.

$$(6wx^9z^2)(2w^4x^6z^5)(-3w^3x^8z^2)$$



 **Problem 25.9**

Multiply.

$$(2a^5b)(-4b^2c^3)(5ac^4)$$



## The Quotient Rule

We can use the principles we applied to reducing fractions to work with exponents.



When reducing fractions, recall that we are actually creating 1's in place of our canceled factors since dividing a (nonzero) number by itself is 1.

Example:

$$\begin{aligned} \frac{12}{18} &= \frac{2 \cdot 2 \cdot 3}{2 \cdot 3 \cdot 3} = \frac{\overset{1}{\cancel{2}} \cdot 2 \cdot \overset{1}{\cancel{3}}}{\cancel{2} \cdot \cancel{3} \cdot 3} \\ &= \frac{1 \cdot 2 \cdot 1}{1 \cdot 1 \cdot 3} = \frac{2}{3} \end{aligned}$$

### Problem 25.10

Simplify.

$$\frac{a^5}{a^3}$$

By writing out the meaning of the exponents, we see that we have factors that are in the numerator and the denominator of our fraction. We recall from reducing fractions that these factors cancel and equal 1.

$$\frac{a^5}{a^3} = \frac{a \cdot a \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a}}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}} = \frac{a^2}{1} = a^2$$

In simplified form,  $\frac{a^5}{a^3} = a^2$ .



When there are coefficients, you simply reduce them as you would any other fraction.

### Problem 25.11

Simplify.

$$\frac{6y^7}{3y^4}$$



### Problem 25.12

Simplify.

$$\frac{12x^4y^3}{3xy^2}$$



In the next problem, there are more factors in the denominator than the numerator.

**Problem 25.13**

Simplify.

$$\frac{a^2}{a^5}$$

By again, writing out the meaning of the exponents, we see that we have factors that are in the numerator and the denominator of our fraction. We recall from reducing fractions that these factors cancel and equal 1.

$$\frac{a^2}{a^5} = \frac{\cancel{a} \cdot \cancel{a}}{\cancel{a} \cdot \cancel{a} \cdot a \cdot a \cdot a} = \frac{1}{a^3}$$

In simplified form,  $\frac{a^2}{a^5} = \frac{1}{a^3}$ .



When there is more than one variable, there can also be variables left in the numerator and denominator of our fractions.

**Problem 25.14**

Simplify.

$$\frac{16x^4y}{-64x^2y^4}$$

**Problem 25.15**

Simplify.

$$\frac{18a^7b}{8a^2b^3}$$



The patterns we have observed in the above problems are summarized below.

**The Quotient Rule**

To divide two expressions with the same base, keep the same base and subtract the exponents.

$$\frac{x^n}{x^m} = x^{n-m} \quad \text{if } n > m \quad \text{and} \quad \frac{x^n}{x^m} = \frac{1}{x^{m-n}} \quad \text{if } n < m$$

**Note:** We are not allowed to divide by 0. As a result, anytime we write an expression with a variable in the denominator, we will assume it is not zero or else our expression wouldn't make sense.

We can combine what we learned with the product rule to do problems that require a few more steps and increased care.

** Problem 25.16**

Simplify.

$$\frac{(8y^2)(2y)}{-4y^5}$$

** Problem 25.17**

Simplify.

$$\frac{(6x^3y^2)(3y)}{-15xy^5}$$

** Problem 25.18**

Simplify.

$$\frac{(-15x^7y^6z)(2y^6z^3)}{-25xy^5z^6}$$

** Problem 25.19**

Simplify.

$$\frac{(4ab)(-9a^{12}b)}{(-30a^6b^4)(2a^4b^7)}$$

The Quotient Rule actually helps us understand a few special cases of exponents that often confuse students.

### Zero Exponent Rule

In this next problem, everything cancels.

#### Problem 25.20

Simplify.

$$\frac{a^2}{a^2}$$

We already know the answer is 1, of course, since everything cancels. But we can apply the Quotient Rule to this problem and observe the following.

$$\frac{a^2}{a^2} = a^0$$

But we recall when canceling fractions, we are actually creating 1's.

$$\frac{a^2}{a^2} = \frac{1}{1} = 1$$

What we have just determined is that we should interpret  $a^0 = 1$ .

■

#### The Zero Exponent Rule

Every base (except 0) raised to the 0 power is 1.

$$x^0 = 1 \quad x \neq 0$$

$0^0$  is undefined since we can't divide by 0 as we did above.

#### Problem 25.21

Simplify.

$$\frac{x^4y}{x^4y^4}$$

■

#### Problem 25.22

Simplify.

$$\frac{24a^5bc^0}{18a^7b^5}$$

■





## GPS 26: Rules of Exponents, Part 2

### Negative Exponent Rule

We know that when we subtract a bigger number from a smaller number, the result is negative. Let's see how this works with exponents in the next problem.

#### Problem 26.1

Simplify.

$$\frac{a^3}{a^5}$$

We already know the answer, of course.

$$\frac{a^3}{a^5} = \frac{1}{a^2}$$

But instead of doing this, let's apply the Quotient Rule.

$$\frac{a^3}{a^5} = a^{3-5} = a^{-2}$$

We have just determined how we should interpret the negative exponent.

$$a^{-2} = \frac{1}{a^2}$$

■

We can do a similar procedure to determine how we should interpret negative exponents in the denominator.

#### The Negative Exponent Rule

For every nonzero base,

$$x^{-n} = \frac{1}{x^n} \quad \text{and} \quad \frac{1}{x^{-n}} = x^n$$

Basically the Negative Exponent Rule indicates that a negative exponent means we can move the factor from the numerator to the denominator (or from the denominator to the numerator) and change the exponent to its opposite.

#### Problem 26.2

Simplify. Write expressions with only positive exponents.

$$\frac{-2x^{-3}y^4}{3z^{-5}}$$

Negative exponents can be eliminated by moving the factor and taking the opposite exponent. The positive exponents don't move. Also, the exponents aren't on the coefficients so they remain where they are, even though one coefficient is negative.

$$\frac{-2x^{-3}y^4}{3z^{-5}} = \frac{-2y^4z^5}{3x^3}$$

**Problem 26.3**

Simplify. Write expressions with only positive exponents.

$$z^{-2}$$

**Problem 26.4**

Simplify. Write expressions with only positive exponents.

$$\frac{a^{-3}}{4}$$

**Problem 26.5**

Simplify. Write expressions with only positive exponents.

$$z^6 z^{-9}$$

**Problem 26.6**

Simplify. Write expressions with only positive exponents.

$$\frac{9x^{-9}y^{-3}}{6z^{-7}}$$

**Problem 26.7**

Simplify. Write expressions with only positive exponents.

$$\frac{4a^3 b^{-6}}{8a^{-8} b^{-5}}$$

**Problem 26.8**

Simplify. Write expressions with only positive exponents.

$$\frac{-45x^{-4}y^3}{9x^{-5}y^{-4}}$$

In the next few rules, there will be parentheses. Recall that we often use parentheses for *grouping*.

### The Power to a Power Rule

#### Problem 26.9

Simplify.

$$(x^3)^2$$

The parentheses tell us that we have 2 groups of  $x^3$  multiplied.

$$(x^3)^2 = \overbrace{(x^3)(x^3)}^{2 \text{ groups of } 3 \text{ } x\text{'s} \text{ equals } 6 \text{ } x\text{'s } \textit{total}} = x^6$$

In simplified form,  $(x^3)^2 = x^6$ .

■

#### Problem 26.10

Simplify.

$$(x^4)^3$$

■

#### Problem 26.11

Simplify.

$$(x^5)^4$$

■

Since the last few problems involved having multiple groups, each having equal number of objects, it should not surprise you to see multiplication.

#### The Power to a Power Rule

To raise a power to a power, keep the same base and multiply the exponents.

$$(x^a)^b = x^{ab}$$

## The Product Raised to a Power Rule

### Problem 26.12

Simplify.

$$(5x^2y)^3$$

The parentheses tell us that we have 3 groups of  $5x^2y$ .

$$\begin{aligned} (5x^2y)^3 &= (5x^2y)(5x^2y)(5x^2y) \\ &= (5 \cdot 5 \cdot 5)(x^2 \cdot x^2 \cdot x^2)(y \cdot y \cdot y) \quad \text{by rearranging the order} \\ &= 5^3x^6y^3 \\ &= 125x^6y^3 \end{aligned}$$

In simplified form,  $(5x^2y)^3 = 125x^6y^3$ .

■

### Problem 26.13

Simplify.

$$(ab^3)^4$$

■

### Problem 26.14

Simplify.

$$(2a)^3$$

■

The pattern is easy to spot. Every factor inside the parentheses gets raised to the power outside the parentheses.

#### The Product Raised to a Power Rule

$$(xy)^n = x^n y^n$$

 **Problem 26.15**

Simplify.

$$(4ab^3)^2$$

 **Problem 26.16**

Simplify.

$$(-3a^2b^4c^0)^3$$

 **Problem 26.17**

Simplify.

$$(-5x^4y^5)^2(xy)$$

 **Problem 26.18**

Simplify.

$$(-r^5t)^3$$

 **Problem 26.19**

Simplify.

$$(-q^3u^5)^4$$

 **Problem 26.20**

Simplify.

$$\frac{(3x)^5}{(3x^3)^2}$$

## The Quotient to a Power Rule

The same ideas for products raised to a power work for quotients.

### Problem 26.21

Simplify.

$$\left(\frac{x}{y}\right)^3$$

The parentheses tell us that we have 3 groups of  $x/y$ .

$$\begin{aligned}\left(\frac{x}{y}\right)^3 &= \left(\frac{x}{y}\right)\left(\frac{x}{y}\right)\left(\frac{x}{y}\right) \\ &= \frac{x^3}{y^3} \quad \text{by multiplication of fractions rule}\end{aligned}$$

In simplified form  $\left(\frac{x}{y}\right)^3 = \frac{x^3}{y^3}$ .

■

### Problem 26.22

Simplify.

$$\left(\frac{a}{b}\right)^4$$

■

### Problem 26.23

Simplify.

$$\left(\frac{2}{x}\right)^3$$

■

The pattern is easy to spot. The denominator and the numerator each get raised to the power outside the parentheses.

#### The Quotient Raised to a Power Rule

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

 **Problem 26.24**

Simplify.

$$\left(\frac{2}{y^6}\right)^4$$

 **Problem 26.25**

Simplify.

$$\left(\frac{-a^3}{b^2}\right)^3$$



We have learned all the rules. In the remaining problems, we will apply them all.



You should interpret the instructions to “Simplify” to mean:

- Do not use a variable more than once in the answer.
- Make sure all exponents are positive. No zero exponents. No negative exponents.
- Eliminate all parentheses.
- Reduce all fractions, and multiply all coefficients.

 **Problem 26.26**

Simplify.

$$(-2x^5)(3x^4)$$

 **Problem 26.27**

Simplify.

$$(-14w^4z^5)(-3wz^3)$$



**Problem 26.28**

Simplify.

$$\frac{w^4 x^5}{w^4 x^{-3}}$$

**Problem 26.29**

Simplify.

$$\frac{x^5 x^7}{(x^3)^2}$$

**Problem 26.30**

Simplify.

$$(2a^3 b)(-6b^5 c^6)(3ac^2)$$

**Problem 26.31**

Simplify.

$$\frac{30x^6 y^5}{6xy^5}$$

**Problem 26.32**

Simplify.

$$\frac{8x^6 y}{-56x^4 y^8}$$

**Problem 26.33**

Simplify.

$$\frac{(-10p^8 q^3)(15q^7)}{-6pq^5}$$

**Problem 26.34**

Simplify.

$$\frac{8a^7b^5c^0}{20a^6b^5}$$

**Problem 26.35**

Simplify.

$$(m^4n)^3$$

**Problem 26.36**

Simplify.

$$(-3r^4)^2$$

**Problem 26.37**

Simplify.

$$\left(\frac{3x}{7y}\right)^2$$

**Problem 26.38**

Simplify.

$$(3y^{-4})^2$$

**Problem 26.39**

Simplify.

$$(6x^{-3})^{-2}$$

**Problem 26.40**

Simplify.

$$\frac{6a^4b^{-7}}{48a^{-4}b^{-6}}$$

**Problem 26.41**

Simplify.

$$\left(\frac{(2x^2)^3}{(2x^3)^2}\right)^2$$

# Polynomials and Operations

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## GPS 27: Introduction to Polynomials and Operations

Here's an example of a polynomial.

$$7x^3 + 6x^2 - 3x + 4$$

This polynomial is the sum of four terms:  $7x^3$ ,  $6x^2$ ,  $-3x$ , and 4. Each term has a coefficient and a variable with a whole number exponent.

Term	Coefficient	Variable	Exponent	Degree	Special Name
$7x^3$	7	$x^3$	3	3	Cubic term
$6x^2$	6	$x^2$	2	2	Quadratic term
$-3x$	-3	$x$	1	1	Linear term
4	4	none	0	0	Constant term

The **degree of a polynomial** is the highest degree of any of its terms. This means the polynomial above has degree 3. We also like to give names to polynomials based on how many terms are added together.

A **monomial** has only one term.

$$2x, \quad 3y^2, \quad 6z^3$$

A **binomial** has two terms.

$$x^2 - 4, \quad 2x + 3y, \quad x^3 + 1$$

A **trinomial** has three terms.

$$x^2 + 4x + 2, \quad 2x + 3y + 5, \quad 5y^3 - 125y^2 + 5y$$

If there are more than three terms, we just call it a polynomial.

## Adding Polynomials

Adding polynomials is the same as combining like terms. This means we group the terms with the same variables *and* exponents together and combine their coefficients.

### Problem 27.1

Simplify completely.

$$(2x^2 - 3x + 6) + (-5x^2 + 7x + 5)$$

We will simply rearrange the terms so that like terms are together and combine.

$$\begin{aligned}(2x^2 - 3x + 6) + (-5x^2 + 7x + 5) &= (2x^2 - 5x^2) + (-3x + 7x) + (6 + 5) \\ &= -3x^2 + 4x + 11\end{aligned}$$

In simplified form,  $(2x^2 - 3x + 6) + (-5x^2 + 7x + 5) = -3x^2 + 4x + 11$ .

■

### Problem 27.2

Simplify completely.

$$(8x^2 + 6x - 4) + (2x^2 - 5x - 2)$$

### Problem 27.3

Simplify completely.

$$(-3x^2 - 6x - 7) + (-2x^2 + 6x - 3)$$

## Subtracting Polynomials

One way to think about subtraction is by adding the opposite. We can use this same property we are accustomed to using with numbers even when we are working with polynomials.



### Subtraction

To subtract  $b$  from  $a$ , add the *opposite* of  $b$  to  $a$ .

$$a - b = a + (-b)$$

For example:

$$8 - 3 = 8 + (-3)$$

### Problem 27.4

Simplify completely.

$$(6x^2 - x + 7) - (3x^2 - 7x + 2)$$

We will change every coefficient in the second polynomial to its opposite and add.

$$\begin{aligned} (6x^2 - x + 7) - (3x^2 - 7x + 2) &= (6x^2 - x + 7) + (-3x^2 + 7x - 2) \\ &= (6x^2 - 3x^2) + (-x + 7x) + (7 - 2) \\ &= 3x^2 + 6x + 5 \end{aligned}$$

In simplified form,  $(6x^2 - x + 7) - (3x^2 - 7x + 2) = 3x^2 + 6x + 5$ .



### Problem 27.5

Simplify completely.

$$(6x^3 - 3x^2 + 5x - 15) - (-5x^3 + 13x^2 - 4x - 10)$$

**Problem 27.6**

Simplify completely.

$$(-3x^2 - 5x + 10) - (3x^2 - 5x - 15)$$

**Problem 27.7**

Simplify completely.

$$(2x - 14) - (-9x + 6)$$

**Problem 27.8**

Simplify completely.

$$(7t^2 + 2t) - (-7t^2 - 7t + 4)$$

**Problem 27.9**

Simplify completely.

$$(6x + 5) + (2x - 8) - (3x + 5)$$

## Dividing by a Monomial

When we learned to add fractions, we used an important fact that will help us divide a polynomial by a monomial.



### Fraction Addition and Subtraction

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

We can replace the addition with subtraction, and it is still true.

$$\frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}$$

We are more accustomed to seeing these equalities with the left and right sides of this equation switched, but the order makes no difference. We will also use this abbreviated version of the Quotient Rule from our Rules of Exponents.

### The Quotient Rule

To divide two expressions with the same base, keep the same base and subtract the exponents.

$$\frac{x^n}{x^m} = x^{n-m}$$

### Problem 27.10

Simplify completely.

$$\frac{12x^6 - 6x^3 + 4x^2}{2x^2}$$

We use the property from fraction addition to break this into three smaller fractions and apply the quotient rule.

$$\begin{aligned} \frac{12x^6 - 6x^3 + 4x^2}{2x^2} &= \frac{12x^6}{2x^2} - \frac{6x^3}{2x^2} + \frac{4x^2}{2x^2} \\ &= 6x^4 - 3x + 2 \end{aligned}$$

In simplified form,  $\frac{12x^6 - 6x^3 + 4x^2}{2x^2} = 6x^4 - 3x + 2$ .





**Problem 27.11**

Simplify completely.

$$\frac{81y^6 - 45y^5 - 27y^3}{9y}$$

**Problem 27.12**

Simplify completely.

$$\frac{12x^5 + 6x^4 + 3x^2}{3x^2}$$

**Problem 27.13**

Simplify completely.

$$\frac{35x^6 - 20x^4 - 5x^2}{5x^2}$$

**Problem 27.14**

Simplify completely.

$$\frac{-56a^6 - 49a^5 + 42a^4}{-7a^3}$$

**Problem 27.15**

Simplify completely.

$$\frac{48x^7 - 12x^4 - 6x^3}{-6x^3}$$

**Problem 27.16**

Simplify completely.

$$\frac{39b^9 - 21b^5 + 3b^2}{-3b^2}$$

## GPS 28: Multiplying Polynomials

When working with polynomials, there is really only one method of multiplication you have to master, and we have met this method before.

### The Distributive Property

$$a(b + c) = ab + ac$$

In addition to the distributive property, we will use the Product Rule for exponents.

### The Product Rule

To multiply two expressions with the same base, keep the same base and add the exponents.

$$x^n \cdot x^m = x^{n+m}$$

### Problem 28.1

Multiply.

$$2x^2(3x^2 + 4x - 1)$$

We will distribute  $2x^2$  to every term in the second polynomial and use our rules of exponents to simplify.

$$\begin{aligned} 2x^2(3x^2 + 4x - 1) &= (2x^2)(3x^2) + (2x^2)(4x) - (2x^2)(1) \\ &= 6x^4 + 8x^3 - 2x^2 \end{aligned}$$

In simplified form,  $2x^2(3x^2 + 4x - 1) = 6x^4 + 8x^3 - 2x^2$ .

■

### Problem 28.2

Multiply.

$$7x^3(4x^2 - x + 6)$$

■

### Problem 28.3

Multiply.

$$-3xy^3(9x^2y - 8xy^2 + 7x - 4y)$$

Sometimes the term we need to distribute is on the right instead of the left. It doesn't matter since we can multiply in any order we like.

**Problem 28.4**

Multiply.

$$(3x^3 - 5x^2y + 7x)(-x^3y)$$

We do exactly as before. We distribute and use the rules of exponents.

$$\begin{aligned}(3x^3 - 5x^2y + 7x)(-x^3y) &= (3x^3)(-x^3y) - (5x^2y)(-x^3y) + (7x)(-x^3y) \\ &= -3x^6y + 5x^5y^2 - 7x^4y\end{aligned}$$

■

 **Problem 28.5**

Multiply.

$$(9x^3 + x^2 - 2x)(4xy)$$

 **Problem 28.6**

Multiply.

$$(x^3y^2 - 5xy + 8)(-3xy^2)$$

In all of the previous problems, one of the factors was a monomial. We are now going to learn to multiply any two polynomials. We will use the exact same technique of distributing, but this time we will need to distribute twice before we get to the final answer.

**Problem 28.7**

Multiply.

$$(2x + 7)(3x^2 + 4x + 5)$$

This problem will require us to distribute *twice*.

First, we will distribute the entire binomial  $(2x + 7)$  to each term in the second polynomial.

$$\begin{array}{c} \text{Diagram: } (2x + 7)(3x^2 + 4x + 5) \\ \text{Arrows show } (2x + 7) \text{ being distributed to } 3x^2, 4x, \text{ and } 5. \\ (3x^2)(2x + 7) + (4x)(2x + 7) + (5)(2x + 7) \end{array}$$

Next, we distribute the term outside of each  $(2x + 7)$ .

$$\begin{array}{c} \text{Diagram: } (3x^2)(2x + 7) + (4x)(2x + 7) + (5)(2x + 7) \\ \text{Arrows show } 3x^2, 4x, \text{ and } 5 \text{ being distributed to } 2x \text{ and } 7. \\ (3x^2)(2x) + (3x^2)(7) + (4x)(2x) + (4x)(7) + (5)(2x) + (5)(7) \quad \leftarrow \end{array}$$

We use our rules of exponents to simplify this expression.

$$6x^3 + 21x^2 + 8x^2 + 28x + 10x + 35$$

Finally, we must combine the like terms to reach the final answer.

$$6x^3 + 29x^2 + 38x + 35$$

■

We will guide you through the next problem to help you keep track of what to do at each step.

**Problem 28.8**

Multiply.

$$(3x + 4)(2x^2 + 5x + 6)$$

First, distribute the  $(3x + 4)$  as one big piece. We have provided parentheses for you to fill in.

$$( \quad ) ( \quad ) + ( \quad ) ( \quad ) + ( \quad ) ( \quad )$$

Next, distribute the term outside each  $(3x + 4)$ . This time, you will have to write your own parentheses.

Use your rules of exponents to simplify each term.

Finally, combine like terms to arrive at your final answer.

■

Try a couple more without our guidance.

**Problem 28.9**

Multiply.

$$(4x + 3)(5x^2 + 2x + 6)$$

■

 **Problem 28.10**

Multiply.

$$(8x + 6)(7x^2 + 9x + 4)$$

■

Before moving further, let's take a closer look at Problem 28.7.

$$(2x + 7)(3x^2 + 4x + 5)$$

We want to compare the original problem to the expression with the arrow off to the right.

$$(3x^2)(2x) + (3x^2)(7) + (4x)(2x) + (4x)(7) + (5)(2x) + (5)(7)$$

Notice that the term  $2x$  is multiplied by every term in the second polynomial  $3x^2 + 4x + 5$ .

$$(3x^2)(2x) + (3x^2)(7) + (4x)(2x) + (4x)(7) + (5)(2x) + (5)(7)$$

Next, notice that the term  $7$  is also multiplied by every term in the second polynomial.

$$(3x^2)(2x) + (3x^2)(7) + (4x)(2x) + (4x)(7) + (5)(2x) + (5)(7)$$

In summary, we observe that you must multiply *every term of the first polynomial by every term of the second polynomial*. You may have heard of a four letter word that many use for multiplying *binomials*. But like most four letter words, its proper use is very limited, and students often get themselves into trouble using it inappropriately.

The most difficult part of this process is keeping track of all the terms and their products. In the next problem, we show a nice way to organize all the terms.

**Problem 28.11**

Multiply.

$$(5x - 1)(2x^2 - 3x + 6)$$

We start by making a table to arrange each polynomial. We put one term in each row and each column as below. Pay careful attention to the negatives.

	$2x^2$	$-3x$	$6$
$5x$			
$-1$			

In each row, fill in the same term across every column. Put the term in parentheses since you will be multiplying.

	$2x^2$	$-3x$	$6$
$5x$	$(5x)$	$(5x)$	$(5x)$
$-1$	$(-1)$	$(-1)$	$(-1)$

Next, in each column, fill in the same term down every row. Again, put them in parentheses.

	$2x^2$	$-3x$	$6$
$5x$	$(5x)(2x^2)$	$(5x)(-3x)$	$(5x)(6)$
$-1$	$(-1)(2x^2)$	$(-1)(-3x)$	$(-1)(6)$

In each cell, use the rules of exponents to simplify each product.

	$2x^2$	$-3x$	$6$
$5x$	$10x^3$	$-15x^2$	$30x$
$-1$	$-2x^2$	$3x$	$-6$

Finally, we must combine like terms. Like terms are *often (but not always!)* nicely arranged in diagonal patterns.

	$2x^2$	$-3x$	$6$
$5x$	$10x^3$	$-15x^2$	$30x$
$-1$	$-2x^2$	$3x$	$-6$

This brings us to our final answer.

$$10x^3 - 17x^2 + 33x - 6$$

■

After practicing this a few times, you won't have to draw the table more than once.

In the next problem, we will start by drawing and labeling the table for you.

 **Problem 28.12**

Multiply.

$$(6x - 2)(3x^2 - 7x + 5)$$

We drew and labeled the table for you. Fill in each cell with the proper terms. Use parentheses.

	$3x^2$	$-7x$	5
$6x$			
$-2$			

Next, fill in the products from the table above.

	$3x^2$	$-7x$	5
$6x$			
$-2$			

Finally, combine like terms to get the final answer.

■

In the next several problems, you must draw and label your own table.

 **Problem 28.13**

Multiply.

$$(3x - 4)(5x^2 + 8x - 9)$$

■





The diagonal terms are ***not always*** like terms. Sometimes diagonal terms cannot be combined, and sometimes the terms you can combine aren't diagonal! Be careful and don't just start adding up diagonals without thinking.

 **Problem 28.14**

Multiply.

$$(2x - 5)(x + 3y)$$

---

 **Problem 28.15**

Multiply.

$$(3x - 4)(3x + 4)$$

---

 **Problem 28.16**

Multiply.

$$(x^2 - x + 9)(x - 2)$$

There is one problem type of problem that often confuses students.

**Problem 28.17**

Multiply.

$$(2x + 3)^2$$

We must pay careful attention to our rules of exponents. Properly applying the rules of exponents gives us the following.

$$(2x + 3)^2 = (2x + 3)(2x + 3)$$

We must write the binomial twice, and then multiply as in the previous problems.

	$2x$	$3$
$2x$	$(2x)(2x)$	$(2x)(3)$
$3$	$(3)(2x)$	$(3)(3)$

We simplify each product.

	$2x$	$3$
$2x$	$4x^2$	$6x$
$3$	$6x$	$9$

Finally, we combine like terms.

$$4x^2 + 12x + 9$$

■

 **Problem 28.18**

Multiply.

$$(3x - 2)^2$$

 **Problem 28.19**

Multiply.

$$(7x + 1)^2$$

---

 **Problem 28.20**

Multiply.

$$(x + 2)(x + 6)$$

---

 **Problem 28.21**

Multiply.

$$(y - 2)(y + 2)$$

 **Problem 28.22**

Multiply.

$$(2a - b)(3a + 4b)$$

---

 **Problem 28.23**

Multiply.

$$(4x - 3)^2$$

---

 **Problem 28.24**

Multiply.

$$(b^2 - 4b + 2)(b^2 + 6b - 5)$$

 **Problem 28.25**

Multiply.

$$(4w + 3)(5w - 8)$$

---

 **Problem 28.26**

Multiply.

$$(3x - 5)(2x - 5)$$

---

 **Problem 28.27**

Multiply.

$$(3x + y)(2a - b)$$

 **Problem 28.28**

Multiply.

$$(2x - 3)(3x^2 - 4x + 3)$$

---

 **Problem 28.29**

Multiply.

$$(7x + 2)(a - 2)$$

---

 **Problem 28.30**

Multiply.

$$(3x - 2y)(3x + 2y)$$



## GPS 29: Removing the Greatest Common Factor

### Factoring Polynomials

In the previous sections, we learned lots of ways to combine and simplify polynomials. In the next few sections, we will learn some ways of taking them apart. In particular, we will learn how to factor polynomials.

Recall the meaning of the word *factor* as you have seen it before now.

$$\begin{array}{ccc}
 2 \cdot 5 = 10 & (3x^2)(6x) = 18x^3 & (x+2)(x+3) = x^2 + 5x + 6 \\
 \begin{array}{c} \swarrow \searrow \\ \text{factors} \end{array} \quad \begin{array}{c} \uparrow \\ \text{product} \end{array} & \begin{array}{c} \swarrow \searrow \\ \text{factors} \end{array} \quad \begin{array}{c} \uparrow \\ \text{product} \end{array} & \begin{array}{c} \swarrow \searrow \\ \text{factors} \end{array} \quad \begin{array}{c} \uparrow \\ \text{product} \end{array}
 \end{array}$$

In the past you often started with two or more factors and multiplied them together to find the product. Factoring is the opposite of multiplying in the sense that you start with the product, but then attempt to break the product into two or more factors. We will see some reasons why we will do this soon enough. For now, think of factoring as breaking a large product into smaller pieces. These smaller pieces will often be easier to work with in our future applications rather than working with the one large product.



Students often confuse the words *factor* and *multiple* of whole numbers. Note that a factor cannot be larger than the original number. A number is actually a factor (and a multiple) of itself.

The first technique we will learn is one of the building blocks we must master in order to factor more complex polynomials in future sections.

### Problem 29.1

Factor.

$$18x + 12$$

The *largest* number we can divide both 18 and 12 by is 6. To remove the 6, we will divide both  $18x$  by 6 and 12 by 6.

$$\begin{aligned}
 18x + 12 &= 6 \left( \frac{18x}{6} + \frac{12}{6} \right) \\
 &= 6(3x + 2)
 \end{aligned}$$

We should ***always*** check our factoring is correct by multiplying. We can check by distributing the 6. Let's check.

$$\begin{aligned}
 6(3x + 2) &= 6(3x) + 6(2) \\
 &= 18x + 12 \quad \checkmark
 \end{aligned}$$

We see that after multiplying, we are back to the original problem.





Removing the greatest common factor is like distributing in reverse.  
Recall that distributing looks like this

$$a(b + c) = ab + ac$$

Removing the greatest common factor looks exactly the opposite.

$$ab + ac = a(b + c)$$

 **Problem 29.2**

Factor. Check your answer by multiplying.

$$27p + 18$$

 **Problem 29.3**

Factor. Check your answer by multiplying.

$$24y - 40$$

■ In the previous problems, there were no variables in the second term. When there are variables in all the terms, you may also be able to factor out the variable. You should ask yourself, “What is the largest number of each variable I can factor from every term?”

**Problem 29.4**

Factor.

$$8x^3 + 6x^2$$

We start by looking at the coefficients 8 and 6. The largest number we can factor from both 8 and 6 is 2. We now look at the variable  $x$ . In the first term we have  $x^3$ , and in the second term we have  $x^2$ . Since we must factor the same number from each term, the most we can remove is  $x^2$ . As a result, the greatest common factor is  $2x^2$ . We will divide  $8x^3$  by  $2x^2$  and  $6x^2$  by  $2x^2$

$$\begin{aligned} 8x^3 + 6x^2 &= 2x^2 \left( \frac{8x^3}{2x^2} + \frac{6x^2}{2x^2} \right) \\ &= 2x^2(4x + 3) \end{aligned}$$

We always check factoring by multiplying.

$$\begin{aligned} 2x^2(4x + 3) &= 2x^2(4x) + 2x^2(3) \\ &= 8x^3 + 6x^2 \quad \checkmark \end{aligned}$$

**Problem 29.5**

Factor. Check your answer by multiplying.

$$15x^4 + 10x^3$$

■

**Problem 29.6**

Factor. Check your answer by multiplying.

$$30y^3 - 24y$$

■

**Problem 29.7**

Factor.

$$18x^2y^3 - 12xy^5$$

This problem is no more difficult than the previous problems. In this problem, we will first consider the coefficients, then we will consider each variable one at a time. First we note that the largest number we can factor out of 18 and 12 is 6. Next, we look at the  $x$ 's. The first term has an  $x^2$  but the second term has only  $x$ . As a result, the largest number of  $x$ 's we can factor is  $x$ . Finally, we look at the  $y$ 's. The first term has  $y^3$  and the second term has  $y^5$ . Hence we can only factor  $y^3$  from both terms. All combined, the greatest common factor is  $6xy^3$ .

$$\begin{aligned} 18x^2y^3 - 12xy^5 &= 6xy^3 \left( \frac{18x^2y^3}{6xy^3} - \frac{12xy^5}{6xy^3} \right) \\ &= 6xy^3(3x - 2y^2) \end{aligned}$$

We again check by multiplying.

$$\begin{aligned} 6xy^3(3x - 2y^2) &= (6xy^3)(3x) - (6xy^3)(2y^2) \\ &= 18x^2y^3 - 12xy^5 \quad \checkmark \end{aligned}$$

■

**Problem 29.8**

Factor. Check your answer by multiplying.

$$14x^3y - 21xy^3$$

**Problem 29.9**

Factor. Check your answer by multiplying.

$$40x^4y^2 + 24x^2y$$

■ In all the previous examples, there were only two terms. But we can use the same technique when there are three or more terms. ■

**Problem 29.10**

Factor.

$$21x^2y^2z^2 + 14x^3yz - 35x^2y$$

We start by observing that the largest number we factor out of 21, 14 and 35 is 7. The most number of  $x$ 's we can factor out of each term is  $x^2$ . We can only factor out  $y$  from each term. We note we cannot factor out any  $z$ 's since the last term does not have any  $z$ 's. Combining these observations, the greatest common factor is  $7x^2y$ .

$$\begin{aligned} 21x^2y^2z^2 + 14x^3yz - 35x^2y &= 7x^2y \left( \frac{21x^2y^2z^2}{7x^2y} + \frac{14x^3yz}{7x^2y} - \frac{35x^2y}{7x^2y} \right) \\ &= 7x^2y(3yz^2 + 2xz - 5) \end{aligned}$$

Finally, we check by multiplying.

$$\begin{aligned} 7x^2y(3yz^2 + 2xz - 5) &= (7x^2y)(3yz^2) + (7x^2y)(2xz) - (7x^2y)(5) \\ &= 21x^2y^2z^2 + 14x^3yz - 35x^2y \quad \checkmark \end{aligned}$$

■

**Problem 29.11**

Factor. Check your answer by multiplying.

$$28x^6 - 12x^4 + 20x^2$$

■

**Problem 29.12**

Factor. Check your answer by multiplying.

$$12x^2y^2 - 18xy^3 - 36x^2y$$

■

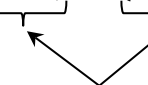
In one of our last and most important situations, the factor we wish to remove is actually a binomial rather than just a monomial.

**Problem 29.13**

Factor.

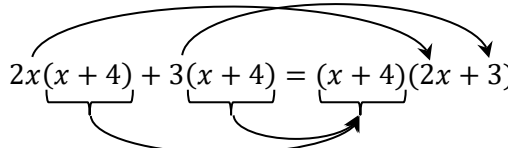
$$2x(x + 4) + 3(x + 4)$$

This problem looks different, but we shouldn't let our mind trick us into thinking this problem is hard. We notice that a common factor of each term is  $x + 4$ .

$$2x(x + 4) + 3(x + 4)$$


Both terms have  
 $x + 4$

When we factor out the  $x + 4$  from the first term, there is a  $2x$  left. When we factor out the  $x + 4$  from the second term, there is a  $3$  left.

$$2x(x + 4) + 3(x + 4) = (x + 4)(2x + 3)$$


Our final factored form is  $(x + 4)(2x + 3)$ .

■

**✎ Problem 29.14**

Factor.

$$7x(x - 7) - 3(x - 7)$$

**✎ Problem 29.15**

Factor.

$$5a(x + 2y) - b(x + 2y)$$

**✎ Problem 29.16**

Factor.

$$2x(x - 3) + (x - 3)$$

**✎ Problem 29.17**

Factor.

$$3x(2x + 7) - (2x + 7)$$

### GPS 30: Factoring by Grouping

In this section, we will add one additional step to our previous problems. We start by reviewing a problem at the end of the last section.

#### Problem 30.1

Factor.

$$3x(2a - 5b) - 2y(2a - 5b)$$

We notice that a common factor of each term is  $2a - 5b$ .

$$3x\underbrace{(2a - 5b)} - 2y\underbrace{(2a - 5b)}$$

Both terms have  
 $2a - 5b$

When we factor out the  $2a - 5b$  from the first term, there is a  $3x$  left. When we factor out the  $2a - 5b$  from the second term, there is a  $-2y$  left.

$$3x(2a - 5b) - 2y(2a - 5b) = (2a - 5b)(3x - 2y)$$

Our final factored form is  $(2a - 5b)(3x - 2y)$ .

■

#### Problem 30.2

Factor.

$$5a(x - 2y) + 3b(x - 2y)$$

■

#### Problem 30.3

Factor.

$$7c(3w + 2z) - 2d(3w + 2z)$$

■

Notice that in these problems, there was an obvious factor to remove in each piece. In the next problem, we will have to first determine what the common factor is by grouping terms together with a common factor.

### Problem 30.4

Factor.

$$ab - 3a + 6b - 18$$

We start by breaking these four terms into two groups.

$$\boxed{ab - 3a} + \boxed{6b - 18}$$

We will factor out the greatest common factor from each group. In the first group, we factor out an  $a$ . In the second group, we factor out a 6.

$$\boxed{ab - 3a} + \boxed{6b - 18} = a(b - 3) + 6(b - 3)$$

Now this expression looks like what we have been working with already. The last step is to factor out the  $b - 3$ .

$$a(b - 3) + 6(b - 3) = (b - 3)(a + 6)$$

We must pause to check our work. To check factoring, we multiply.

	$a$	$6$
$b$	$ab$	$6b$
$-3$	$-3a$	$-18$

There are no like terms to combine, so the product is

$$(b - 3)(a + 6) = ab + 6b - 3a - 18 \quad \checkmark$$

Notice the terms are in a different order than the original problem, but the terms are all the same and all have the same sign as the original. The factored form is  $(b - 3)(a + 6)$ .

■

#### Order of the terms when checking

When checking factoring problems, you may get a polynomial with the terms written in a different order than your original problem. In this case, make sure you have all the same terms and each term has the same sign as in the original problem. If they are all the same, then they are equal.

**For example:**

$$ab - 3a + 6b - 18 \quad \text{and} \quad ab + 6b - 3a - 18$$

are equal because the terms  $\boxed{ab}$ ,  $\boxed{-3a}$ ,  $\boxed{6b}$  and  $\boxed{-18}$  are in both polynomials.



**Order of the factors in the answer**

Recall that when multiplying numbers, the order of the factors does not matter. The same is true for polynomials. You must have the same exact factors, but in any order you wish.

**For example:**

Both of the following two answers are correct.

$$ab - 3a + 6b - 18 = (b - 3)(a + 6) \quad \checkmark$$

$$ab - 3a + 6b - 18 = (a + 6)(b - 3) \quad \checkmark$$

 **Problem 30.5**

Factor. Check your answer by multiplying.

$$4ax + 20bx + ay + 5by$$





**Problem 30.6**

Factor.

$$8ax - 20bx - 6ay + 15by$$

We begin this problem much the same as the last one.

$$\boxed{8ax - 20bx} - \boxed{6ay + 15by}$$

The first of these two groups is easy to factor. We will factor out a  $4x$  from each term. But there is a tiny difference between this problem and the previous ones. This problem has a subtraction joining the two groups

$$\boxed{8ax - 20bx} - \boxed{6ay + 15by}$$

This means in the second group, we must factor out a  $-3y$  and not just a  $3y$ .

The sign is the *opposite* because of the subtraction.

$$\boxed{8ax - 20bx} - \boxed{6ay + 15by} = 4x(2a - 5b) - 3y(2a - 5b)$$

$$= (2a - 5b)(4x - 3y)$$

We are never done without first checking our answer.

	$4x$	$-3y$
$2a$	$8ax$	$-6ay$
$-5b$	$-20bx$	$15by$

There are again no like terms to combine. We get

$$8ax - 6ay - 20bx + 15by$$

Notice the terms are in a different order than the original problem, but the terms are all the same and all have the same sign as the original.

■

 **Problem 30.7**

Factor. Check your answer by multiplying.

$$4cw - 6dw - 10cz + 15dz$$

---

 **Problem 30.8**

Factor. Check your answer by multiplying.

$$20ax - 36bx - 35ay + 63by$$

**✎ Problem 30.9**

Factor. Check your answer by multiplying.

$$2ax - 2bx - ay + by$$

---

**✎ Problem 30.10**

Factor. Check your answer by multiplying.

$$2cw - 2w + c - 1$$

 **Problem 30.11**

Factor. Check your answer by multiplying.

$$zd - z + d - 1$$

---

 **Problem 30.12**

Factor. Check your answer by multiplying.

$$ab - a - b + 1$$

**✎ Problem 30.13**

Factor. Check your answer by multiplying.

$$x^3 - x^2 + x - 1$$

---

**✎ Problem 30.14**

Factor. Check your answer by multiplying.

$$3ax + bx - 9a - 3b$$

 **Problem 30.15**

Factor. Check your answer by multiplying.

$$10ax + 2bx + 15ay + 3by$$

---

 **Problem 30.16**

Factor. Check your answer by multiplying.

$$6ac - 12ad - 7bc + 14bd$$



### GPS 31: Factoring Trinomials by Grouping – Part 1

So far we have only used the Factor by Grouping technique on polynomials with four terms. In this section, we will see how to apply this technique to a trinomial which you recall has only three terms.

#### Problem 31.1

Factor. Check your answer by multiplying.

$$6x^2 + 17x + 5$$

This polynomial has only three terms, and so we cannot start by dividing into two groups of two terms as we have done previously. Instead, our strategy will be to break the  $+17x$  into two pieces to give us a total of four pieces we can factor by grouping.

$$6x^2 \boxed{+17x} + 5$$

$$6x^2 \boxed{+??} \boxed{+??} + 5$$

Our first goal is to find two terms that add to  $+17x$ . But recall in the next step of Factor by Grouping, we will remove a common factor from each group. This means each new term must also have a factor in common with 6 and 5. To ensure this, we will calculate the grouping number which is  $(6)(5) = 30$ . In summary, we must find two numbers that add to 17 and that multiply to 30.

We start by looking for two numbers that multiply to 30, and then check to see if their sum is 17. From looking at the table to the right, we see that 2 and 15 are the numbers we need. We can use these two numbers to break up our original trinomial.

Multiply to 30		Sum of the numbers	
1	30	31	x
2	15	17	✓
3	10	13	x
5	6	11	x

$$6x^2 \boxed{+17x} + 5$$

$$6x^2 \boxed{+2x} \boxed{+15x} + 5$$

We break this new polynomial into two groups, and then complete the factoring.

$$\boxed{6x^2 + 2x} + \boxed{15x + 5}$$

$$2x(3x + 1) + 5(3x + 1)$$

$$(3x + 1)(2x + 5)$$

As always, we check by multiplying.

	$2x$	$5$
$3x$	$6x^2$	$15x$
$1$	$2x$	$5$

After combining like terms, we get our original polynomial back:  $6x^2 + 17x + 5$ . ✓



We will guide you through your first problem next.



 **Problem 31.2**

Factor. Check your answer by multiplying.

$$6x^2 + 19x + 10$$

Our goal is to break the  $+19x$  into two pieces.

$$6x^2 \boxed{+19x} + 10$$

↓      ↓

$$6x^2 \boxed{+??} \boxed{+??} + 10$$

To help determine the two numbers, find the Grouping Number.

- a) What is the leading coefficient (the coefficient of the  $x^2$ )? \_\_\_\_\_  
 b) What is the constant term (the term without any  $x$ 's)? \_\_\_\_\_

The Grouping Number is the product of the two numbers you just found.

The **Grouping Number** is: \_\_\_\_\_.

In the table, list all pairs of numbers whose product is the Grouping Number. Make sure you list all pairs. You will use every row if you found them all!

What is coefficient of the linear term (this is the middle term with only the  $x$ )? \_\_\_\_\_

Pick the pair of numbers whose sum is coefficient of the middle term.

Multiply to	Sum of the numbers

Fill in the boxes below using the pair you found. **Don't forget to include the  $x$ 's and the plus signs!**

$$6x^2 \boxed{+19x} + 10$$

↓      ↓

$$6x^2 \boxed{\phantom{+19x}} \boxed{\phantom{+19x}} + 10$$

Form two groups. The first group will be the first two terms, and the second group will be the last two terms. Then factor the greatest common factor from each group. When you are done, there should be a binomial in common to each group.

Factor the binomial from each group to arrive at the final answer.

Check your answer by multiplying.

■

Before moving to another problem, let's write down the procedure we just followed.

### Factoring a Trinomial by Grouping

- 1) Calculate the **Grouping Number**. The Grouping Number is the product of the leading coefficient (the coefficient of the term with the highest exponent) and the constant term (the term with no variables).
- 2) Find the coefficient of the linear term (this is the term with only  $x$ ). We need two numbers to break the linear term into two pieces.
  - a. The two numbers should multiply to the Grouping Number.
  - b. The two numbers should add to the middle term's coefficient.
- 3) Break the linear term into two pieces using the two numbers you just found.
- 4) Form two groups using the first two terms and the last two terms.
- 5) Factor the greatest common factor from each group.
- 6) Factor the common binomial from each group.
- 7) Check your answer by multiplying.

#### **Problem 31.3**

Factor. Check your answer by multiplying.

$$6x^2 + 13x + 6$$

Grouping Number: \_\_\_\_\_ (Multiply to this)

Linear Coefficient: \_\_\_\_\_ (Add to this)

#### **Problem 31.4**

Factor. Check your answer by multiplying.

$$3x^2 + 8x + 5$$

Grouping Number: \_\_\_\_\_

Linear Coefficient: \_\_\_\_\_

**Problem 31.5**

Factor. Check your answer by multiplying.

$$5x^2 + 21x + 4$$

Grouping Number: \_\_\_\_\_

Linear Coefficient: \_\_\_\_\_

**Problem 31.6**

Factor. Check your answer by multiplying.

$$x^2 + 6x + 8$$

Grouping Number: \_\_\_\_\_

Linear Coefficient: \_\_\_\_\_

 **Problem 31.7**

Factor. Check your answer by multiplying.

$$x^2 + 14x + 24$$

 **Problem 31.8**

Factor. Check your answer by multiplying.

$$2x^2 + 11x + 5$$



## GPS 32: Factoring Trinomials by Grouping – Part 2

### Problem 32.1

Factor. Check your answer by multiplying.

$$2x^2 - 11x + 12$$

We start by calculating the Grouping Number. The Grouping Number is  $(2)(12) = 24$ . The coefficient of the linear term is  $-11$ . We must find two numbers whose product is 24 and whose sum is  $-11$ . Recall that if the product of two numbers is positive, both numbers must be positive or both numbers must be negative. In this case, since the sum is negative, we know both numbers must be negative.

We start by looking for all pairs of two negative numbers that multiply to 24 and then check to see if their sum is  $-11$ . From looking at the table to the right, we see that  $-3$  and  $-8$  are the numbers we need. We can use these two numbers to break up our original trinomial.

Multiply to 24		Sum of the numbers	
-1	-24	-25	x
-2	-12	-14	x
-3	-8	-11	✓
-4	-6	-10	x

$$\begin{array}{c}
 2x^2 \boxed{-11x} + 12 \\
 \swarrow \quad \searrow \\
 2x^2 \boxed{-3x} \boxed{-8x} + 12
 \end{array}$$

We break this new polynomial into two groups, and then complete the factoring.

$$\begin{array}{l}
 2x^2 - 3x - 8x + 12 \\
 x(2x - 3) - 4(2x - 3) \\
 (2x - 3)(x - 4)
 \end{array}$$

Our final factored polynomial is  $(2x - 3)(x - 4)$ .

As always, we check by multiplying.

	$x$	$-4$
$2x$	$2x^2$	$-8x$
$-3$	$-3x$	$-12$

After combining like terms, we get our original polynomial back:  $2x^2 - 11x + 12$ . ✓

■



If the **product** of two numbers is positive, then the numbers must both be positive *or* both be negative.

- If their sum is positive, then both numbers must be positive.
- If their sum is negative, then both numbers must be negative.

**✎ Problem 32.2**

Factor. Check your answer by multiplying.

$$3x^2 - 14x + 15$$

**✎ Problem 32.3**

Factor. Check your answer by multiplying.

$$4x^2 - 8x + 3$$

 **Problem 32.4**

Factor. Check your answer by multiplying.

$$x^2 - 8x + 12$$

 **Problem 32.5**

Factor. Check your answer by multiplying.

$$x^2 - 12x + 36$$





### GPS 33: Factoring Trinomials by Grouping – Part 3

#### Problem 33.1

Factor. Check your answer by multiplying.

$$10x^2 + x - 2$$

We start by calculating the Grouping Number. The Grouping Number is  $(10)(-2) = -20$ . The coefficient of the linear term is  $+1$ . We must find two numbers whose product is  $-20$  and whose sum is  $+1$ . Recall that if the product of two numbers is negative, then one number must be positive and one number must be negative. In this case, since the sum is positive, we know that the “larger” (ignoring the signs) of the two numbers must be positive and the “smaller” must be negative.

We start by looking for all pairs of numbers that multiply to  $-20$  and then check to see if their sum is  $1$ . We assign the “larger” (ignoring the sign of the number) to be positive and the “smaller” to be negative. From looking at the table to the right, we see that  $-4$  and  $5$  are the numbers we need. We can use these two numbers to break up our original trinomial.

Multiply to -20		Sum of the numbers	
-1	20	19	×
-2	10	8	×
-4	5	1	✓

$$\begin{aligned}
 &10x^2 \boxed{+x} - 2 \\
 &\quad \swarrow \quad \searrow \\
 &10x^2 \boxed{-4x} \boxed{+5x} - 2 \\
 &2x(5x - 2) + 1(5x - 2) \\
 &(5x - 2)(2x + 1)
 \end{aligned}$$

Our final factored polynomial is  $(5x - 2)(2x + 1)$ .

As always, we check by multiplying.

	$2x$	$1$
$5x$	$10x^2$	$5x$
$-2$	$-4x$	$-2$

After combining like terms, we get our original polynomial back:  $10x^2 + x - 2$ . ✓



If the **product** of two numbers is negative, then one number must be positive and one number must be negative.

- If their sum is positive, then the “larger” (ignoring the signs) of the two numbers must be positive.
- If their sum is negative, then the “larger” of the two numbers must be negative.

**Problem 33.2**

Factor. Check your answer by multiplying.

$$3x^2 + 19x - 14$$

**Problem 33.3**

Factor. Check your answer by multiplying.

$$x^2 + 2x - 48$$

 **Problem 33.4**

Factor. Check your answer by multiplying.

$$x^2 + x - 2$$

 **Problem 33.5**

Factor. Check your answer by multiplying.

$$2x^2 + 5x - 3$$



## GPS 34: Factoring Trinomials by Grouping – Part 4

### Problem 34.1

Factor. Check your answer by multiplying.

$$4x^2 - 3x - 10$$

We start by calculating the Grouping Number. The Grouping Number is  $(4)(-10) = -40$ . The coefficient of the linear term is  $-3$ . We must find two numbers whose product is  $-40$  and whose sum is  $-3$ . Recall that if the product of two numbers is negative, then one number must be positive and one number must be negative. In this case, since the sum is negative, we know that the “larger” (ignoring the signs) of the two numbers must be negative and the “smaller” must be positive.

We start by looking for all pairs of numbers that multiply to  $-40$  and then check to see if their sum is  $-3$ . We assign the “larger” (ignoring the sign of the number) to be negative and the “smaller” to be positive. From looking at the table to the right, we see that  $5$  and  $-8$  are the numbers we need. We can use these two numbers to break up our original trinomial.

	Multiply to $-40$	Sum of the numbers	
1	$-40$	$-39$	×
2	$-20$	$-18$	×
4	$-10$	$-6$	×
5	$-8$	$-3$	✓

$$\begin{array}{c}
 4x^2 \boxed{-3x} - 10 \\
 \swarrow \quad \searrow \\
 4x^2 \boxed{+5x} \boxed{-8x} - 10 \\
 x(4x + 5) - 2(4x + 5) \\
 (4x + 5)(x - 2)
 \end{array}$$

Our final factored polynomial is  $(4x + 5)(x - 2)$ .

As always, we check by multiplying.

	$x$	$-2$
$4x$	$4x^2$	$-8x$
$5$	$5x$	$-10$

After combining like terms, we get our original polynomial back:  $4x^2 - 3x - 10$ . ✓

■

 **Problem 34.2**

Factor. Check your answer by multiplying.

$$x^2 - 4x - 21$$

 **Problem 34.3**

Factor. Check your answer by multiplying.

$$3x^2 - 25x - 18$$

 **Problem 34.4**

Factor. Check your answer by multiplying.

$$x^2 + 12x + 36$$

 **Problem 34.5**

Factor. Check your answer by multiplying.

$$x^2 - 3x - 10$$



 **Problem 34.6**

Factor. Check your answer by multiplying.

$$25x^2 - 15x + 2$$

 **Problem 34.7**

Factor. Check your answer by multiplying.

$$3x^2 + 10x - 8$$

 **Problem 34.8**

Factor. Check your answer by multiplying.

$$x^2 - 2x + 1$$

 **Problem 34.9**

Factor. Check your answer by multiplying.

$$12x^2 + x - 1$$

**✍ Problem 34.10**

Factor. Check your answer by multiplying.

$$6x^2 - 7x - 10$$

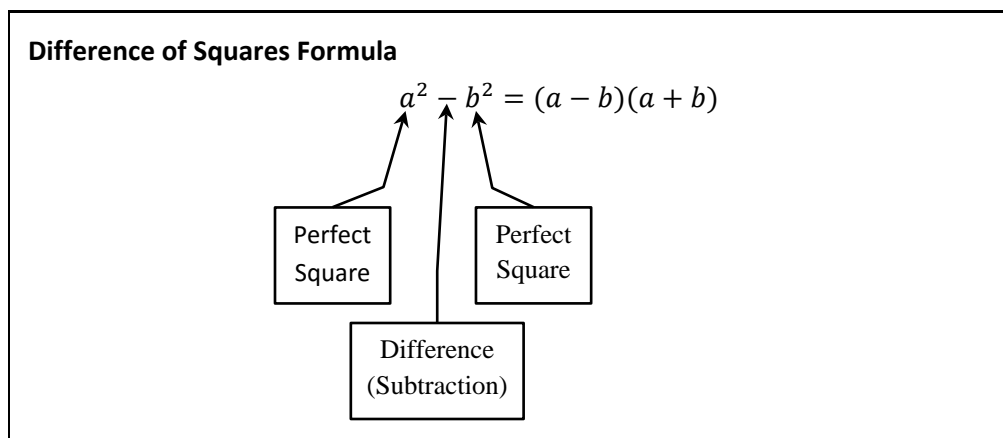
**✍ Problem 34.11**

Factor. Check your answer by multiplying.

$$4x^2 + 20x + 25$$

## GPS 35: Factoring a Difference of Squares

There is one type of polynomial that appears quite often in factoring problems.



### Problem 35.1

Factor. Check your answer by multiplying.

$$9x^2 - 16$$

In order to use the Difference of Squares Formula, we must verify this is a difference of squares.

We check 3 things:

- 1) There are two terms with a subtraction between them.
- 2) The first term  $9x^2$  is a perfect square. We see that  $(3x)^2 = 9x^2$ .
- 3) The second term 16 is a perfect square. We see that  $4^2 = 16$ .

The Difference of Squares Formula applies.

$$\begin{aligned} 9x^2 - 16 &= (3x)^2 - 4^2 \\ &= (3x - 4)(3x + 4) \end{aligned}$$

In factored form, the answer is  $(3x - 4)(3x + 4)$ .

As always, we must now check that our factorization is correct by multiplying.

	3x	4
3x	$9x^2$	12x
-4	-12x	-16

Notice that the terms on the diagonal are *opposites*, and hence their sum is zero. This leaves our original polynomials  $9x^2 - 16$ . ✓



We will guide you through your first difference of squares.

### Problem 35.2

Factor. Check your answer by multiplying.

$$25y^2 - 4$$

First we check that this is a difference of squares.

- 1) Are there two terms with a subtraction between them?
- 2) Is the first term a perfect square? If so, fill in the blank:  $( \quad )^2 = 25y^2$
- 3) Is the second term a perfect square? If so, fill in the blank:  $( \quad )^2 = 4$

If you answered yes to all three (if you didn't answer yes, check your answers!), then this is a difference of squares.

The Difference of Squares Formula applies. Fill in the blanks below using the blanks you filled in above.

$$25y^2 - 4 = ( \quad )^2 - ( \quad )^2$$

$$= ( \quad - \quad ) ( \quad + \quad )$$

One  
Minus

One  
Plus

The answer is \_\_\_\_\_.

Now check your answer by multiplying.

■

On the next pages, you can try some more on your own.

 **Problem 35.3**

Factor. Check your answer by multiplying.

$$x^2 - 81$$

 **Problem 35.4**

Factor. Check your answer by multiplying.

$$x^2 - 1$$

 **Problem 35.5**

Factor. Check your answer by multiplying.

$$36t^2 - 49$$

 **Problem 35.6**

Factor. Check your answer by multiplying.

$$25p^2 + 16$$

 **Problem 35.7**

Factor. Check your answer by multiplying.

$$100q^2 - 1$$

 **Problem 35.8**

Factor. Check your answer by multiplying.

$$49u^2 - 144$$



## GPS 36: Multistep Factoring

For now, we have learned all the different techniques for factoring. In the following problems we will combine the techniques we have already learned. When we factor a polynomial, we usually break it into two factors which are also polynomials. We can sometimes *factor a factor* again. This is like breaking a huge rock into two big rocks, and then breaking the big rocks into even smaller rocks.

In this section, you will see the instructions change from “Factor” to “Factor completely.” When you are told to “Factor completely,” you must factor every factor as many times as possible until you can no longer factor it any more.

### Problem 36.1

Factor *completely*.

$$2x^3 - 32x$$

#### 1. Factor out the Greatest Common Factor

When factoring completely, one always starts by factoring out the greatest common factor. The greatest common factor of this polynomial is  $2x$ .

$$2x^3 - 32x = 2x(x^2 - 16)$$

#### 2. Factor each factor.

There are two factors. The first factor is  $2x$  which cannot be factored any more. The second factor is  $x^2 - 16$  which we recognize is a difference of squares. We now factor the difference of squares.

$$\begin{aligned} 2x^3 - 32x &= 2x(x^2 - 16) \\ &= 2x(x + 4)(x - 4) \end{aligned}$$

Don't forget to bring down  
the factors from the  
previous step.

#### 3. Check if any factor can be factored again.

We have factored this polynomial into three factors  $2x$ ,  $x + 4$ , and  $x - 4$ . None of these factors can be factored again, so this is our final answer.

In completely factored form,

$$2x^3 - 32x = 2x(x + 4)(x - 4)$$

■



 **Problem 36.2**Factor *completely*.

$$3x^3y - 3xy^3$$

1. **Factor out the Greatest Common Factor**

2. **Factor each factor.**

3. **Check if any factor can be factored again.**

The completely factored form is \_\_\_\_\_

 **Problem 36.3**Factor *completely*.

$$2x^2y + 10xy + 12y$$



 **Problem 36.4**Factor *completely*.

$$10ab + 14ab^2$$

 **Problem 36.5**Factor *completely*.

$$3x^2 - 21x + 36$$

 **Problem 36.6**Factor *completely*.

$$5x^2 - 20x - 105$$

 **Problem 36.7**Factor *completely*.

$$6x^2y - 50xy - 36y$$

**Problem 36.8**Factor *completely*.

$$4x^2 + 8x + 4$$

**Problem 36.9**Factor *completely*.

$$6x^2 + 24x - 72$$

**Problem 36.10**Factor *completely*.

$$60x^2y + 110xy + 30y$$

**Problem 36.11**Factor *completely*.

$$5x^2 - 14x - 3$$

## GPS 37: Solving Quadratic Equations by Factoring

In this section, we will see one reason why we have learned to factor.

### Problem 37.1

Find all solutions to the equation.

$$2x^2 + 7x + 3 = 0$$

We start by factoring the polynomial  $2x^2 + 7x + 3$ . The Grouping Number is  $(2)(3) = 6$  and the linear coefficient is 7. We can easily spot the two numbers we need to split the middle term is 6 and 1.

$$\begin{aligned} 2x^2 + 7x + 3 &= 0 \\ 2x^2 + 6x + x + 3 &= 0 \\ 2x(x + 3) + 1(x + 3) &= 0 \\ (x + 3)(2x + 1) &= 0 \end{aligned}$$

We now make use of the Zero Product Property which says, "If the product of two numbers is 0, then one of the numbers must be equal to 0." In our case, this translates to one of the factors must be zero. We proceed by breaking our one equation into two simpler equations.

$$\begin{array}{ccc} & (x + 3)(2x + 1) = 0 & \\ \swarrow & & \searrow \\ x + 3 = 0 & \text{or} & 2x + 1 = 0 \end{array}$$

Both of these equations are easy to solve.

$$\begin{array}{ll} x + 3 = 0 & 2x + 1 = 0 \\ x = -3 & 2x = -1 \\ & x = -\frac{1}{2} \end{array}$$

This equation has two solutions.

$$x = -3 \text{ or } x = -\frac{1}{2}$$

■

We highlight a very important property we needed while working this problem.

#### Zero Product Property

If the product of two numbers is 0, then one of the numbers must be equal to 0.

$$\text{If } a \cdot b = 0, \text{ then } a = 0 \text{ or } b = 0.$$

 **Problem 37.2**

Find all solutions to the equation.

$$2x^2 + 11x + 12 = 0$$

1. Factor the polynomial.
2. Set each factor equal to zero and solve the new simpler equations.
3. Write down *both* solutions.

 **Problem 37.3**

Find all solutions to the equation.

$$x^2 - 16 = 0$$

 **Problem 37.4**

Find all solutions to the equation.

$$5x^2 - 40x = 0$$



In order to use the Zero Product Property, our product must first be equal to zero. Often we must rearrange the equation before factoring and using the Zero Product Property.

**Problem 37.5**

Find all solutions to the equation.

$$3x^2 = 12$$

Before we can factor this equation, we must subtract 12 from both sides so it will equal zero.

$$\begin{aligned} 3x^2 &= 12 \\ 3x^2 - \mathbf{12} &= 12 - \mathbf{12} \\ 3x^2 - 12 &= 0 \end{aligned}$$

Now that the equation is equal to zero, we can factor and solve.

$$\begin{aligned} 3x^2 - 12 &= 0 \\ 3(x^2 - 4) &= 0 \\ 3(x - 2)(x + 2) &= 0 \end{aligned}$$

To factor this polynomial, we first removed the greatest common factor. Then we factored the difference of squares. We now set each factor to 0.

$$\begin{array}{ccccc} 3 \neq 0 & & \text{or} & & x - 2 = 0 & & \text{or} & & x + 2 = 0 \\ \text{A constant factor is} & & & & x = 2 & & & & x = -2 \\ \text{never equal to zero!} & & & & & & & & \end{array}$$

The solutions are  $x = 2$  or  $x = -2$ . Sometimes you will this abbreviated by writing  $x = \pm 2$ .

■

**Problem 37.6**

Find all solutions to the equation.

$$5x^2 = 500$$

**Problem 37.7**

Find all solutions to the equation.

$$x^2 + 3x = 18$$

**Problem 37.8**

Find all solutions to the equation.

$$6x^2 = -3x$$

**Problem 37.9**

Find all solutions to the equation.

$$4x^2 - 4x = x + 6$$

**Problem 37.10**

Find all solutions to the equation.

$$3b^2 = 21b$$

**Problem 37.11**

Find all solutions to the equation.

$$10y^2 + 11y = 6$$

**Problem 37.12**

Find all solutions to the equation.

$$5t^2 = 45$$

**Problem 37.13**

Find all solutions to the equation.

$$x^2 - 15x = -54$$

**Problem 37.14**

Find all solutions to the equation.

$$3x^2 - 4 = 11x$$

**Problem 37.15**

Find all solutions to the equation.

$$x^2 - 7 = -6x$$



**Problem 37.16**

Find all solutions to the equation.

$$4x^2 = 9$$

**Problem 37.17**

Find all solutions to the equation.

$$x^2 - 4x - 12 = 0$$

**Problem 37.18**

Find all solutions to the equation.

$$5x^2 = 8x + 4$$

**Problem 37.19**

Find all solutions to the equation.

$$x^2 - 2 = x$$

# Algebraic Expressions

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## GPS 38: Evaluating Algebraic Expressions

In many instances, we have expressions into which we must substitute values. In this section, we will learn how to substitute given values of variables into the given algebraic expressions. This section will test our knowledge and application of the order of operations. We will see how to do this in the example below.

### Problem 38.1

Given  $a = 2$  and  $b = -1$ , evaluate the expression given below.

$$a^2 + ab + b^2$$

We evaluate this expression by simply substituting the given values into the expression above. In each case, we must remember to first raise numbers to the power, then multiply and finally, add or subtract.

$$\begin{aligned} & a^2 + ab + b^2 \\ (2)^2 + (2)(-1) + (-1)^2 \\ & = 4 - 2 + 1 \\ & = 2 + 1 \\ & = 3 \end{aligned}$$

Replace  $a$  with 2, and  $b$  with  $-1$ .  
Remember to raise to the power first.  
Multiply.  
Add the first two terms.

The value of the expression is **3**.



### Problem 38.2

Given  $a = 1$  and  $b = 2$ , evaluate the expression given below.

$$a^2 - a + b^2$$

Answer = \_\_\_\_\_



### Problem 38.3

Given  $a = -2$  and  $b = 3$ , evaluate the expression given below.

$$2a^2b - 4ab - 3b^2$$

We evaluate this expression by simply substituting the given values into the expression above. In each case, we must remember to first raise numbers to the power, then multiply and finally, add or subtract.

$$\begin{aligned} & 2a^2b - 4ab - 3b^2 \\ = 2(-2)^2(3) - 4(-2)(3) - 3(3)^2 \\ = 2(4)(3) + 24 - 3(9) \\ = 24 + 24 - 27 \\ = 48 - 27 \\ = 21 \end{aligned}$$

Replace  $a$  with  $-2$ , and  $b$  with 3.  
Remember to raise to the power first.  
Multiply.  
Add the first two terms.  
Subtract.

The value of the expression is **21**.





Always put parentheses around the number you are substituting into a function. This will help avoid sign errors—especially with exponents.

**Problem 38.4**

Given  $x = -1$  and  $y = 3$ , evaluate the expression given below.

$$x^2y + xy + y^2$$

Answer = \_\_\_\_\_



**Problem 38.5**

Given  $p = -2$  and  $q = 3$ , evaluate the expression given below.

$$2p^2 - 3pq + 2q^2$$

Answer = \_\_\_\_\_



**Problem 38.6**

Given  $a = -4$  and  $b = 3$ , evaluate the expression given below.

$$2a^2b + ab - b^2$$

Answer = \_\_\_\_\_



**Problem 38.7**

Given  $x = -1$  and  $y = 3$ , evaluate the expression given below.

$$-xy + x^2$$

Answer = \_\_\_\_\_



**Problem 38.8**

Given  $a = -1$  and  $b = 2$ , evaluate the expression given below.

$$-3a^2 - 2ab + 3b^2$$

Answer = \_\_\_\_\_



**Problem 38.9**

Given  $p = -3$  and  $q = -2$ , evaluate the expression given below.

$$-p^2 - q^2 - pq$$

Answer = \_\_\_\_\_



# Square Roots and Operations

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## GPS 39: Introduction to Square Roots

### What is a square root?

Squaring a number means to multiply the number by itself. For example,  $3^2 = 3 \times 3 = 9$ . Finding the square root is the opposite of squaring a number.

#### Square Root

To find the square root of a number, we ask,

What nonnegative number times itself equals our number?

For example, the square root of 9 is 3 since we just saw that  $3^2 = 9$ . We write this using symbols as  $\sqrt{9} = 3$ .

Unfortunately, most whole numbers don't have whole number square roots. Without a calculator, it is difficult to find the square root of most numbers. Instead, we learn to recognize common squares and approximate answers when we must.

A whole number whose square root is also a whole number is called a *perfect square*.

### Common Square Roots and Perfect Squares

Square Roots	Perfect Squares
$\sqrt{0} = 0$	$0^2 = 0$
$\sqrt{1} = 1$	$1^2 = 1$
$\sqrt{4} = 2$	$2^2 = 4$
$\sqrt{9} = 3$	$3^2 = 9$
$\sqrt{16} = 4$	$4^2 = 16$
$\sqrt{25} = 5$	$5^2 = 25$
$\sqrt{36} = 6$	$6^2 = 36$
$\sqrt{49} = 7$	$7^2 = 49$
$\sqrt{64} = 8$	$8^2 = 64$
$\sqrt{81} = 9$	$9^2 = 81$
$\sqrt{100} = 10$	$10^2 = 100$
$\sqrt{121} = 11$	$11^2 = 121$
$\sqrt{144} = 12$	$12^2 = 144$

Sometimes we need help breaking apart large numbers into smaller numbers that are easier to compute. The following rule is helpful for this purpose.

**The Product Rule**

When two nonnegative numbers are multiplied under the square root, we can split the term into two square roots being multiplied.

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

**Problem 39.1**

Simplify.

$$\sqrt{16 \cdot 9}$$

We use the above rule to “split” the square root. That is,  $\sqrt{16 \cdot 9} = \sqrt{16} \cdot \sqrt{9}$ .

But, we know that  $\sqrt{16} = 4$  and that  $\sqrt{9} = 3$ . So, we have

$$\begin{aligned}\sqrt{16 \cdot 9} &= \sqrt{16} \cdot \sqrt{9} \\ &= 4 \cdot 3 \\ &= 12\end{aligned}$$

■

 **Problem 39.2**

Simplify.

$$\sqrt{25 \cdot 16}$$

 **Problem 39.3**

Simplify.

$$\sqrt{36 \cdot 49}$$

**What if one of the numbers is not a perfect square?**

**Problem 39.4**

Simplify.

$$\sqrt{25 \cdot 2}$$


We follow our steps from the previous problem. However, we keep in mind that we do not know the square root of 2. So, we have

$$\begin{aligned}\sqrt{25 \cdot 2} &= \sqrt{25}\sqrt{2} \quad (\text{splitting the square root}) \\ &= 5\sqrt{2} \quad (\text{replacing } \sqrt{25} \text{ by } 5)\end{aligned}$$


Since we do not know the exact value of  $\sqrt{2}$ , we leave it in the final answer as it is.

$$\sqrt{25 \cdot 2} = 5\sqrt{2}$$

■

 **Problem 39.5**  
Simplify.

$$\sqrt{36 \cdot 5}$$

 **Problem 39.6**  
Simplify.

$$\sqrt{3 \cdot 16}$$

### What about Whole Numbers that aren't Perfect Squares?

**Problem 39.7**  
Simplify.

$$\sqrt{12}$$

The first thing we notice is that 12 is not a perfect square. But let's look closer at its factors.

$$\sqrt{12} = \sqrt{\underbrace{4} \cdot \underbrace{3}}$$

We **know** the square root of the first number.

4 is the largest factor of 12 that is a **perfect square**.

We **do not know** the square root of the second number.

We can break 12 into two factors.

$$\begin{aligned} \sqrt{12} &= \sqrt{4 \cdot 3} \\ &= \sqrt{4} \cdot \sqrt{3} && \text{We break the square root into two pieces.} \\ &= 2\sqrt{3} && \text{We replace } \sqrt{4} \text{ with 2.} \end{aligned}$$

In simplified form,

$$\sqrt{12} = 2\sqrt{3}$$



 **Problem 39.8**

Simplify.

$$\sqrt{18}$$

1. Think of a way you can write 18 as the product of two numbers so that the following is true:
  - a. You know how to take the square root of the first number. It's a perfect square.
  - b. The second number is not divisible by a perfect square.

$$\sqrt{18} = \sqrt{(\quad) \cdot (\quad)}$$

2. Now break the square root into two pieces using the two numbers you just found.
  
3. Take the square root of the perfect square.

The final answer is \_\_\_\_\_.

**Finding the best Perfect Squares**

To determine how to break up the square root, you can make a table like we did when learning to factor. First make a list of all pairs of numbers that multiply to the number under the square root. Then find all perfect squares in your list. Use the largest perfect square.

$$\begin{aligned}\sqrt{96} &= \sqrt{16} \cdot \sqrt{6} \\ &= 4\sqrt{6}\end{aligned}$$


If you use a smaller perfect square, you will have to break down the square root again.

$$\begin{aligned}\sqrt{96} &= \sqrt{4} \cdot \sqrt{24} \\ &= 2\sqrt{24} \\ &= 2 \cdot \sqrt{4} \cdot \sqrt{6} \\ &= 2 \cdot 2 \cdot \sqrt{6} \\ &= 4\sqrt{6}\end{aligned}$$


	Multiply to 96	Perfect Squares
1	96	1
2	48	
3	32	
4	24	4
6	16	<b>16</b>
8	12	

1 is never useful for breaking apart square roots.


16 is the largest!

 **Problem 39.9**  
Simplify.


$$\sqrt{8}$$

 **Problem 39.10**  
Simplify.

$$\sqrt{50}$$

 **Problem 39.11**  
Simplify.

$$\sqrt{20}$$

 **Problem 39.12**  
Simplify.

$$\sqrt{48}$$

**Problem 39.13**  
Simplify.

$$3\sqrt{28}$$


Now, we keep 3 outside and simplify the  $\sqrt{28}$ . So, we have

$$\begin{aligned} 3\sqrt{28} &= 3\sqrt{4 \cdot 7} \quad \text{We know the square root of 4.} \\ &= 3\sqrt{4}\sqrt{7} \\ &= 3 \cdot 2\sqrt{7} \quad \text{Replace } \sqrt{4} \text{ by 2. Note the multiplication sign.} \\ &= 6\sqrt{7} \end{aligned}$$


In simplified form,  $3\sqrt{28} = 6\sqrt{7}$






 **Problem 39.14**  
Simplify.


$$3\sqrt{24}$$

 **Problem 39.15**  
Simplify.

$$-6\sqrt{45}$$

 **Problem 39.16**  
Simplify.

$$-2\sqrt{63}$$

 **Problem 39.17**  
Simplify.

$$-\sqrt{200}$$

We can also take the square root of fractions.

**The Quotient Rule**

When two nonnegative numbers are divided under the square root, we can split them into two square roots being divided.

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Since we are not allowed to divide by zero, we assume that  $b$  is not zero.

**Problem 39.18**

Simplify.

$$\sqrt{\frac{4}{9}}$$

We can break this into two square roots.

$$\begin{aligned}\sqrt{\frac{4}{9}} &= \frac{\sqrt{4}}{\sqrt{9}} \\ &= \frac{2}{3}\end{aligned}$$

■

 **Problem 39.19**

Simplify.

$$\sqrt{\frac{16}{25}}$$

 **Problem 39.20**

Simplify.

$$\sqrt{\frac{1}{64}}$$

We now combine what we have learned and apply it to problems that have both, multiplication and division.

**Problem 39.21**

Simplify.

$$\frac{\sqrt{24}}{10}$$

At first, it may be tempting to try to attempt to reduce this fraction since 24 and 10 are both divisible by 2. But since 10 is *not* under a square root, we cannot do this. We must only reduce when *both* numbers are under a square root or when both numbers are not under a square root.

$$\begin{aligned} \frac{\sqrt{24}}{10} &= \frac{\sqrt{4 \cdot 6}}{10} \text{ we simplify the square root first} \\ &= \frac{\sqrt{4}\sqrt{6}}{10} \\ &= \frac{2\sqrt{6}}{10} \text{ The fraction } \frac{2}{10} \text{ can be simplified since neither number is under a square root} \\ &= \frac{\sqrt{6}}{5} \end{aligned}$$

In final simplified form,

$$\frac{\sqrt{24}}{10} = \frac{\sqrt{6}}{5}$$

■

**Problem 39.22**

Simplify.

$$\frac{\sqrt{32}}{4}$$

**Problem 39.23**

Simplify.

$$\frac{5\sqrt{18}}{6}$$

**Problem 39.24**

Simplify.

$$\frac{\sqrt{6}\sqrt{15}}{\sqrt{5}}$$

Here, we can use our Product and Quotient Rules to first bring everything under a single root. So,

$$\frac{\sqrt{6}\sqrt{15}}{\sqrt{5}} = \sqrt{\frac{6 \cdot 15}{5}}$$

Now, we can reduce the fraction inside, since 15 and 5 have a common factor. So,

$$\begin{aligned} \frac{\sqrt{6}\sqrt{15}}{\sqrt{5}} &= \sqrt{\frac{6 \cdot 15}{5}} \\ &= \sqrt{6 \cdot 3} \\ &= \sqrt{18} \\ &= \sqrt{9 \cdot 2} \\ &= \sqrt{9}\sqrt{2} \\ &= 3\sqrt{2} \end{aligned}$$

In final simplified form,  $\frac{\sqrt{6}\sqrt{15}}{\sqrt{5}} = 3\sqrt{2}$ .

■

**Problem 39.25**


Simplify.

$$\frac{\sqrt{7}\sqrt{42}}{\sqrt{3}}$$


**Problem 39.26**

Simplify.

$$\frac{\sqrt{6}\sqrt{18}}{\sqrt{2}}$$

 **Problem 39.27**  
Simplify.

$$\frac{\sqrt{50}}{\sqrt{72}}$$

 **Problem 39.28**  
Simplify.

$$\frac{\sqrt{12}}{\sqrt{48}}$$

## GPS 40: Operations with Square Roots

We will consider problems with addition and subtraction. Here we must take care to only combine terms that have the same square roots.

### Addition and Subtraction of Radicals

If  $a$ ,  $b$  and  $c$  are numbers, then we can add and subtract terms with the same square root.

$$a\sqrt{c} + b\sqrt{c} = (a + b)\sqrt{c}$$

$$a\sqrt{c} - b\sqrt{c} = (a - b)\sqrt{c}$$

This is just like adding and subtracting like-terms.

### Problem 40.1

Simplify.

$$2\sqrt{3} + 5\sqrt{3}$$

Both of these terms have  $\sqrt{3}$  in common. Thus we can add their coefficients.

$$\begin{aligned} 2\sqrt{3} + 5\sqrt{3} &= (2 + 5)\sqrt{3} \\ &= 7\sqrt{3} \end{aligned}$$

■

### Problem 40.2

Simplify.

$$10\sqrt{2} - 6\sqrt{2}$$

### Problem 40.3

Simplify.

$$4\sqrt{5} + 6\sqrt{5} - 7\sqrt{5} + 8\sqrt{5}$$

**Problem 40.4**

Simplify.

$$4\sqrt{5} + 2\sqrt{6} - 7\sqrt{5} + 8\sqrt{6}$$

**Problem 40.5**

Simplify.

$$6\sqrt{7} + 12\sqrt{11} - 5\sqrt{11} + 7\sqrt{7}$$

**Problem 40.6**

Simplify.

$$2\sqrt{18} + 3\sqrt{32}$$

At first glance it appears as though we cannot combine these terms because they are not like terms. However, we can take a moment to simplify each radical.

$$\begin{aligned} 2\sqrt{18} + 3\sqrt{32} &= 2\sqrt{9 \cdot 2} + 3\sqrt{16 \cdot 2} \quad \text{Simplify each term like in the previous section.} \\ &= 2\sqrt{9}\sqrt{2} + 3\sqrt{16}\sqrt{2} \\ &= 2 \cdot 3 \cdot \sqrt{2} + 3 \cdot 4 \cdot \sqrt{2} \\ &= 6\sqrt{2} + 12\sqrt{2} \quad \text{After simplifying, we see these are like terms.} \\ &= 18\sqrt{2} \end{aligned}$$

Our final answer is

$$2\sqrt{18} + 3\sqrt{32} = 18\sqrt{2}$$

■

**Problem 40.7**

Simplify.

$$7\sqrt{12} - 2\sqrt{75}$$

**Problem 40.8**

Simplify.

$$6\sqrt{28} + 2\sqrt{63}$$

**Problem 40.9**

Simplify.

$$2\sqrt{20} + \sqrt{5} - 2\sqrt{45}$$

**Problem 40.10**

Simplify.

$$3\sqrt{24} + 6\sqrt{40} - 2\sqrt{54} - 5\sqrt{90}$$



We remind ourselves that the same rules of multiplication and the distributive property still apply.

**Problem 40.11**

Simplify completely.

$$\sqrt{2}(\sqrt{6} + \sqrt{2})$$

We start by distributing the  $\sqrt{2}$ .

$$\begin{aligned}\sqrt{2}(\sqrt{6} + \sqrt{2}) &= \sqrt{2}\sqrt{6} + \sqrt{2}\sqrt{2} \\ &= \sqrt{12} + \sqrt{4}\end{aligned}$$

But now we must simplify each radical.

$$\begin{aligned}\sqrt{12} + \sqrt{4} &= \sqrt{4 \cdot 3} + 2 \\ &= 2\sqrt{3} + 2\end{aligned}$$

We cannot combine them further. This is the final answer.

$$\sqrt{2}(\sqrt{6} + \sqrt{2}) = 2\sqrt{3} + 2$$

■

 **Problem 40.12**

Simplify completely.

$$\sqrt{6}(\sqrt{8} + \sqrt{6})$$

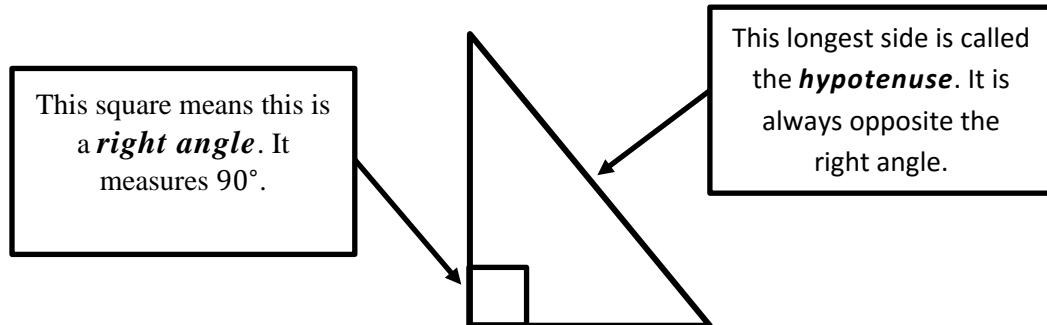
 **Problem 40.13**

Simplify completely.

$$\sqrt{2}(3\sqrt{10} - \sqrt{14})$$

## GPS 41: Pythagorean Theorem

**Right triangles** have one right angle ( $90^\circ$  angle). The side opposite the right angle is called the **hypotenuse**.



There is a special relationship between the sides in a right triangle. This unique property only applies to right triangles. You have almost certainly encountered it before.

**The Pythagorean Theorem**  
In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.

$$a^2 + b^2 = c^2$$

! The hypotenuse is *always* by itself.

We will combine this with one final property of square roots.

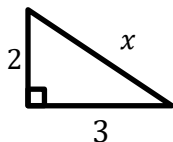
**The Square Root of a Nonnegative Squared**  
If  $x$  is any nonnegative number, then

$$\sqrt{x^2} = x$$

By taking the square root of  $x^2$ , we can get back to  $x$  itself.

**Problem 41.1**

What is the value of  $x$  in the right triangle?



We can use the Pythagorean Theorem with  $a = 2$  and  $b = 3$ . The hypotenuse is  $x$ .

$$a^2 + b^2 = c^2$$

$$2^2 + 3^2 = x^2 \quad \text{We use } x \text{ for the side we do not know.}$$

$$4 + 9 = x^2$$

$$13 = x^2 \quad \text{We now take the square root of both sides.}$$

$$\sqrt{13} = \sqrt{x^2}$$

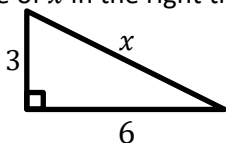
$$\sqrt{13} = x$$

The length of the missing side is  $\sqrt{13}$  units.



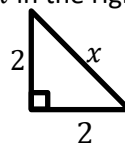
 **Problem 41.2**

What is the value of  $x$  in the right triangle?



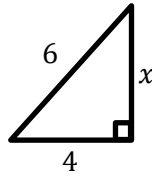
 **Problem 41.3**

What is the value of  $x$  in the right triangle?



**Problem 41.4**

What is the value of  $x$  in the right triangle?



We can use the Pythagorean Theorem with  $a = 4$  and  $c = 6$ .

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 4^2 + x^2 &= 6^2 \\ 16 + x^2 &= 36 \end{aligned}$$

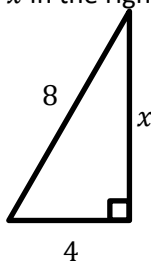
Note that the  $x^2$  is not by itself on one side of the equals sign. We cannot just take the square root. First we must isolate  $x^2$  on one side of the equals sign. To do this, we subtract 16 from both sides.

$$\begin{aligned} 16 + x^2 &= 36 \\ 16 - \mathbf{16} + x^2 &= 36 - \mathbf{16} \\ x^2 &= 20 \quad \text{Now we see the } x^2 \text{ is by itself.} \\ x &= \sqrt{20} \quad \text{We take the square root of both sides.} \\ x &= \sqrt{4 \cdot 5} \quad \text{We should simplify, when possible.} \\ x &= 2\sqrt{5} \end{aligned}$$

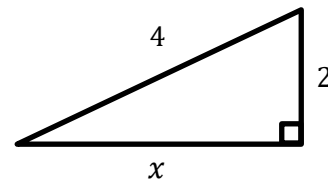
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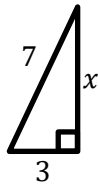
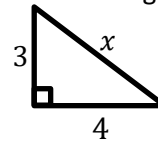
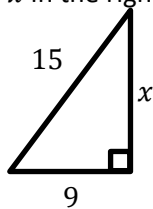
**Problem 41.5**

What is the value of  $x$  in the right triangle?

**Problem 41.6**

What is the value of  $x$  in the right triangle?



**Problem 41.7**What is the value of  $x$  in the right triangle?**Problem 41.8**What is the value of  $x$  in the right triangle?**Problem 41.9**What is the value of  $x$  in the right triangle?**Problem 41.10**What is the value of  $x$  in the right triangle?